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Extremal graphs for the distinguishing index

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The distinguishing index $D'(G)$ of a graph G is the least number of colours in an edge colouring, not necessarily proper, such that the identity is the only automorphism preserving it. This invariant is not defined for graphs having K_2 as a component or more than one isolated vertex, and all other graphs we call admissible. In this paper we consider the following Turán-type problem: given an integer $d \geq 2$, what is a maximum size of an admissible graph G of order n such that $D'(G) > d$? The main result is the following theorem. If $d \geq 2$ and G is an admissible graph of order $n \geq d+4$ and size

$$\|G\| > \binom{n-d-1}{2} + d + 1,$$

then $D'(G) \leq d$ except for a few small graphs of order at most 7. We also exhibit all extremal graphs of order $n \geq 9$, that is, graphs G with $\|G\| = \binom{n-d-1}{2} + d + 1$ and $D'(G) > d$.

1. Introduction and main result

For a simple, finite graph $G = (V, E)$, let $c : E \longrightarrow \{1, \ldots, d\}$ be an edge colouring, not necessarily proper. We say that c *breaks an automorphism* $\varphi \in \mathrm{Aut}(G)$ if φ does not preserve c, that is, there exists an edge that is mapped by φ into an edge with a different colour. If a colouring c breaks every nontrivial automorphism of G, then we say that c is a *distinguishing d-colouring*. Following [6], the least integer d for which a distinguishing d-colouring exists is called the *distinguishing index* of a graph G, and denoted by $D'(G)$. Clearly, $D'(G)$ is defined for all graphs G without K_2 as a component and with at most one isolated vertex. We call such graphs *admissible*, and the set of all of them we denote by $\mathcal{G}$. Investigations of this graph invariant, introduced by us in 2015, have been later undertaken by a dozen of other authors (e.g., in [1,10–12,14]).

There is a general upper bound for the distinguishing index of a connected graph.

Theorem 1.1. (Kalinowski, Pilśniak [6]) *If G is a connected graph of order $n \geq 3$, then*

$$D'(G) \leq \Delta(G)$$

except for three small cycles C_3, C_4, C_5.

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The second author characterized all graphs for which the equality is achieved. A tree is called *symmetric* (respectively, *bisymmetric*) if it has a central vertex v_0 (resp. a central edge e_0), all pendant vertices are at the same distance to v_0 (resp. e_0), and all vertices that are not pendant are of the same degree.

Theorem 1.2. (Pilśniak [13]) *If G is a connected graph, then $D'(G) = \Delta(G)$ if and only if either G is a cycle of length at least six or a symmetric or a bisymmetric tree or K_4 or $K_{3,3}$.*

For complete graphs, it was shown in [6] that $D'(K_n) = 3$ if $n = 3, 4, 5$, and $D'(K_n) = 2$ if $n \geq 6$. This is a consequence of the fact that $D'(K_n) = 2$ if and only if there exists an asymmetric graph (i.e. a graph with a trivial automorphism group) of order n, and such graphs exist for $n \geq 6$. Clearly, if we delete some edges from the complete graph K_n with $n \geq 6$, then the resulting graph may still have the distinguishing index at most two. In this paper, we investigate the following question which is typical in extremal graph theory. What is the maximum size of a connected graph with $D'(G) > 2$? We also answer a more general question for the property $\{G \in \mathcal{G} : D'(G) > d\}$, for any $d \geq 2$. Moreover, we determine extremal graphs, i.e. graphs G with $D'(G) \geq d$ and maximum size.

More formally, we define $f(n, d)$ as the maximum possible size of a graph G of order n with $D'(G) > d$, that is,

$$f(n, d) = \max\{\|G\| \,:\, G \in \mathcal{G},\ |G| = n,\ D'(G) > d\}.$$

In other words, $f(n, d)$ is the least number such that the inequality $\|G\| > f(n, d)$ implies $D'(G) \leq d$.

It follows from Theorem 1.1 that C_3 is the only graph with the distinguishing index equal to its order, hence $f(n, n-1)$ is defined only for $n = 3$. Trivially, $f(n, 1) = \binom{n}{2}$ for $D'(K_n) > 1, n \geq 3$. If $d \geq 2$, then the number $f(n, d)$ is defined for all $n \geq d + 2$ since $D'(K_{1,n-1}) = n - 1 > d$.

It should be noted, however, that the property $\{G \in \mathcal{G} : D'(G) \leq d\}$ is not monotone, as it is typical for Turán-type problems. To see this, consider an asymmetric tree T and insert an edge e between two pendant vertices. Then $D'(T + e) = 2 > D'(T) = 1$. This property is neither increasing since $D'(K_{1,n-1} + e) = n - 3 < D'(K_{1,n-1}) = n - 1$ for $n \geq 5$.

To formulate our main result, we use the following standard notation. Let G_1 and G_2 be two graphs with disjoint vertex sets. By $G_1 \cup G_2$ we denote their *union*, i.e. $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. By $G_1 + G_2$ we denote their *join*, i.e. the union $G_1 \cup G_2$ together with all the edges joining the sets $V(G_1)$ and $V(G_2)$.

Clearly, $D'(K_1 + (K_{n-d-2} \cup \overline{K_{d+1}})) = d + 1$ since every pendant edge has to have a distinct colour (see Fig. 1). The aim of this paper is to prove the following.

Theorem 1.3. (Main result) *Let n, d be a pair of integers such that $n \geq 8$ for $d = 2$, and $n \geq d + 4$ for $d \geq 3$. Then*

$$f(n, d) = \binom{n-d-1}{2} + d + 1.$$

Moreover, $K_1 + (K_{n-d-2} \cup \overline{K_{d+1}})$ is a unique connected graph of order n and size $f(n, d)$ with the distinguishing index greater than d, unless $d = 2$ and $n = 8$. The only disconnected G of order n and size $\|G\| = f(n, d)$ with $D'(G) > d$ is $K_{n-3} \cup K_3$ for $d = 2$ and $n \geq 8$.

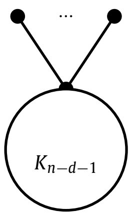

Fig. 1. The extremal graph $K_1 + (K_{n-d-2} \cup \overline{K_{d+1}})$ with the distinguishing index $d + 1$.

Our paper is organized as follows. In Section 2 we recall some known results we use further in the paper. The proof of Theorem 1.3 for $d = 2$ is given in Section 3. Using different arguments, we prove Theorem 1.3 for $d \geq 3$ in Section 4. The sharpness of our main result is discussed in Section 5. In particular, we determine the values of $f(n, d)$ for $n \leq d + 3$. We terminate the last section with a conjecture.

2. Preliminary results

In this section we recall some known results that are used further in the paper.

We say that a graph G is *almost spanned* by a subgraph H if $G - v$ is spanned by H for some $v \in V(G)$. The following results of Pilśniak [13] provide very useful sufficient conditions for a graph to have the distinguishing index at most two.

Lemma 2.1. (Pilśniak [13]) *If a graph G is spanned or almost spanned by an asymmetric subgraph H, then*

$$D'(G) \leq 2.$$

A graph is *traceable* if it contains a Hamilton path.

Theorem 2.2. (Pilśniak [13]) *If G is a traceable graph of order $n \geq 7$, then $D'(G) \leq 2$.*

A graph G is called $K_{1,s}$*-free* if G does not contain $K_{1,s}$ as an induced subgraph. Recently, Gorzkowska, Kargul, Musiał and Pal proved the following theorem in [3].

Theorem 2.3. (Gorzkowska, Kargul, Musiał, Pal [3]) *Let G be a connected graph of order $n \geq 6$ and let $s \geq 3$. If G is $K_{1,s}$-free, then*

$$D'(G) \leq s - 1.$$

Some results from our recent joint paper with Imrich and Woźniak are used in Section 4.

Theorem 2.4. (Imrich, Kalinowski, Pilśniak, Woźniak [5]) *If G is a connected graph without pendant edges, then*

$$D'(G) \leq \left\lceil \sqrt{\Delta(G)} \right\rceil + 1.$$

Given a vertex a of a graph H, by $\mathrm{Aut}(H)_a$ we denote the stabilizer of the vertex a, i.e.

$$\mathrm{Aut}(H)_a = \{\varphi \in \mathrm{Aut}(H) : \varphi(a) = a\}.$$

For two vertices a, b, we denote $\mathrm{Aut}(H)_{a,b} = \mathrm{Aut}(H)_a \cap \mathrm{Aut}(H)_b$.

Lemma 2.5. (Imrich, Kalinowski, Pilśniak, Woźniak [5]) *Let a, b be two vertices of a connected graph H of maximum degree Δ, such that*

$$\mathrm{dist}(a, v) + \mathrm{dist}(v, b) = \mathrm{dist}\,(a, b)$$

for every vertex $v \in V(H)$. Then H admits an edge colouring with $\lceil\sqrt{\Delta}\rceil$ colours breaking every nonidentity automorphism of $\mathrm{Aut}(H)_{a,b}$.

In the proof of Theorem 1.3, we also apply some known results which provide sufficient conditions for the existence of long paths or cycles. The first one is a Turán-type result for traceable graphs.

Theorem 2.6. (Erdős, Gallai [2]) *If G is a graph of order n and*

$$\|G\| > \binom{n-2}{2} + 2,$$

then G is traceable.

Let $c(G)$ denote the *circumference* of a graph G, i.e. the length of a longest cycle in G.

Theorem 2.7. (Woodall [15]) *Let G be a graph of order n and let $1 \leq r \leq \frac{n-1}{2}$. Then $c(G) > n - r$ whenever*

$$\|G\| > \binom{n-r}{2} + \binom{r+1}{2}.$$

An amplification of Woodall's theorem for 2-connected graphs was proved by Kopylov in [9]. It is worth mentioning that in 2016 this result was discussed and strengthened by Füredi, Kostochka and Verstraëte in [4]. Define

$$h(n, c, k) = \binom{c-k}{2} + k(n - c + k).$$

Theorem 2.8. (Kopylov [9]) *Let $4 \leq c \leq n - 1$, and $k = \left\lfloor \frac{c-1}{2} \right\rfloor$. If G is a 2-connected graph of order n such that*

$$\|G\| > \max\{h(n, c, 2), h(n, c, k)\},$$

then $c(G) \geq c$.

Applying the approach of Kemnitz and Schiermeyer [8], the following upper bound for the size of the subgraph induced by vertices of a longest cycle in a nonhamiltonian graph G can be easily derived from the proof of Lemma 14 in [7]. For completeness, we attach the proof.

Proposition 2.9. (Kalinowski, Pilśniak, Schiermeyer, Woźniak [7]) *Let C be a longest cycle in a graph $G=(V,E)$ of order n and let*

$$k=\max\{|N(u)\cap V(C)|\;:\;u\in V(G)\setminus V(C)\}.$$

If $k\geq 2$ and $c(G)<n$, then

$$\|G[V(C)]\|\leq\binom{c(G)}{2}-\frac{k}{2}(c(G)-k-1).$$

Proof. Fix an orientation of C. For two vertices $x,y\in V(C)$, denote by $C[x,y]$ the path of C from x to y along this orientation. If $x\in V(C)$, then x^+ denotes its successor along the orientation of C. For two sets $A,B\subseteq V$, denote $E(A,B)=\{xy\in E\;:\;x\in A,\,y\in B\}$. For a subgraph F of G and $x\in V(G)$, let $d_F(x)=|N(x)\cap V(F)|$.

Let u be a vertex of G outside of C with k neighbours $u_1,\dots,u_k$ on C. Then the set $X=\{u_1^+,\dots,u_k^+\}$ is independent since C is a longest cycle in G. Moreover, for any pair u_i^+,u_j^+ with $i\neq j$ and any $z\in C[u_i^{++},u_j]$ we have $u_i^+z^+\notin E$ or $zu_j^+\notin E$, for the same reason. Let $C_1=C[u_i^{++},u_j]$, $C_2=C[u_j^{++},u_i]$. Then

$$\begin{aligned}&d_C(u_i^+)+d_C(u_j^+)=d_{C_1}(u_i^+)+d_{C_1}(u_j^+)+d_{C_2}(u_i^+)+d_{C_2}(u_j^+)\leq\\&|V(C_1)|+1+|V(C_2)|+1=|V(C)|.\end{aligned}$$

Summing up this inequality for all $\binom{k}{2}$ possible pairs of vertices and dividing by $k-1$ we obtain

$$|E(X,V(C)\setminus X)|=\sum_{i=1}^{k}d_C(u_i^+)\leq\frac{k}{2}c(G).$$

As X is an independent set, we have

$$\begin{aligned}&\|G[V(C)]\|=|E(X,V(C)\setminus X)|+|E(V(C)\setminus X,V(C)\setminus X)|\leq\frac{k}{2}c(G)+\binom{c(G)-k}{2}=\\&\frac{1}{2}(c(G)^2+k^2-kc(G)-c(G)+k)=\binom{c(G)}{2}-\frac{k}{2}(c(G)-k-1).\quad\square\end{aligned}$$

3. Proof of main result for d = 2

We split the proof of Theorem 1.3 into two parts, because the method we apply in the next section for $d\geq 3$ cannot be used directly for $d=2$. In this section, we confirm the case $d=2$ by proving the following.

Theorem 3.1. *If G is a graph of order $n\geq 8$ and size*

$$\|G\|>\binom{n-3}{2}+3,$$

then $D'(G)\leq 2$. Moreover, if $n\geq 9$, then $K_1+(K_{n-4}\cup\overline{K_3})$ and $K_{n-3}\cup K_3$ are the only graphs of order n and size $\binom{n-3}{2}+3$ with the distinguishing index greater than two.

Proof. Let $G=(V,E)$ be an admissible graph of order $n\geq 8$ and size $\|G\|\geq\binom{n-3}{2}+3$.

Clearly, if G is disconnected, then either $G=K_{n-3}\cup K_3$ with $D'(K_{n-3}\cup K_3)=3$, or $G=H\cup K_1$ for some graph H (because the sequence $\binom{k}{2}+\binom{n-k}{2}$, $k=2,\dots,\lceil n/2\rceil$, is strictly decreasing). In the latter case

$$\|H\|\geq\binom{|H|-2}{2}+3,$$

hence, H is traceable by Theorem 2.6, therefore $D'(G)\leq 2$ by Theorem 2.2.

Let then G be connected. In view of Theorem 2.2, we may assume that G is not traceable. Hence, $c(G)\leq n-2$. It is easy to see that Theorem 2.7 implies $c(G)\geq n-4$. Let C be a longest cycle of G and let $v_1,\dots,v_{c(G)}$ be consecutive vertices of C. We consider three cases corresponding to the circumference $c(G)$.

Case $c(G)=n-2$.

Let $V \setminus V(C) = \{u_1, u_2\}$. Without loss of generality, we may assume that $v_1u_1 \in E$. Then the tree $T = C + v_1u_1 - v_3v_4$ is asymmetric because the length of C is at least six. As T is an almost spanning subgraph of G, Lemma 2.1 implies $D'(G) \leq 2$.

Case $c(G) = n - 3$.

Let $V \setminus V(C) = \{u_1, u_2, u_3\}$. If $G[\{u_1, u_2, u_3\}]$ contains an edge, then without loss of generality we may assume that $v_1u_1, u_1u_2 \in E$. The tree $T = C + v_1u_1 + u_1u_2 - v_2v_3$ is an asymmetric almost spanning subgraph of G. Hence, $D'(G) \leq 2$ by Lemma 2.1. Then suppose that the set $\{u_1, u_2, u_3\}$ is independent.

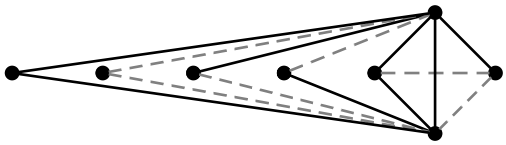

Fig. 2. The graph $K_2 + (K_2 \cup \overline{K_4})$, where dashed edges induce its almost spanning asymmetric tree.

Let $n = 8$ and $\|G\| > \binom{n-3}{2} + 3 = 13$. We have $c(G) = 5$, hence no vertex of C is a cut vertex of G, otherwise G would be traceable. Consequently, G is 2-connected, the set of the three vertices outside C is independent, each of them is adjacent to the same pair of vertices of C because C is a longest cycle of G of length 5. By Proposition 2.9, $G[V(C)]$ has at most $\binom{5}{2} - (5-3) = 8$ edges, so $\|G\| = 14$. It follows that $G = K_2 + (K_2 \cup \overline{K_4})$. This graph has an almost spanning asymmetric tree (see Fig. 2), so $D'(G) \leq 2$. Thus the theorem is true for $n = 8$.

Let now $n \geq 9$ and $\|G\| \geq \binom{n-3}{2} + 3$. Assume first that there are two vertices, say u_2, u_3, such that for every $\varphi \in \mathrm{Aut}(G)$ with $\varphi(u_2) = u_3$, there is a vertex $v \in V(C)$ with $\varphi(v) \neq v$. We colour the edges of the asymmetric tree $T = C - v_3v_4 + v_1u_1$ with colour 1, and all other edges of G with colour 2, thus obtaining a distinguishing 2-colouring of G since u_2, u_3 are the only vertices without incident edges coloured with 1.

Suppose now that every permutation of the vertices u_1, u_2, u_3 can be extended to an automorphism φ of G by setting $\varphi(v) = v$ for every $v \in V(C)$. Consequently, the sets $N(u_i), i \in \{1, 2, 3\}$, coincide and do not contain any two consecutive vertices on C since C is a longest cycle in G. Let $k = d(u_i), i \in \{1, 2, 3\}$.

If $k = 1$, then G is isomorphic to the extremal graph $K_1 + (K_{n-4} \cup \overline{K_3})$ with $D'(G) = 3$, and we are done.

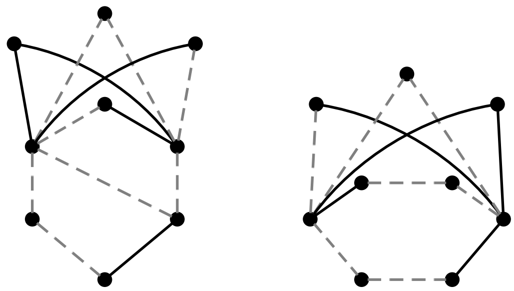

Fig. 3. Two graphs of order 9 and circumference 6 with asymmetric almost spanning subgraphs (dashed).

For $k \in \{2, 3\}$, we use Proposition 2.9. Setting $c(G) = n - 3$, we obtain

$$\|G\| \leq \binom{n-3}{2} - \frac{k}{2}(n - k - 4) + 3k.$$

On the other hand, $\|G\| \geq \binom{n-3}{2} + 3$, therefore $n \leq k + 10 - \frac{6}{k}$. Hence, if $k = 2$, then $n = 9$, and it is easily seen that G has to have a subgraph isomorphic to one of two graphs depicted in Fig. 3, each having an almost spanning asymmetric subgraph. Thus, $D'(G) \leq 2$. If $k = 3$, then $n \leq 11$ by the above inequality. Hence, $6 \leq c(G) \leq 8$ since C is a longest cycle. Therefore, without loss of generality, we may assume that $u_iv_1, u_iv_3 \in E$, $i \in \{1, 2, 3\}$. Let also $u_iv_j \in E, i \in \{1, 2, 3\}$, for some $j \in \{5, \ldots, n-4\}$. In this case, the subgraph $H = C + u_1v_1 + u_2v_1 + u_2v_3 + u_2v_j - v_3v_4$ is an asymmetric almost spanning subgraph of G.

Finally, let $k \geq 4$. Assuming $v_1u_1 \in E$, we colour the edges of the tree $C + v_1u_1 - v_3v_4$ and all edges incident with u_2 with colour 1, while all remaining edges of G get colour 2. The subgraph H coloured with 1 is an almost spanning asymmetric subgraph of G. To see this, it is enough to observe that u_2 is the only vertex of degree at least k of H without an incident pendant edge.

Case $c(G) = n - 4$.

Theorem 2.7 implies that the circumference of G can be equal to $n - 4$ only if $8 \leq n \leq 11$ because $\binom{n-4}{2} + \binom{5}{2} < \binom{n-3}{2} + 3$ for $n \geq 12$.

We infer from Proposition 2.9 that $\|G\| \leq \binom{n-4}{2} - \frac{k}{2}(n - k - 5) + 4k + m_k$, where k is a maximum number of neighbours on C of a vertex outside C, and m_k denotes the number of edges in a subgraph G_0 induced by the four vertices outside the longest cycle C. Clearly, $k \leq \frac{n-4}{2}$ since $c(G) = n-4$, so $k \leq 3$ for $n \leq 11$. Observe that $m_k \leq 3$ since otherwise, by Theorem 2.6, the subgraph G_0, and thus the whole graph G, would be traceable. Moreover, $m_k = 0$ whenever the vertices outside C have common neighbours at distance two on C, and this is always the case for $k=3$ since $n \leq 11$, as well as for $k=2$ and $n=8,9$.

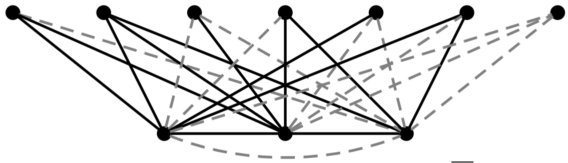

Fig. 4. A distinguishing 2-colouring of $K_3 + \overline{K_7}$.

Furthermore, $\|G\| \geq \binom{n-3}{2} + 3$ by assumption. Thus, we have to check whether these two constraints on the size of G are consistent for six possible cases of a pair n, k. This holds only in two cases. Namely, for $n=8, k=2$, we get a graph $G = K_2 + \overline{K_6}$ (with $D'(G) = 3$, but this does not contradict the theorem), and for $n=10, k=3$, we get a graph $K_3 + \overline{K_7}$ whose distinguishing 2-colouring is shown in Fig. 4. $\square$

4. Proof of main result for d ≥ 3

In this section, we prove Theorem 1.3 for $d \geq 3$, that is, we prove the following.

Theorem 4.1. *Let $d \geq 3$ and let G be an admissible graph of order $n \geq d+4$ and size*

$$\|G\| \geq \binom{n-d-1}{2} + d + 1.$$

Then $D'(G) \leq d$ unless $G = K_1 + (K_{n-d-2} \cup \overline{K_{d+1}})$.

The proof is preceded by three simple observations and a key proposition that settles the claim for 2-connected graphs.

Two edge colourings c_1, c_2 of a graph G are called *nonisomorphic* if for every nontrivial automorphism φ of G, there exist an edge e such that $c_2(\varphi(e)) \neq c_1(e)$.

Lemma 4.2. *If G is a traceable graph of order $n \geq 4$, then G admits at least two nonisomorphic distinguishing 3-colourings of edges.*

Proof. Let P be a Hamilton path of G. We colour one pendant edge of P with colour 1, the other edges of P with 2, and all remaining edges outside of P with colour 3. Clearly, this colouring is distinguishing. Colour 1 is used only once, while colour 2 is used at least twice since the length of P is at least three. To obtain another colouring, we exchange colours 1 and 2. $\square$

Lemma 4.3. *If G is a connected graph of order $n \leq 7$ and without pendant edges, then $D'(G) \leq 3$.*

Proof. Assume first that G is a 2-connected graph of order $n \leq 7$. Clearly, $D'(K_3) = 3$. The claim holds for traceable graphs by Lemma 4.2. If G is not traceable, then $4 \leq c(G) \leq n-2$. Let $v_1, \ldots, v_{c(G)}$ be consecutive vertices of a longest cycle C.

If $c(G) = 4$, then G has two or three vertices outside of C, and they are adjacent to the same pair, say v_1, v_3, of nonconsecutive vertices of C. Thus, there are four or five paths of length 2 joining v_1 to v_3 which we colour with distinct pairs of colours $(1,1), (1,2), (2,2), (1,3), (2,3)$. If $c(G) = 5$, then there are two vertices outside of C, each adjacent either to v_1, v_3 or to v_1, v_4. Thus, there are at most three vertices of degree 2 that can be permuted, and we can easily distinguish them and the vertices of C with three colours.

Suppose that the connectivity of G equals 1. We have just shown that the distinguishing index of each block of G is not greater than 3. The only question appears if there are isomorphic blocks in G. This is possible only if G consists either of two or three triangles, or of two isomorphic blocks of order four: $K_4, K_4 - e, C_4$. It is easy to check that $D'(G) \leq 3$ in each of these cases. $\square$

Lemma 4.4. *If G is a connected graph of order $n \geq 5$, then $D'(G) \leq n-3$ except for $K_{1,n-1}, K_5, C_5$.*

Proof. If $\Delta(G) \leq n-2$, then Theorem 1.1 and Theorem 1.2 imply that $D'(G) < n-2$ unless $G = C_5$.

Let $\Delta(G) = n-1$. Then G is spanned by a star $K_{1,n-1}$ with a central vertex v_1 and pendant vertices $v_2, \ldots, v_n$. For $n=5$, it is an easy exercise to check that each such graph G satisfies $D'(G) \leq 2$ unless $G \in \{K_{1,4}, K_5\}$.

If $n \geq 6$, then we can use Theorem 2.3 to infer that $D'(G) \leq n-3$ whenever G is $K_{1,n-2}$-free. Assume then that $G \neq K_{1,n-1}$ contains an induced star $K_{1,n-2}$ with pendant vertices $v_3, \ldots, v_{n-1}$. Hence, G is isomorphic to a graph obtained from the star $K_{1,n-2}$ by adding p edges $v_2v_3, \dots, v_2v_{p+2}$, where $1 \le p \le n-2$. Suppose first that $p \le n-3$. Then G can be decomposed into two subgraphs $G_1 = G[\{v_1, v_2, \dots, v_{p+2}\}] \cong K_2 + \overline{K_p}$, and $G_2 = G[\{v_1, v_{p+3}, \dots, v_n\}] \cong K_{1,n-p-2}$. No edge of G_1 can be mapped by an automorphism of G onto an edge of G_2, and vice versa, hence we can use the same colours for both subgraphs. By Theorem 2.4,

$$D'(G_1) \le \left\lceil \sqrt{\Delta(G_1)} \right\rceil + 1 = \left\lceil \sqrt{p+1} \right\rceil + 1 \le \left\lceil \sqrt{n-2} \right\rceil + 1 \le n-3.$$

The edges of G_2 can be coloured with $n-2-p \le n-3$ distinct colours. Finally, for $p = n-2$, we have $G \cong K_2 + \overline{K_{n-2}}$ and $G - v_1v_2 \cong K_{2,n-2}$. Again, Theorem 2.4 implies

$$D'(G - v_1v_2) \le \left\lceil \sqrt{\Delta(G - v_1v_2)} \right\rceil + 1 = \left\lceil \sqrt{n-2} \right\rceil + 1 \le n-3.$$

Thus $D'(G) \le n-3$ since a colour of the edge v_1v_2 does not matter. □

Now, we prove Theorem 4.1 for 2-connected graphs.

Proposition 4.5. *If $d \ge 3$ and G is a 2-connected graph of order $n \ge d+4$ and size*

$$\|G\| \ge \binom{n-d-1}{2} + d + 1,$$

then $D'(G) \le d$.

Proof. Let $d \ge 3$ and let $G = (V, E)$ be a graph satisfying the assumptions. The set of colours we use is $\{1, \dots, d\}$. The only 2-connected graph with circumference 3 is K_3, therefore $c(G) \ge 4$ since $n \ge d+4 \ge 7$. On the other hand, if $c(G) \ge n-1$, then G is traceable, hence $D'(G) \le 2$ by Theorem 2.2. Thus we may assume that $4 \le c(G) \le n-2$. Let C be a longest cycle of G.

Suppose first that $c(G) = 4$. Clearly, either $G = K_{2,n-2}$ or $G = K_2 + \overline{K_{n-2}}$ since G is a 2-connected graph without cycles of length 5 or longer. These two graphs differ only by one edge and $\|K_2 + \overline{K_{n-2}}\| = \|K_{2,n-2}\| + 1 = 2n-3$. Each of them contains exactly $n-2$ vertices of degree 2 that can be permuted arbitrarily. If we show that $n-2 < d^2$, then we get a distinguishing d-colouring of G by colouring the pair of edges incident to each vertex of degree 2 with a distinct pair of colours including $(1,2)$ but excluding $(2,1)$ to break also any automorphism switching the two vertices of higher degree. Thus, we have to justify that the inequality $2n-3 \ge \binom{n-d-1}{2} + d + 1$ implies $n-2 < d^2$. Let $a_n = 2n-3$ and $b_n = \binom{n-d-1}{2} + d + 1$. We have $a_{n+1} - a_n = 2$ while $b_{n+1} - b_n = n-d-1 \ge 3$, hence the sequence $a_n - b_n$ is decreasing. Therefore, it suffices to verify that $a_n - b_n < 0$ for $n-2 = d^2$. Indeed,

$$a_{d^2+2} - b_{d^2+2} = -\frac{d}{2}(d^3 - 2d^2 - 2d + 1) < 0$$

since $d^3 - 2d^2 - 2d > 0$ for $d > 1 + \sqrt{3}$.

Assume now that $c(G) \ge 5$. Denote $R = V \setminus V(C)$ and $r = |R|$.

Claim 4.6. $r \le (d-1)^2 + 1$, *that is*, $c(G) \ge n - (d-1)^2 - 1$.

Proof. As $\|G\| \ge \binom{n-d-1}{2} + d + 1$, to this end, due to Theorem 2.8 of Kopylov, it suffices to show that

$$h(n, c, k) < \binom{n-d-1}{2} + d + 1 \tag{1}$$

for $c = n - (d-1)^2 - 1$ and for both $k = 2$ and $k = \lfloor \frac{c-1}{2} \rfloor$. Recall that $h(n, c, k) = \binom{c-k}{2} + k(n-c+k)$.

For $k = 2$, we have

$$h(n, n-(d-1)^2-1, 2) = \binom{n-d^2+2d-4}{2} + 2(d^2 - 2d + 4)$$
$$= \frac{1}{2}\left[n^2 - (2d^2 - 4d + 9)n + d^4 - 4d^3 + 17d^2 - 26d + 36\right].$$

The right-hand side of the inequality (1) equals

$$\binom{n-d-1}{2} + d + 1 = \frac{1}{2}[n^2 - (2d+3)n + d^2 + 5d + 4].$$

Thus, we obtain the inequality

$$-(2d^2-4d+9)n+d^4-4d^3+17d^2-26d+36<-(2d+3)n+d^2+5d+4$$

which holds for

$$n>\frac{d^4-4d^3+16d^2-31d+32}{2(d^2-3d+3)}=\frac{1}{2}(d^2-d+10)+\frac{d+1}{d^2-3d+3}.$$

On the other hand, according to our assumptions,

$$n=r+c(G)\geq(d-1)^2+1+5=d^2-2d+7.$$

Clearly, $\frac{d+1}{d^2-3d+3}\leq\frac{4}{3}$ and $d^2-2d+7>\frac{1}{2}(d^2-d+13)$ for $d\geq 3$ because the quadratic trinomial d^2-3d+1 has two zeros less than 3. Therefore, $\|G\|>h(n,n-(d-1)^2-1,2)$ for every graph G in question.

Take now $k=\lfloor\frac{c-1}{2}\rfloor=\lfloor\frac{n-(d-1)^2-2}{2}\rfloor$. If $c-1$ is even, then

$$\begin{aligned}&h\left(n,c,\left\lfloor\frac{c-1}{2}\right\rfloor\right)=\binom{c-\frac{c-1}{2}}{2}+\frac{c-1}{2}\left(n-c+\frac{c-1}{2}\right)\\&=\frac{1}{8}\left(-c^2+4cn-4n+1\right).\end{aligned}$$

Thus for $c=n-(d-1)^2-1$, we obtain

$$\begin{aligned}&h\left(n,n-d^2+2d-2,\left\lfloor\frac{n-d^2+2d-3}{2}\right\rfloor\right)\\&=\frac{1}{8}\left[-(n-d^2+2d-2)^2+4n(n-d^2+2d-2)-4n+1\right]\\&=\frac{1}{8}\left[3n^2-(2d^2-4d+8)n-d^4+4d^3-8d^2+8d-3\right].\end{aligned}$$

Hence, the inequality (1) holds whenever

$$\begin{aligned}&\frac{1}{8}\left[3n^2-(2d^2-4d+8)n-d^4+4d^3-8d^2+8d-3\right]\\&<\frac{1}{2}\left[n^2-(2d+3)n+d^2+5d+4\right],\end{aligned}$$

that is, when

$$n^2+(2d^2-12d-4)n+d^4-4d^3+12d^2+12d+19>0.$$

This is a quadratic inequality with respect to n which is true for any $d\geq 3$ and any n, because its discriminant $\Delta(d)$ is negative for $d\geq 3$. Indeed,

$$\Delta(d)=(2d^2-12d-4)^2-4(d^4-4d^3+12d^2+12d+19)=4(-8d^3+20d^2+12d-55),$$

which is a decreasing function for $d\geq 3$ since its derivative $\Delta'(d)=16(-6d^2+10d+3)$ has two zeros, both less than 3, hence $\Delta'(d)<0$ for $d\geq 3$. Moreover, $\Delta(3)=-60<0$.

If $c-1$ is odd, then

$$\begin{aligned}&h\left(n,c,\left\lfloor\frac{c-1}{2}\right\rfloor\right)=\binom{c-\frac{c-2}{2}}{2}+\frac{c-2}{2}\left(n-c+\frac{c-2}{2}\right)\\&=\frac{1}{8}\left(-c^2+4cn-8n+2c+8\right)\\&=\frac{1}{8}\left[-(n-d^2+2d-2)^2+4n(n-d^2+2d-2)-8n+2(n-d^2+2d-2)+8\right]\\&=\frac{1}{8}\left[3n^2-(2d^2-4d+10)n-d^4+4d^3-10d^2+12d\right].\end{aligned}$$

The inequality $h\left(n,n-d^2+2d-2,\left\lfloor\frac{n-d^2+2d-3}{s}\right\rfloor\right)<\binom{n-d-1}{2}+d+1$ is equivalent to the following quadratic inequality

$$n^2+(2d^2-12d-2)n+d^4-4d^3+14d^2+8d+16>0.$$

Again, it holds for $d\geq 3$ and any n, because its discriminant $\Delta(d)=4(-8d^3+20d^2+4d-15)$ is negative for $d\geq 3$. To see this, note that $\Delta(3)=-156<0$, and $\Delta(d)$ is a decreasing function for $d\geq 3$ because its derivative $\Delta'(d)=16(-6d^2+10d+1)$ has two zeros less than 3. Thus the Claim is proved. □

To complete the proof of Proposition 4.5, we show the existence of a distinguishing d-colouring of the graph G. Let $v_1 \dots v_{c(G)}$ be consecutive vertices of a longest cycle C.

First, consider the case $c(G) = 5$. Every vertex outside of C is of degree 2 and is adjacent to a pair of vertices of C sharing at least one vertex, say v_1 (see Fig. 5). Therefore, there are at most $n-4$ vertices of degree 2 that can be permuted. Note that $n - 4 \le (d-1)^2 + 1 + 5 - 4 < d^2$, therefore we can colour each pair of edges incident to such a vertex of degree 2 by a different pair of colours including $(1, 2)$, and not using $(2, 1)$.

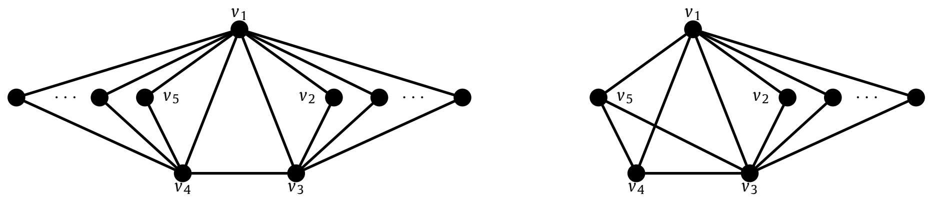

Fig. 5. Every 2-connected graph of circumference $c(G) = 5$ is a subgraph of one of these two graphs.

Suppose now that $c(G) \ge 6$. Without loss of generality, we may assume that there is a vertex $u_1 \in V \setminus V(C)$ adjacent to the vertex v_1 of C. The subgraph $T = C + v_1u_1 - v_3v_4$ is an asymmetric tree. We colour the edges of T with colour d and the remaining edges of $G[V(T)]$ we colour arbitrarily, say with colour 1. We do not use colour d anymore, so each vertex of T will be fixed by every automorphism of G preserving such a colouring.

Each vertex $u \in V \setminus V(T)$ lies on a path between two vertices of T, because G is 2-connected. Let P be a shortest such path, and let $a, b \in V(T)$ be the end vertices of P. Consider the subgraph H of G induced by the edges of all yet uncoloured paths between a and b of the same length as P. Observe that the edge ab, if existed, would be already coloured, hence ab is not an edge of H. If there were a vertex $u \in V(H)$ adjacent to both a and b, then the length of P would equal 2, so the degree of every vertex in $V(H) \setminus \{a, b\}$ would be 2. Consequently, $\Delta(H) \le (d-1)^2$ since there are at most $(d-1)^2$ vertices outside of T. By Lemma 2.5, the subgraph H admits an edge colouring with colours $1, \dots, d-1$ breaking all automorphisms from $\mathrm{Aut}(H)_{a,b}$. Consequently, each vertex of H will be fixed by every automorphism preserving our colouring. We colour each uncoloured edge of $G[V(H) \cup V(T)]$ arbitrarily, say with colour 1.

If $V \setminus (V(H) \cup V(T)) \neq \emptyset$, then we repeat this procedure replacing T with $G[V(H) \cup V(T)]$. Thus we recursively obtain a distinguishing colouring of edges of G. $\square$

To complete the proof of Theorem 4.1 for graphs which are not 2-connected, we make use of the following lemma.

Lemma 4.7. *Let G be a graph of order n such that $\|G\| \ge \binom{n-d-1}{2} + d + 1$ with $d \ge 3$. Let G_1, G_2 be two proper subgraphs of G of orders n_1, n_2, respectively, such that $E(G) = E(G_1) \cup E(G_2)$ and $|V(G_1) \cap V(G_2)| \le 1$. If $n_1 \ge d + 3$, then*

$$\|G_1\| \ge \binom{n_1 - d}{2} + d.$$

Proof. Let G be a graph of order n satisfying the assumptions of the lemma. It follows that either G is disconnected or a unique common vertex of G_1 and G_2 is a cut vertex of G.

If $|V(G_1) \cap V(G_2)| = 1$, then $n_2 = n - n_1 + 1$. If $V(G_1) \cap V(G_2) = \emptyset$, then $n_2 = n - n_1$. Hence in both cases, $\|G_2\| \le \binom{n-n_1+1}{2}$ and $n \ge n_1 + 1$. Consequently, $\|G_1\| \ge \binom{n-d-1}{2} + d + 1 - \binom{n-n_1+1}{2}$. The inequality

$$\binom{n-d-1}{2} + d + 1 - \binom{n-n_1+1}{2} \ge \binom{n_1-d}{2} + d$$

holds for every $n \ge n_1 + 1$. $\square$

Lemma 4.7 asserts that G_1 fulfils the assumptions of Theorem 4.1 with $d-1$ if $|G_1| \ge d+3$. Clearly, the claim holds also for the subgraph G_2 whenever $|G_2| \ge d + 3$.

Proof of Theorem 4.1. Let $G \neq K_1 + (K_{n-d-2} \cup \overline{K_{d+1}})$ be an admissible graph of order $n \ge d+4$ and size $\|G\| \ge \binom{n-d-1}{2} + d + 1$ for some $d \ge 3$. The proof goes by induction on the number s of blocks of G. Observe that G contains at least one 2-connected block because $\|G\| \ge n$. If $s = 1$, then G is a 2-connected graph, so the claim follows directly from Proposition 4.5.

For the induction step, assume that the claim holds for any $d \ge 3$, and for every admissible graph with at most $s-1$ blocks satisfying the assumptions of Theorem 4.1. Let G have $s \ge 2$ blocks. If $\delta(G) = 1$, then G has at most $d+1$ pendant edges, and it is easy to see that there is a pendant vertex u such that $G - u \neq K_1 + (K_{n-d-2} \cup \overline{K_d})$. Let $G_1 = G - u$ and let $G_2 = K_2$ be the pendant edge incident to u. Then G_1 satisfies the assumptions of Lemma 4.7, therefore G_1 has a distinguishing colouring with colours $1, \ldots, d-1$, by the induction hypothesis. We put colour d on the pendant edge incident to u.

Assume then that G does not have pendant edges. We choose a largest possible block G_2 of G, containing at most one cut vertex of G. That is, G_2 is either a pendant block with exactly one cut vertex z of G, or G_2 is a 2-connected component of G. We divide the graph G into two subgraphs G_1, G_2 of orders n_1, n_2 respectively, such that $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$ and the intersection $V(G_1) \cap V(G_2)$ is either an empty set or equals $\{z\}$. Then the number of blocks of each of the subgraphs G_1, G_2 is less than that of G. It follows that, for each $i \in \{1, 2\}$, Lemma 4.7 implies that $\|G_i\| \geq \binom{n_i-d}{2} + d$ unless $n_i \leq d+2$. Therefore, we consider three cases.

Case 1. Let $n_i \geq d+3$ if $d \geq 4$, and $n_i \geq 8$ if $d = 3$, for both $i \in \{1, 2\}$. Then $\|G_i\| \geq \binom{n_i-d}{2} + d, i \in \{1, 2\}$, by Lemma 4.7. If $d \geq 4$, then $D'(G_i) \leq d-1, i \in \{1, 2\}$, by the induction hypothesis. Also, if $d = 3$, then $D'(G_i) \leq d-1 = 2, i \in \{1, 2\}$, by Theorem 3.1. We colour the edges of G_1 with colours $1, \ldots, d-1$ and we necessarily use colour d in a colouring of the block G_2. Thus we obtain a distinguishing colouring of the whole graph G with d colours, breaking also a possible isomorphism of the block G_2 and a subgraph of G_1.

Case 2. Let $d = 3$ and $n_j \leq 7$ for some $j \in \{1, 2\}$. Thus we cannot use Theorem 3.1 for the graph G_j in the induction step. For each $i \in \{1, 2\}$, we have $D'(G_i) \leq 3$, due to Lemma 4.3 if $n_i \leq 7$, or due to Theorem 3.1 if $n_i \geq 8$ (when we even have $D'(G_i) \leq 2$). If $D'(G_1) \neq D'(G_2)$, then G has a distinguishing 3-colouring, where colour 3 appears exclusively in the subgraph with the larger distinguishing index.

Only if $n_i \leq 7$ for both $i \in \{1, 2\}$, it is possible that $D'(G_1) = D'(G_2) = 3$. In this case, we have to break an automorphism of G which exchanges the block G_2 with an isomorphic block of G_1, if it exists. The constraints for n_i and for the size of G easily imply that G_1 contains two blocks isomorphic to G_2 only if $G = K_1 + 3K_2$, i.e. G consists of three triangles sharing a vertex. Clearly, $D'(K_1 + 3K_2) = 3$. Otherwise, G_1 contains at most one block G_2' isomorphic to G_2. The only possible case when G_2' is a proper subgraph of G_1 appears if $G_2 = K_4$ and G_1 consists of K_4 and K_3 with a common vertex. But K_4 has two nonisomorphic distinguishing 3-colourings due to Lemma 4.2, and we are done. Finally, suppose that G_1 and G_2 are isomorphic. It follows from Theorem 2.6 that they are traceable, because $n \geq 2n_1 - 1$ and $2\left[\binom{n_1-2}{2} + 2\right] < \binom{2n_1-5}{2} + 4$ for $n_1 \leq 7$. Again, we infer from Lemma 4.2 that $D'(G) \leq 3$.

Consequently, Theorem 4.1 holds for $d = 3$.

Case 3. Let $d \geq 4$ and $n_j \leq d+2$ for some $j \in \{1, 2\}$. Then $D'(G_j) \leq d-1$ by Lemma 4.4 for $n_j \geq 5$, or by Theorem 1.1 for $n_j \leq 4$. Analogously, $D'(G_i) \leq d-1$, for $i \neq j$, if $n_i \leq d+2$. Also, if $n_i \geq d+3$ then $D'(G_i) \leq d-1$ by the induction hypothesis. Therefore, we can distinguishingly colour the graph G with d colours, using colour d only in G_2.

This completes the proof of Theorem 4.1, and thus of Theorem 1.3. $\square$

5. Remarks

In this section, we discuss the relevance of the assumptions of Theorem 1.3, that is, we determine the values $f(n, d)$ not covered by our result. We end with a conjecture.

Let $d = 2$. We clearly have $f(n, 2) = \|K_n\| = \binom{n}{2}$ for $n = 3, 4, 5$. Further, observe that $D'(K_2 + \overline{K_{n-2}}) = 3$ for $n = 6, 7, 8$, hence $f(n, 2) \geq 2n - 3 > \binom{n-3}{2} + 3$ for $n = 6, 7$, and $K_{n-4} + K_1 + \overline{K_3}$ is not a unique connected graph of order $n = 8$ and size $\binom{n-3}{2} + 3$ with the distinguishing index greater than two.

Let $d \geq 3$. Trivially, $f(n, n-2) = \|K_n\| = \binom{n}{2}$ for $n = 3, 4$, and $f(n, n-2) = \|K_{1,n-1}\| = n-1$, for $n \geq 5$.

For $d = n-3$, it follows from Lemma 4.4 that $f(5, 2) = \|K_5\| = 10$, $f(6, 3) = \|K_3 \cup K_3\| = 6$, and $f(n, n-3) = \|K_{1,n-1}\| = n-1$ for $n \geq 7$. This is why we assume $d \leq n-4$ in Theorem 1.3.

We terminate with the following question. What is a maximum size of a graph G with $D'(G) > d$ and with bounded minimum degree? In other words, for suitable positive integers n, δ, d, we search for the value

$$g(n, d, \delta) = \max\{\|G\| : G \in \mathcal{G}, \ |G| = n, \ \delta(G) \geq \delta, \ D'(G) > d\}.$$

The vertex set of the graph $K_\delta + (K_{n-d^\delta-\delta-1} \cup \overline{K_{d^\delta+1}})$ consists of a clique K of order $n - d^\delta - 1$ and an independent set A of $d^\delta + 1$ vertices, each adjacent to the same set $v_1, \ldots, v_\delta$ of vertices of the clique K. For each vertex $u \in A$, the sequence $(v_1 u, \ldots, v_\delta u)$ of edges incident to u has to be coloured by a distinct sequence of colours in a distinguishing colouring. With d colours, we have only d^δ such sequences. Hence, $K_\delta + (K_{n-d^\delta-\delta-1} \cup \overline{K_{d^\delta+1}}) > d$. We believe that this is a unique connected extremal graph, whence we believe that the following conjecture is true.

Conjecture 5.1. *Let $d \geq 2$, $\delta \geq 2$ and $n \geq d^\delta + \delta + 1$. Then*

$$g(n, d, \delta) = \binom{n - d^\delta - 1}{2} + \delta(d^\delta + 1).$$

Moreover, $K_\delta + (K_{n-d^\delta-\delta-1} \cup \overline{K_{d^\delta+1}})$ is a unique connected graph of order n, minimum degree at least δ, size $g(n, d, \delta)$, and the distinguishing index greater than d.

Declaration of competing interest

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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