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Paired domination stability in graphs

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Abstract

A set S of vertices in a graph G is a paired dominating set if every vertex of G is adjacent to a vertex in S and the subgraph induced by S contains a perfect matching (not necessarily as an induced subgraph). The paired domination number, $\gamma_{\rm pr}(G)$, of G is the minimum cardinality of a paired dominating set of G. A set of vertices whose removal from G produces a graph without isolated vertices is called a non-isolating set. The minimum cardinality of a non-isolating set of vertices whose removal decreases the paired domination number is the $\gamma_{\rm pr}^-$-stability of G, denoted $\mathrm{st}_{\gamma_{\rm pr}}^-(G)$. The paired domination stability of G is the minimum cardinality of a non-isolating set of vertices in G whose removal changes the paired domination number. We establish properties of paired domination stability in graphs. We prove that if G is a connected graph with $\gamma_{\rm pr}(G) \geq 4$, then $\mathrm{st}_{\gamma_{\rm pr}}^-(G) \leq 2\Delta(G)$ where $\Delta(G)$ is the maximum degree in G, and we characterize the infinite family of trees that achieve equality in this upper bound.

Keywords: Paired domination, paired domination stability.

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1 Introduction

In 1983 Bauer, Harary, Nieminen and Suffel [3] introduced and studied the concept of domination stability in graphs. Stability for other domination type parameters has been studied in the literature. For example, total domination stability, 2-rainbow domination stability, exponential domination stability, Roman domination stability are studied in [1, 2, 12, 15, 16], among other papers. In this paper we study the paired version of domination stability.

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Two vertices u and v are *neighbors* if they are adjacent, that is, if $uv \in E$. A *dominating set* of G is a set D of vertices such that every vertex in $V(G) \setminus D$ has a neighbor in D. The minimum cardinality of a dominating set is the *domination number*, $\gamma(G)$, of G. Domination is well studied in the literature. A recent book on domination in graphs can be found in [10]. A small sample of papers on domination critical graphs can be found in [3, 4, 5, 6, 9, 17, 18]. Adopting the notation coined by Bauer et al. [3], the γ^--*stability* (γ^+-*stability*, resp.) of G, denoted by $\gamma^-(G)$ ($\gamma^+(G)$, resp.), is the minimum number of vertices whose removal decreases (increases, resp.) the domination number. The minimum number of vertices whose removal decreases or increases the domination number is the *domination stability*, $\mathrm{st}_\gamma(G)$, of G, and so $\mathrm{st}_\gamma(G) = \min\{\gamma^-(G), \gamma^+(G)\}$.

We refer to a graph without isolated vertices as an *isolate-free graph*. Unless otherwise stated, let G be an isolate-free graph. A *total dominating set*, abbreviated TD-set, of G is a set D of vertices of G such that every vertex, including vertices in the set D, has a neighbor in D. The minimum cardinality of a TD-set of G is the *total domination number*, $\gamma_t(G)$, of G. We call a TD-set of G of cardinality $\gamma_t(G)$ a γ_t-*set of* G. A vertex v is *totally dominated* by a set D in G if the vertex v has a neighbor in D. We refer the reader to the book [14] for fundamental concepts on total domination in graphs. Total domination critical graphs are studied, for example, in [7, 13]. The total version of domination stability was first studied by Henning and Krzywkowski [12].

A *paired dominating set*, abbreviated PD-set, of an isolate-free graph G is a dominating set S of G with the additional property that the subgraph $G[S]$ induced by S contains a perfect matching M (not necessarily induced). With respect to the matching M, two vertices joined by an edge of M are *paired* and are called *partners* in S. The *paired domination number*, $\gamma_{\mathrm{pr}}(G)$, of G is the minimum cardinality of a PD-set of G. We call a PD-set of G of cardinality $\gamma_{\mathrm{pr}}(G)$ a γ_{pr}-*set of* G. We note that the paired domination number $\gamma_{\mathrm{pr}}(G)$ is an even integer. For a recent survey on paired domination in graphs, we refer the reader to the book chapter [8].

Every PD-set is a TD-set, implying that $\gamma_t(G) \le \gamma_{\mathrm{pr}}(G)$. A *non-isolating set* of vertices in G is a set $S \subseteq V$ such that the graph $G - S$ is isolate-free, where $G - S$ is the graph obtained from G by removing S and all edges incident with vertices in S. Let $\mathrm{NI}(G)$ denote the set of all non-isolating sets of vertices of G.

Adopting the standard notation for domination stability given in [3, 12], the γ_{pr}^--stability

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(resp., $\gamma^+_{\rm pr}$-stability) of G, denoted by $\mathrm{st}^-_{\gamma_{\rm pr}}(G)$ (resp., $\mathrm{st}^+_{\gamma_{\rm pr}}(G)$) is the minimum cardinality of a set in $\mathrm{NI}(G)$ whose removal decreases (increases, resp.) the paired domination number. Thus,

$$\mathrm{st}^-_{\gamma_{\rm pr}}(G) = \min_{S\in \mathrm{NI}(G)} \{\, |S| \colon \gamma_{\rm pr}(G-S) < \gamma_{\rm pr}(G)\}$$

and

$$\mathrm{st}^+_{\gamma_{\rm pr}}(G) = \min_{S\in \mathrm{NI}(G)} \{\, |S| \colon \gamma_{\rm pr}(G-S) > \gamma_{\rm pr}(G)\}.$$

If there is no set in $\mathrm{NI}(G)$ whose removal increases the paired domination number, then we define $\mathrm{st}^+_{\gamma_{\rm pr}}(G) = \infty$. For example, $\mathrm{st}^-_{\gamma_{\rm pr}}(P_5) = 1$ while $\mathrm{st}^+_{\gamma_{\rm pr}}(P_5) = \infty$. The *paired domination stability*, $\mathrm{st}_{\gamma_{\rm pr}}(G)$, of G is the minimum cardinality of a set in $\mathrm{NI}(G)$ whose removal increases or decreases the paired domination number. Thus,

$$\mathrm{st}_{\gamma_{\rm pr}}(G) = \min_{S\in \mathrm{NI}(G)} \{\, |S| \colon \gamma_{\rm pr}(G-S) \neq \gamma_{\rm pr}(G)\} = \min\{\mathrm{st}^-_{\gamma_{\rm pr}}(G), \mathrm{st}^+_{\gamma_{\rm pr}}(G)\}.$$

Let G be a graph and let $S \in \mathrm{NI}(G)$. If $\gamma_{\rm pr}(G-S) < \gamma_{\rm pr}(G)$ and $|S| = \mathrm{st}^-_{\gamma_{\rm pr}}(G)$, then we call S a $\mathrm{st}^-_{\gamma_{\rm pr}}$*-set of* G. If $\gamma_{\rm pr}(G-S) > \gamma_{\rm pr}(G)$ and $|S| = \mathrm{st}^+_{\gamma_{\rm pr}}(G)$, then we call S a $\mathrm{st}^+_{\gamma_{\rm pr}}$*-set of* G. If $\gamma_{\rm pr}(G-S) \neq \gamma_{\rm pr}(G)$ and $|S| = \mathrm{st}_{\gamma_{\rm pr}}(G)$, then we call S a $\mathrm{st}_{\gamma_{\rm pr}}$*-set of* G.

Defining the *null graph* K_0, which has no vertices, as a graph, we have the following results due to Bauer et al. [3] and Rad et al. [15] for the γ^--stability of a graph.

Theorem 1.1 ([3, 15]). *If G is an isolate-free graph of order n, then the following holds.*

(a) $\mathrm{st}_\gamma(G) \le \delta(G) + 1$.

(b) *If* $G \not\cong K_n$, *then* $\mathrm{st}_\gamma(G) \le n - 1$.

Considering the null graph, the paired domination stability of a non-trivial graph is always defined. If G is a graph of order n and $\gamma_{\rm pr}(G) = 2$, then $\mathrm{st}^-_{\gamma_{\rm pr}}(G) = n$ since removing all vertices from the graph G produces the null graph with paired domination number zero.

For notation and graph theory terminology we generally follow [14]. In particular, for $r, s \ge 1$, a *double star* $S(r, s)$ is the tree with exactly two vertices that are not leaves, one of which has r leaf-neighbors and the other s leaf-neighbors. A *rooted tree* is a tree T in which we specify one vertex r called the *root*. For each vertex v of T different from r, its *parent* is the neighbor of v on the unique (r, v)-path, while every other neighbor of v is a *child* of v in T. If w is a vertex of T different from v and the (unique) (r, w)-path contains v, then w is a *descendant* of v in T. We note that every child of v is a descendant of v. The *diameter* $\mathrm{diam}(G)$ of G is the maximum distance among all pairs of vertices of G. A *diametral path* in G is a shortest path between two vertices in G of length equal to $\mathrm{diam}(G)$. For an integer $k \ge 1$, $[k] = \{1, \ldots, k\}$.

2 Main results

Our first aim is to show that the paired domination stability of a graph can be very different from its total domination stability studied in [12].

Theorem 2.1. *For $k \ge 1$ an arbitrary integer, the following holds.*

(a) *There exist connected graphs G such that* $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) - \mathrm{st}^-_{\gamma_t}(G) = k$.

(b) *There exist connected graphs H such that* $\mathrm{st}^-_{\gamma_t}(H) - \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(H) = k$.

Our second aim is to establish properties of paired domination stability in graphs. Thereafter, we establish upper bounds on the paired domination stability and the γ^-_{pr}-stability of a graph. For this purpose, we shall need the following family of trees defined by Henning and Krzywkowski [12]. For integers $k \geq 2$ and $\Delta \geq 2$, the authors in [12] define $T_{k,\Delta}$ as the "graph obtained from the disjoint union of k double stars $S(\Delta-1, \Delta-1)$ by adding $k-1$ edges between the leaves of these double stars so that the resulting graph is a tree with maximum degree Δ." Let $\mathcal{F}_{k,\Delta}$ be the family of all such trees $T_{k,\Delta}$, and let

$$\mathcal{F}_\Delta = \bigcup_{k\geq 2} \mathcal{F}_{k,\Delta}.$$

The following result establishes an upper bound on the γ^-_{pr}-stability of a tree, and characterizes the trees with maximum possible γ^-_{pr}-stability.

Theorem 2.2. *If T is a tree with maximum degree Δ satisfying $\gamma_{\mathrm{pr}}(T) \geq 4$, then the following hold.*

(a) $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \leq 2\Delta$, *with equality if and only if* $T \in \mathcal{F}_\Delta$.

(b) $\mathrm{st}_{\gamma_{\mathrm{pr}}}(T) \leq 2\Delta - 1$, *and this bound is sharp for all* $\Delta \geq 2$.

For general graphs, we establish the following upper bound on the γ^-_{pr}-stability in terms of the maximum degree of the graph.

Theorem 2.3. *If G is a connected graph with $\gamma_{\mathrm{pr}}(G) \geq 4$, then $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) \leq 2\Delta(G)$, and this bound is sharp.*

As an immediate consequence of Theorem 2.3, we have the following upper bound on the paired domination stability of a graph.

Corollary 2.4. *If G is a connected graph with $\gamma_{\mathrm{pr}}(G) \geq 4$, then $\mathrm{st}_{\gamma_{\mathrm{pr}}}(G) \leq 2\Delta(G)$.*

3 Paired stability versus domination and total stability

In this section, we show that paired domination stability and the domination stability of a graph can be very different. By Theorem 1.1, for every nontrivial graph G, we have $\mathrm{st}_\gamma(G) \leq \delta(G) + 1$. In particular, $\mathrm{st}_\gamma(T) \leq 2$ for every nontrivial tree T. This is in contrast to the paired domination stability, where for any given $\Delta \geq 2$, we show that there exist a family of trees T with maximum degree Δ satisfying $\mathrm{st}_{\gamma_{\mathrm{pr}}}(T) = 2\Delta - 1$.

For $\Delta = 2$, the authors in [12] define $\mathcal{H}_\Delta$ as the family of all paths of order at least 7 and congruent to 3 modulo 4, that is, $\mathcal{H}_\Delta = \{P_n \mid n \equiv 3 \pmod 4 \text{ and } n \geq 7\}$. For integers $\Delta \geq 3$ and $\Delta \geq k \geq 2$, they define $H_{k,\Delta}$ as the graph "obtained from the disjoint union of k double stars $S(\Delta-1, \Delta-1)$ by selecting one leaf from each double star and identifying these k leaves into one new vertex" and they define the family

$$\mathcal{H}_\Delta = \bigcup_{k\geq 2} H_{k,\Delta}.$$

We determine next the paired domination stability of a tree in the family $\mathcal{H}_\Delta$.

Proposition 3.1. *For $\Delta \ge 3$, if $T \in \mathcal{H}_\Delta$, then* $\mathrm{st}_{\gamma_{\rm pr}}(T) = 2\Delta - 1$.

Proof. For integers $\Delta \ge k \ge 2$ where $\Delta \ge 3$, consider a tree $T \in \mathcal{H}_{k,\Delta}$. By definition of the family $\mathcal{H}_{k,\Delta}$, the tree T is constructed from the disjoint union of k double stars $S_1, \dots, S_k$, each isomorphic to $S(\Delta-1, \Delta-1)$, by selecting one leaf from each double star and identifying these k chosen leaves into one new vertex, which we call v_c. Let x_i and y_i be the two central vertices of the double star S_i for $i \in [k]$, where x_i is adjacent to v_c in T. Let $D = \cup_{i=1}^{k}\{x_i, y_i\}$. Since $\Delta \ge 3$, every vertex in D is a support vertex of T, implying that every PD-set in T contains the set D and therefore $\gamma_{\rm pr}(T) \ge |D| = 2k$. Since the set D is a PD-set of T (with the vertices x_i and y_i paired for all $i \in [k]$), we have $\gamma_{\rm pr}(T) \le |D| = 2k$. Consequently, $\gamma_{\rm pr}(T) = 2k$ and D is the unique $\gamma_{\rm pr}$-set of T.

Let S be a $\mathrm{st}_{\gamma_{\rm pr}}$-set of T. Thus, S is a set in $\mathrm{NI}(T)$ with $|S| = \mathrm{st}_{\gamma_{\rm pr}}(T)$ satisfying $\gamma_{\rm pr}(T - S) \ne \gamma_{\rm pr}(T) = 2k$. We show that $|S| \ge 2\Delta - 1$. Suppose, to the contrary, that $|S| \le 2\Delta - 2$. If the set S contains both x_i and y_i for some $i \in [k]$, then since S is a non-isolating set of T every leaf neighbor of x_i and y_i is also in S, implying that $|S| \ge 2\Delta - 1$, a contradiction. Hence, the set S contains at most one of x_i and y_i for every $i \in [k]$. Let D^* be a $\gamma_{\rm pr}$-set of $T - S$, and so $|D^*| \ne 2k$.

Suppose that $v_c \in S$. In this case, if $|S| = 1$, then the paired domination numbers of T and $T - S$ are the same, a contradiction. Hence, $|S| \ge 2$. If neither x_i nor y_i belong to S for some $i \in [k]$, then by the minimality of the non-isolating set S, no vertex of T_i different from v_c belongs to S, and so $|D^* \cap V(T_i)| = 2$. If S contains y_i but not x_i for some $i \in [k]$, then every leaf neighbor of y_i is in S and by the minimality of the set S, no leaf neighbor of x_i belongs to S, and so $|D^* \cap V(T_i)| = 2$. Analogously, if S contains x_i but not y_i for some $i \in [k]$, then $|D^* \cap V(T_i)| = 2$. This is true for all $i \in [k]$, implying that $|D^*| = \sum_{i=1}^{k} |D^* \cap V(T_i)| = 2k$, a contradiction. Hence, $v_c \notin S$.

As observed earlier, the set S contains at most one of x_i and y_i for every $i \in [k]$. If $y_i \in S$ and $y_j \in S$ for some $i, j \in [k]$ where $i \ne j$, then $|S| \ge 2\Delta$, a contradiction. If $y_i \in S$ and $x_j \in S$ for some $i, j \in [k]$ where $i \ne j$, then $|S| \ge 2\Delta - 1$, a contradiction. If $x_i \in S$ and $x_j \in S$ for some $i, j \in [k]$ where $i \ne j$, then $|S| \ge 2\Delta - 2$. In this case, by the minimality of S we have $S = (N[x_i] \cup N[x_j]) \setminus \{v_c, y_i, y_j\}$ and $|S| = 2\Delta - 2$. But then $T - S$ consists of three components, namely two stars isomorphic to $K_{1,\Delta-1}$ and one component belonging to the family $T \in \mathcal{H}_{k-2,\Delta}$ with paired domination number $2(k-2)$. Thus, $\gamma_{\rm pr}(T - S) = 2 + 2 + 2(k-2) = 2k$, a contradiction. Therefore, $\mathrm{st}_{\gamma_{\rm pr}}(T) = |S| \ge 2\Delta - 1$, as claimed.

Conversely, if we take $S = N(x_1) \cup N(y_1) \setminus \{v_c\}$, then $S \in \mathrm{NI}(T)$ and $T - S \in \mathcal{H}_{k-1,\Delta}$. Thus, $\gamma_{\rm pr}(T - S) = 2(k-1) < \gamma_{\rm pr}(T)$, and so $\mathrm{st}_{\gamma_{\rm pr}}(T) \le \mathrm{st}^-_{\gamma_{\rm pr}}(T) \le |S| = 2\Delta - 1$. Consequently, $\mathrm{st}_{\gamma_{\rm pr}}(T) = \mathrm{st}^-_{\gamma_{\rm pr}}(T) = 2\Delta - 1$. □

As observed earlier, $\mathrm{st}_{\gamma}(T) \le 2$ for every nontrivial tree T. By Proposition 3.1, paired domination stability therefore differs significantly from domination stability. We show next that the paired domination stability and the total domination stability of a graph can also be very different.

Proposition 3.2. *For $k \ge 1$ an integer, there exist trees T such that* $\mathrm{st}^-_{\gamma_{\rm pr}}(T) - \mathrm{st}^-_{\gamma_t}(T) = k$.

Proof. Let $k \ge 1$ be a given integer, and let $T = T_k$ be obtained from a path P_5 given by $v_1v_2v_3v_4v_5$ by attaching k leaf neighbors to each of v_1, v_2 and v_3 (see Figure 1). We note that $\{v_1, v_2, v_3, v_4\}$ is the unique γ_t-set of T and the unique $\gamma_{\rm pr}$-set of T. In particular, $\gamma_t(T) = \gamma_{\rm pr}(T) = 4$. If $S = \{v_5\}$, then the set S is a non-isolating set of T and $\gamma_t(T - S) = |\{v_1, v_2, v_3\}| = 3 < \gamma_t(T)$, implying that $\mathrm{st}^-_{\gamma_t}(T) = 1$.

We show next that $\mathrm{st}^-_{\gamma_{\rm pr}}(T) = k + 1$. Let S be a non-isolating set of T such that $\gamma_{\rm pr}(T - S) < \gamma_{\rm pr}(T)$. We show that $|S| \ge k + 1$. Suppose, to the contrary, that $|S| \le k$. Let D be a $\gamma_{\rm pr}$-set of $T - S$, and so $|D| = \gamma_{\rm pr}(T - S) = 2$. Let L_i denote the set of leaf neighbors of v_i for $i \in [4]$. If $v_i \in S$ for some $i \in [3]$, then S contains all k leaf neighbors of v_i, and so $|S| \ge k + 1$, a contradiction. Hence, $S \cap \{v_1, v_2, v_3\} = \emptyset$. If $\{v_1, v_3\} \subset D$, then $|D| \ge 4$, a contradiction. If $v_1 \notin D$, then $L_1 \subseteq S$, implying that $S = L_1$ and $|S| = k$. However in this case, $\{v_2, v_3, v_4\} \subset D$. If $v_3 \notin D$, then $L_3 \subseteq S$, implying that $S = L_3$ and $|S| = k$. However in this case, $\{v_1, v_2, v_4\} \subset D$. In both cases, $|D| \ge 4$, a contradiction. Therefore, $|S| \ge k + 1$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \ge k + 1$. Conversely, if $S = L_1 \cup L_4$, then S is a non-isolating set of T such that $\gamma_{\rm pr}(T - S) = |\{v_2, v_3\}| < \gamma_{\rm pr}(T)$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \le |S| = k + 1$. Consequently, $\mathrm{st}^-_{\gamma_{\rm pr}}(T) = k + 1$. Thus, $\mathrm{st}^-_{\gamma_{\rm pr}}(T) - \mathrm{st}^-_{\gamma_t}(T) = k$. □

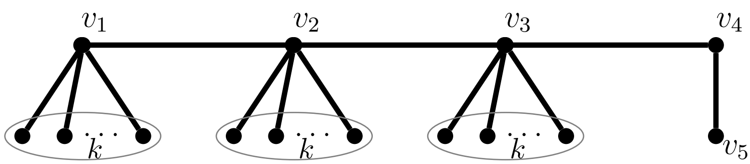

Figure 1: A tree from the family T_k in the proof of Proposition 3.2.

Proposition 3.3. *For $k \ge 1$ an integer, there exist trees T such that $\mathrm{st}^-_{\gamma_t}(T) - \mathrm{st}^-_{\gamma_{\rm pr}}(T) = k$.*

Proof. Let $k \ge 1$ be a given integer, and let $\ell \ge 2k + 1$ be an integer. For $i \in [k]$, let Q_i be obtained from a path $v_{i_1}v_{i_2}v_{i_3}v_{i_4}v_{i_5}$ by attaching ℓ leaf neighbors to each of v_{i_3}, v_{i_4} and v_{i_5}, and let L_{i_3}, L_{i_4} and L_{i_5} be the resulting sets of leaf neighbors of v_{i_3}, v_{i_4} and v_{i_5}, respectively. Let Q be obtained from a path $v_1v_2v_3$ by attaching ℓ leaf neighbors to each of v_1 and v_2, and attaching k leaf neighbors to v_3. Let L_i be the resulting set of leaf neighbors of v_i for $i \in [3]$. Let T be obtained from the disjoint union of the paths $Q, Q_1, \dots, Q_k$ by adding the k edges $v_3v_{i_1}$ for $i \in [k]$. Let A be the set of support vertices of T, and so $|A| = 3(k + 1)$.

Every TD-set of T contains all its support vertices, implying that $\gamma_t(T) \ge |A|$. Since the set A is a TD-set of T, we have $\gamma_t(T) \le |A|$. Consequently, $\gamma_t(T) = |A| = 3(k + 1)$. Every PD-set of T contains the set A and at least one additional vertex from each path Q_i that is a neighbor of v_{i_3} or v_{i_5} for $i \in [k]$, and at least one additional vertex that is a neighbor of v_1 or v_3 since the vertices of every PD-set are paired, implying that $\gamma_{\rm pr}(T) = |A| + k + 1 = 4(k + 1)$.

Let S be a non-isolating set of T such that $\gamma_{\rm pr}(T - S) < \gamma_{\rm pr}(T)$. If $|S| < k$, then every support vertex of T remains a support vertex of $T - S$, implying that $\gamma_{\rm pr}(T - S) \ge \gamma_{\rm pr}(T)$, a contradiction. Hence, $|S| \ge k$. Conversely, if $S^* = L_3$, then the set $A \setminus \{v_3\}$ of all support vertices of $T - S^*$, together with the vertices v_{i_2} for $i \in [k]$, form a PD-set of $T - S^*$, implying that $\gamma_{\rm pr}(T - S^*) \le 4k + 2 < 4k + 4 = \gamma_{\rm pr}(T)$. Hence, $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \le |S^*| = k$. Consequently, $\mathrm{st}^-_{\gamma_{\rm pr}}(T) = k$.

We show next that $\mathrm{st}^-_{\gamma_t}(T) = 2k$. Let $A' = A \setminus \{v_3\}$, and so $|A'| = |A| - 1 = 3k + 2$. Let S be a non-isolating set of T such that $\gamma_t(T - S) < \gamma_t(T)$. We show that $|S| \ge 2k$. Suppose, to the contrary, that $|S| \le 2k - 1$. Let D be a γ_t-set of $T - S$, and so $|D| = \gamma_t(T - S) \le 3k + 2$. Since $|S| < 2k < \ell$ and each vertex in A' has ℓ leaf neighbors in T, we note that every vertex of A' is a support vertex of $T - S$, implying that $A' \subseteq D$, and so $3k + 2 \ge |D| \ge |A'| = 3k + 2$, implying that $D = A'$. In particular, $v_3 \notin D$, implying that all k leaf neighbors of v_3 belong to S; that is, $L_3 \subseteq S$. If $v_{i_1} \notin S$ for some $i \in [k]$, then in order to totally dominate the vertex v_{i_1}, the vertex $v_{i_2} \in D$, contradicting our earlier observation that $D = A'$. Hence, $v_{i_1} \in S$ for all $i \in [k]$, and so $|S| \ge |L_3| + k = 2k$, a contradiction. Therefore, our original supposition that $|S| \le 2k - 1$ is incorrect, implying that $|S| \ge 2k$ and $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \ge 2k$. Conversely, if S^* consists of all $2k$ neighbors of v_3 different from v_2 in T, then S^* is a non-isolating set of T such that $\gamma_t(T - S^*) = |A'| < \gamma_t(T)$, implying that $\mathrm{st}^-_{\gamma_t}(T) \le |S^*| = 2k$. Consequently, $\mathrm{st}^-_{\gamma_t}(T) = 2k$. Thus, $\mathrm{st}^-_{\gamma_t}(T) - \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = k$. □

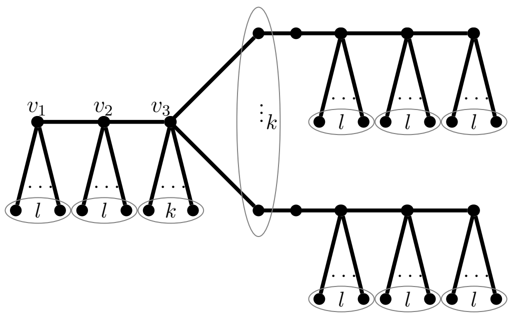

Figure 2: A tree from the family T in the proof of Proposition 3.3.

Theorem 2.1 follows from Propositions 3.2 and 3.3. As further examples, we remark that if P is the Petersen graph, then $\gamma_t(P) = 4$ and $\gamma_{\mathrm{pr}}(P) = 6$. Further, if v is an arbitrary vertex of P, then $\gamma_t(P - v) = 4$, and so $\mathrm{st}^-_{\gamma_t}(P) \ge 2$. Moreover, if S consists of two non-adjacent vertices of P, then $\gamma_t(P - S) = 3$, and so $\mathrm{st}^-_{\gamma_t}(P) \le 2$. Consequently, $\mathrm{st}^-_{\gamma_t}(P) = 2$. However if v is an arbitrary vertex of P, then $\gamma_{\mathrm{pr}}(P - v) = 4$, implying that $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(P) = 1$. Moreover, let G_k be a graph obtained from the Petersen graph by replacing every vertex by a copy of a complete graph K_k for some $k \ge 1$, and adding all edges between two resulting complete graphs that correspond to two vertices of G_k (see Fig. 3). The resulting graph G_k is a $(4k-1)$-regular, $3k$-connected graph that satisfies $\gamma_t(G_k) = 4$ and $\mathrm{st}^-_{\gamma_t}(G_k) = 2k$, and $\gamma_{\mathrm{pr}}(G_k) = 6$ and $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G_k) = k$. This yields the following result.

Proposition 3.4. *For $k \ge 1$ an integer, there exists $(4k-1)$-regular, $3k$-connected graphs G such that $\mathrm{st}^-_{\gamma_t}(G) - \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) = k$.*

4 Properties of paired domination stability

In this section, we present properties of paired domination stability in graphs. We begin with the following property of paired domination in graphs.

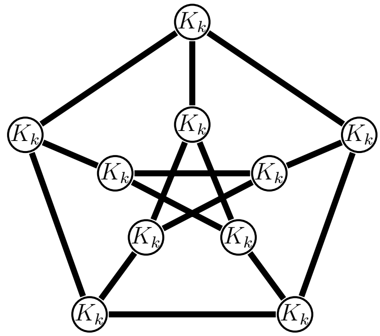

Figure 3: A graph G_k obtained from the Petersen graph by replacing every vertex by K_k.

Proposition 4.1. *Every connected isolate-free graph G contains a spanning tree T such that $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$.*

Proof. Since adding edges to a graph cannot increases its paired domination number, if T is an isolate-free spanning subgraph of a graph G, then $\gamma_{\rm pr}(G) \le \gamma_{\rm pr}(T)$. Let D be a $\gamma_{\rm pr}$-set of G, and so D is a PD-set of G and $|D| = \gamma_{\rm pr}(G)$. Let M be a perfect matching in the subgraph $G[D]$ induced by D. Let T' be a spanning subgraph of G that consists of the edges in M and for each vertex v outside D, an edge of G that joins v to exactly one vertex of the dominating set D. If the resulting spanning subgraph T' is a tree, then we let $T = T'$. Otherwise, if the resulting spanning subgraph T' is a forest with $\ell \ge 2$ components, then we add $\ell - 1$ edges from the edge set of the graph G between these components, avoiding cycles, to construct a tree, which we call T. Since D is a PD-set in the resulting tree T, we note that $\gamma_{\rm pr}(T) \le |D| = \gamma_{\rm pr}(G)$. Since T is an isolate-free spanning subgraph of G, we have $\gamma_{\rm pr}(T) \ge \gamma_{\rm pr}(G)$. Consequently, T is a spanning tree of G satisfying $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$. □

By our earlier convention, if G is a graph of order n and $\gamma_{\rm pr}(G) = 2$, then ${\rm st}^-_{\gamma_{\rm pr}}(G) = n$ since removing all vertices from the graph G produces the null graph with paired domination number zero. We are therefore only interested in the $\gamma^-_{\rm pr}$-stability of graphs with paired domination number at least 4. If G is a graph with $\gamma_{\rm pr}(G) \ge 4$ where x and y are adjacent vertices in G, then $D = V(G) \setminus \{x, y\}$ belongs to the set ${\rm NI}(G)$ and $\gamma_{\rm pr}(G - D) = \gamma_{\rm pr}(K_2) = 2 < \gamma_{\rm pr}(G)$. This yields the following result.

Observation 4.2. Every isolate-free graph G of order n with $\gamma_{\rm pr}(G) \ge 4$ satisfies ${\rm st}^-_{\gamma_{\rm pr}}(G) \le n - 2$.

Proposition 4.3. *If T is a spanning tree of a connected graph G such that $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$, then ${\rm st}^-_{\gamma_{\rm pr}}(T) \ge {\rm st}^-_{\gamma_{\rm pr}}(G)$.*

Proof. Let S be a ${\rm st}^-_{\gamma_{\rm pr}}$-set of T. Thus, S is a set in ${\rm NI}(T)$ with $|S| = {\rm st}^-_{\gamma_{\rm pr}}(T)$ such that $\gamma_{\rm pr}(T - S) < \gamma_{\rm pr}(T)$. Since $\gamma_{\rm pr}(G - S) \le \gamma_{\rm pr}(T - S)$ and $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$, the set S is a non-isolating set of G such that $\gamma_{\rm pr}(G - S) < \gamma_{\rm pr}(G)$. Hence, ${\rm st}^-_{\gamma_{\rm pr}}(G) \le |S| = {\rm st}^-_{\gamma_{\rm pr}}(T)$. □

The following result shows that to determine the $\gamma^-_{\rm pr}$-stability of a graph G, it is not sufficient to only examine spanning trees T of G satisfying $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$.

Proposition 4.4. *For $k \geq 1$ an integer, there exist connected graphs G such that $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) - \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) = k$ for every spanning tree T of G with $\gamma_{\mathrm{pr}}(T) = \gamma_{\mathrm{pr}}(G)$.*

Proof. For $k \geq 1$, let F be obtained from two vertex disjoint copies of $K_{2,k+1}$ by identifying a vertex of degree $k+1$ from each copy. Let u be the resulting identified vertex of degree $2(k+1)$, and let w_1 and w_2 be the two vertices of degree $k+1$ in F. Further, let v_i be a common neighbor (of degree 2) of u and w_i for $i \in [2]$. Let G be obtained from F by adding a leaf neighbor x_i to w_i for $i \in [2]$. Thus, $\mathrm{diam}(G) = 6$ and $x_1w_1v_1uv_2w_2x_2$ is a shortest path in G of length 6. The graph G satisfies $\gamma_{\mathrm{pr}}(G) = 4$. We remark that only connected graphs of $\mathrm{diam}(G) \leq 3$ have $\gamma_{\mathrm{pr}}(G) = 2$. Therefore, $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) \geq 3$. Moreover, the set $S = \{w_1, x_1, x_2\}$ is a non-isolating set of minimum cardinality satisfying $\gamma_{\mathrm{pr}}(G - S) = 2 < \gamma_{\mathrm{pr}}(G)$, and so $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) = 3$. However, the vertex u must have degree 2 in every spanning tree T of G for which $\gamma_{\mathrm{pr}}(T) = \gamma_{\mathrm{pr}}(G) = 4$, implying that the vertices w_1 and w_2 each have $k+1$ leaf neighbors in T. This implies that every non-isolating set of T that decreases the paired domination number contains at least $k+3$ vertices. The set $S = N_T[w_1]$ is a non-isolating set of minimum cardinality satisfying $\gamma_{\mathrm{pr}}(T - S) = 2 < \gamma_{\mathrm{pr}}(T)$, and so $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \leq |S| = k+3$. Consequently, $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = k+3$, and so $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) - \mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) = k$. □

Proposition 4.5. *If S is a $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}$-set of a connected isolate-free graph G with $\gamma_{\mathrm{pr}}(G) \geq 4$, then $\gamma_{\mathrm{pr}}(G - S) = \gamma_{\mathrm{pr}}(G) - 2$.*

Proof. Let S be a $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}$-set of G. Suppose, to the contrary, that $\gamma_{\mathrm{pr}}(G - S) \leq \gamma_{\mathrm{pr}}(G) - 4$. By the connectivity of G, there exists a vertex $u \in S$ that has a neighbor in the set $V(G) \setminus S$. We now consider the set $S' = S \setminus \{u\}$. Let D be a γ_{pr}-set of $G - S$. If u has a neighbor in D, then D is a γ_{pr}-set of $G - S'$, implying that $\gamma_{\mathrm{pr}}(G - S') \leq |D| = \gamma_{\mathrm{pr}}(G - S) \leq \gamma_{\mathrm{pr}}(G) - 4$, contradicting our choice of the set S. Hence, u has no neighbor in D. Let v be an arbitrary neighbor of u that belongs to $V(G) \setminus S$. The set $D \cup \{u, v\}$ is a PD-set of $G - S'$ with u and v paired, and with the pairings of the vertices of D unchanged from their pairings in $G - S$. Hence, $\gamma_{\mathrm{pr}}(G - S') \leq |D| + 2 \leq \gamma_{\mathrm{pr}}(G) - 2$, once again contradicting our choice of the set S. □

5 Paths and cycles

It is well known (see, for example, [11]) that for $n \geq 3$ we have $\gamma_{\mathrm{pr}}(C_n) = \gamma_{\mathrm{pr}}(P_n) = 2\lceil \frac{n}{4} \rceil$. In this section, we determine the paired domination stability of paths and cycles. The proofs require a detailed case analysis, which is straightforward albeit tedious. We therefore omit the proofs in this section. The γ^-_{pr}-stability of a path P_n and a cycle C_n on n vertices is given by the following result.

Theorem 5.1. *If G is a path P_n, for $n \geq 2$, or a cycle C_n, for $n \geq 3$, then*

$$\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(G) = \begin{cases} 1 & \textit{when } n \equiv 1 \pmod{4} \\ 2 & \textit{when } n \equiv 2 \pmod{4} \\ 3 & \textit{when } n \equiv 3 \pmod{4} \\ 4 & \textit{when } n \equiv 0 \pmod{4}. \end{cases}$$

Next we determine the γ^+_{pr}-stability of a path P_n. For $n \leq 10$ with $n \neq 8$ and for $n = 13$, no non-isolating set of vertices in a path P_n exists whose removal increases the paired domination number, and hence, by definition, $\mathrm{st}^+_{\gamma_{\rm pr}}(P_n) = \infty$ for such values of n. It is therefore only of interest to determine the $\gamma^+_{\rm pr}$-stability of a path P_n, where $n \ge 8$ and $n \notin \{9, 10, 13\}$.

Theorem 5.2. *For $n \ge 8$ and $n \notin \{9, 10, 13\}$,*

$$\mathrm{st}^+_{\gamma_{\rm pr}}(P_n) = \begin{cases} 1 & \textit{when } n \pmod 4 \in \{0, 3\} \\ 2 & \textit{when } n \pmod 4 \in \{1, 2\}. \end{cases}$$

As a consequence of Theorems 5.1 and 5.2, the paired domination stability of a path is determined.

Corollary 5.3. *For $n \ge 2$,*

$$\mathrm{st}_{\gamma_{\rm pr}}(P_n) = \begin{cases} 1 & \textit{when } n \pmod 4 \in \{0, 1, 3\} \textit{ and } n \notin \{3, 4, 7\} \\ 2 & \textit{when } n \equiv 2 \pmod 4 \\ 3 & \textit{when } n \in \{3, 7\} \\ 4 & \textit{when } n = 4. \end{cases}$$

We next consider the $\gamma^+_{\rm pr}$-stability of a cycle C_n. As shown in Theorem 5.1, the $\gamma^-_{\rm pr}$-stability of a path and a cycle of the same order are equal. This is not always the case for the $\gamma^+_{\rm pr}$-stability of a path and a cycle. For example, $\mathrm{st}^+_{\gamma_{\rm pr}}(P_{12}) = 1$ and $\mathrm{st}^+_{\gamma_{\rm pr}}(C_{12}) = 2$. Analogously as in the case of paths, for small values of the order of a cycle the $\gamma^+_{\rm pr}$-stability is infinite. Namely, for $n \le 14$ with $n \ne 12$ and $n = 17$ we have that $\mathrm{st}^+_{\gamma_{\rm pr}}(C_n) = \infty$. The following result determines the $\gamma^+_{\rm pr}$-stability of a cycle of large order.

Theorem 5.4. *For $n \ge 12$ and $n \notin \{13, 14, 17\}$,*

$$\mathrm{st}^+_{\gamma_{\rm pr}}(C_n) = \begin{cases} 2 & \textit{when } n \equiv 0 \pmod 4 \\ 3 & \textit{when } n \pmod 4 \in \{2, 3\} \\ 4 & \textit{when } n \equiv 1 \pmod 4). \end{cases}$$

As a consequence of Theorems 5.1 and 5.4, the paired domination stability of a cycle is determined.

Corollary 5.5. *For $n \ge 3$,*

$$\mathrm{st}_{\gamma_{\rm pr}}(C_n) = \begin{cases} 1 & \textit{when } n \equiv 1 \pmod 4 \\ 2 & \textit{when } n \pmod 4 \in \{0, 2\} \textit{ and } n \notin \{4, 8\} \\ 3 & \textit{when } n \equiv 3 \pmod 4 \\ 4 & \textit{when } n \in \{4, 8\}. \end{cases}$$

6 Trees

In this section, we first determine the $\gamma_{\rm pr}$-stability of trees in the family $\mathcal{F}_\Delta$ and a new family $\mathcal{E}_\Delta$.

Lemma 6.1. *For $\Delta \ge 2$, if $T \in \mathcal{F}_\Delta$, then $\mathrm{st}^-_{\gamma_{\rm pr}}(T) = 2\Delta$.*

Proof of Lemma 6.1. Let T be an arbitrary tree in the family $\mathcal{F}_{k,\Delta}$ for some $k \geq 2$ and $\Delta \geq 2$. We show that $\mathrm{st}^{-}_{\gamma_{\rm pr}}(T) = 2\Delta$. The family $\mathcal{F}_{k,2}$ consists of all paths P_{4k} where $k \geq 2$. Therefore by Theorem 5.1, we have $\mathrm{st}^{-}_{\gamma_{\rm pr}}(T) = 4 = 2\Delta$ for each $T \in \mathcal{F}_{k,2}$, which yields the desired result. Hence, we may assume that $\Delta \geq 3$. We show, by induction on $k \geq 2$, that every tree T in the family $\mathcal{F}_{k,\Delta}$ satisfies $\mathrm{st}^{-}_{\gamma_{\rm pr}}(T) = 2\Delta$.

Suppose $k = 2$, and so $T \in \mathcal{F}_{2,\Delta}$ (where recall that $\Delta \geq 3$). The tree T can therefore be constructed from two vertex disjoint double stars T_1 and T_2, where $T_i \cong S(\Delta-1, \Delta-1)$ for $i \in [2]$, by selecting leaves w_1 and w_2 of T_1 and T_2, respectively, and adding the edge w_1w_2 to $T_1 \cup T_2$. Let x_i and y_i be the two vertices of T_i that are not leaves, where x_iw_i is an edge. We note that $y_1x_1w_1w_2x_2y_2$ is a path in T. We note that $\gamma_{\rm pr}(T) = 4$ and the set $\{x_1, x_2, y_1, y_2\}$ is a $\gamma_{\rm pr}$-set of T.

Let S be a $\mathrm{st}^{-}_{\gamma_{\rm pr}}$-set of G. Thus, S is a set in $\mathrm{NI}(G)$ with $|S| = \mathrm{st}^{-}_{\gamma_{\rm pr}}(G)$ such that $\gamma_{\rm pr}(T - S) = 2$. Let R be a $\gamma_{\rm pr}$-set of $T - S$, and so R is a minimum PD-set of $T - S$ (of cardinality 2). Since $T[R] = P_2$, we note that $T - S$ is a tree of diameter at most 3. This implies that at most one of x_i and y_{3-i} belong to $T - S$ for $i \in [2]$. Thus, $|S \cap \{x_i, y_{3-i}\}| \geq 1$ for $i \in [2]$.

Suppose that $y_1 \in S$ and $x_2 \in S$. If $x_1 \in S$, then all leaf neighbors of y_1, x_1 and x_2 belong to S, while if $y_2 \in S$, then all leaf neighbors of y_1, y_2 and x_2 belong to S. In both cases, $|S| \geq 3\Delta - 2 > 2\Delta$.

Suppose that $y_1 \in S$ and $x_2 \notin S$. If $y_2 \in S$, then all leaf neighbors of y_1 and y_2 belong to S, implying that $|S| \geq 2\Delta$. If $y_2 \notin S$, then $x_1 \in S$, implying that S contains all leaf-neighbors of y_1 and x_1, and so $|S| \geq 2\Delta - 1$. However if in this case $|S| = 2\Delta - 1$, implying that $\mathrm{diam}(T - S) \geq 4$, a contradiction. Hence, $|S| \geq 2\Delta$.

Suppose that $y_1 \notin S$ and $x_2 \in S$. Since $T - S$ is a tree, $y_2 \in S$ and all leaf neighbors of y_2 and x_2 belong to S, implying that $|S| \geq 2\Delta - 1$. However if in this case $|S| = 2\Delta - 1$, then S contains x_2 and all leaf neighbors of y_1, implying that $\mathrm{diam}(T - S) \geq 4$, a contradiction. Hence, $|S| \geq 2\Delta$. Therefore, in all three cases we have $|S| \geq 2\Delta$, as desired. This proves the base case when $k = 2$.

For the inductive hypothesis, let $k \geq 3$ and assume that if $T' \in \mathcal{F}_{k',\Delta}$ where $2 \leq k' < k$, then $\mathrm{st}^{-}_{\gamma_{\rm pr}}(T') = 2\Delta$. We now consider a tree T in the family $\mathcal{F}_{k,\Delta}$. Therefore, the tree T can be constructed from k vertex disjoint double stars $H_1, \ldots, H_k$, where $H_i \cong S(\Delta - 1, \Delta - 1)$ for $i \in [k]$, by selecting one leaf y_i from each double star H_i and adding $k - 1$ edges between vertices in $\{y_1, \ldots, y_k\}$ in such a way that the resulting graph is a tree with maximum degree Δ. Let w_i and x_i be the two (adjacent) vertices of H_i that are not leaves for $i \in [k]$, where y_i is a leaf neighbor of x_i for $i \in [k]$. We note that $\gamma_{\rm pr}(T) = 2k$ and the set $\cup_{i=1}^{k}\{w_i, x_i\}$ is the unique $\gamma_{\rm pr}$-set of T.

Let U be the graph of order k whose vertices correspond to the k double stars $H_1, \ldots, H_k$ where two vertices are adjacent in U if and only if the corresponding double stars are joined by an edge in T. We call U the underlying graph of T. By construction, the graph U is a tree, noting that T is a tree. Let $V(U) = \{u_1, \ldots, u_k\}$ where u_i is the vertex of U corresponding to the double star H_i for $i \in [k]$. Renaming the double stars if necessary, we may assume that u_1 is a leaf in U, and that H_1 is joined to H_2 in T. Thus, $y_1y_2 \in E(T)$ and $y_1y_j \notin E(T)$ for $j \in [k] \setminus [2]$. We note that $w_1x_1y_1y_2x_2w_2$ is a path in T. Let $T' = T - V(H_1)$. By construction, the tree T' belongs to the family $\mathcal{F}_{k',\Delta}$ where $k' = k - 1 \geq 2$. By induction, we have $\mathrm{st}^{-}_{\gamma_{\rm pr}}(T') = 2\Delta$.

Let S be a $\mathrm{st}^{-}_{\gamma_{\rm pr}}$-set of T. Thus, S is a set in $\mathrm{NI}(T)$ with $|S| = \mathrm{st}^{-}_{\gamma_{\rm pr}}(T)$ such that $\gamma_{\rm pr}(T - S) \leq \gamma_{\rm pr}(T) - 2 = 2k - 2$. Let Q be a $\gamma_{\rm pr}$-set of $T - S$, and so $|Q| \leq 2k - 2$.

Let $Q' = Q \cap V(T')$ and $S' = S \cap V(T')$. For $i \in [k]$, let $Q_i = Q \cap V(H_i)$ and $S_i = S \cap V(H_i)$. We proceed further with the following claim.

Claim 6.2. $|S| \geq 2\Delta$.

Proof of Claim 6.2. Suppose, to the contrary, that $|S| \leq 2\Delta - 1$.

Subclaim 6.2.1. $|Q_1| \geq 2$.

Proof of Subclaim 6.2.1. Suppose, to the contrary, that $|Q_1| \leq 1$. Suppose that $Q_1 = \emptyset$. In this case, $V(H_1) \setminus \{y_1\} \subseteq S_1$. If $y_1 \in S_1$, then $|S_1| = 2\Delta > |S|$, a contradiction. Hence, $y_1 \notin S_1$, and so $2\Delta - 1 \geq |S| \geq |S_1| = 2\Delta - 1$, implying that $S = S_1$ and $|S| = 2\Delta - 1$. In this case, a $\gamma_{\rm pr}$-set of $T - S$ contains at least one of y_1 and y_2. Since the set $\cup_{i=2}^{k}\{w_i, x_i\}$ is the unique $\gamma_{\rm pr}$-set of T', a $\gamma_{\rm pr}$-set of $T - S$ is therefore not a $\gamma_{\rm pr}$-set of T', and so $\gamma_{\rm pr}(T - S) \geq \gamma_{\rm pr}(T') + 2 = 2(k-1) + 2 = 2k$, a contradiction. Hence, $|Q_1| \geq 1$.

By supposition, $|Q_1| \leq 1$. Consequently, $|Q_1| = 1$, implying that $Q_1 = \{y_1\}$ and $V(H_1) \setminus \{x_1, y_1\} \subseteq S_1$, and so $|S_1| \geq 2\Delta - 2$. If $x_1 \in S_1$, then $|S_1| = 2\Delta - 1$ and we end up in the previous case, which leads to a contradiction. Hence, $x_1 \notin S_1$ and $x_1 \notin Q_1$, implying that $y_2 \in Q$ with the vertices y_1 and y_2 paired in Q, and $|S_1| = 2\Delta - 2$. By supposition, $|S| \leq 2\Delta - 1$. If $|S| = 2\Delta - 2$, then $S = S_1$ and $\gamma_{\rm pr}(T - S) \geq \gamma_{\rm pr}(T') + 2 = 2k$, a contradiction. Hence, $|S| = 2\Delta - 1$, and so the set S contains a vertex $v' \in V(T') \setminus \{y_2\}$. However noting that $\Delta \geq 3$, every non-isolating set of vertices of $T' - y_2$ that decreases the paired domination number cannot contain only one vertex, implying that $\gamma_{\rm pr}(T - S) \geq |\{y_1, y_2\}| + \gamma_{\rm pr}(T' - y_2) = 2 + \gamma_{\rm pr}(T') = 2k$, a contradiction. □

Subclaim 6.2.2. $\{x_1, y_1\} \subseteq Q$.

Proof of Subclaim 6.2.2. Suppose, to the contrary, that $y_1 \notin Q_1$, implying that $S' \in \mathrm{NI}(T')$. Recall that S is a $\mathrm{st}^-_{\gamma_{\rm pr}}$-set of T and $|S'| \leq |S| \leq 2\Delta - 1$. However, $\mathrm{st}^-_{\gamma_{\rm pr}}(T') = 2\Delta$. Therefore, $\gamma_{\rm pr}(T' - S') \geq \gamma_{\rm pr}(T') = 2(k-1)$. Hence, $\gamma_{\rm pr}(T - S) = \gamma_{\rm pr}(T' - S') + |Q_1| \geq 2(k-1) + 2 = 2k$, a contradiction. Hence, $y_1 \in Q_1$.

Suppose, to the contrary, that $x_1 \notin Q_1$. Thus, all $\Delta - 2$ leaf-neighbors of x_1 belong to the set S_1. By Claim 6.2.1, we have $|Q_1| \geq 2$. Hence, the set Q_1 contains w_1 and one of its leaf-neighbor w_1'. We now consider the set $S^* = S \setminus S_1$. Since $S^* \in \mathrm{NI}(T)$ and $(Q \setminus \{w_1'\}) \cup \{x_1\}$ is a PD-set of $T - S^*$, we have $\gamma_{\rm pr}(T - S^*) \leq |Q| = \gamma_{\rm pr}(T - S)$, contradicting our choice of the set S. Hence, $x_1 \in Q_1$. □

Subclaim 6.2.3. $w_1 \notin Q_1$.

Proof of Subclaim 6.2.3. Suppose, to the contrary, that $w_1 \in Q_1$. Hence, $\{w_1, x_1, y_1\} \subseteq Q_1$, and so $S \cap V(H_1) = \emptyset$ by the minimality of S. Thus, $S = S'$ and therefore $|S'| \leq 2\Delta - 1$.

We show firstly that x_1 and y_1 are paired in Q. Suppose, to the contrary, that x_1 and y_1 are not paired in Q. This implies that $y_2 \in Q$, and that y_1 and y_2 are paired in Q. Suppose that $x_2 \notin S$, implying that $S' \in \mathrm{NI}(T')$. By the minimality of the set Q, we have $x_2 \notin Q$. Thus, the set $Q' \cup \{x_2\}$ is a PD-set of $T' - S'$, and so $|Q'| + 1 = |Q' \cup \{x_2\}| \geq \gamma_{\rm pr}(T' - S') \geq \gamma_{\rm pr}(T') = 2(k-1)$. Hence, $|Q| = |Q_1| + |Q'| \geq 3 + (2k-3) = 2k = \gamma_{\rm pr}(T)$, a contradiction. Hence, $x_2 \in S$. We now consider the set $S^* = S \setminus \{x_2\}$. We note that S^* is a non-isolating set of vertices of T, and the set Q is a PD-set of $T - S^*$. Thus,

$\gamma_{\rm pr}(T - S^*) \le |Q| \le 2k - 2$, which contradicts our choice of the set S. Hence, x_1 and y_1 are paired in Q.

Since x_1 and y_1 are paired in Q, the vertex w_1 is paired with one of its leaf neighbors, say w_1'. By the minimality of Q we note that $Q_1 = \{w_1, w_1', x_1, y_1\}$. If $x_2 \in Q$, then the set $Q \setminus \{w_1', y_1\}$ is a PD-set of $T - S$ (with w_1 and x_1 paired), contradicting the minimality of Q. Hence, $x_2 \notin Q$. This in turn implies that $y_2 \notin Q$. If $y_2 \in S$, then once again we contradict the minimality of Q. Therefore, $y_2 \notin S$. We remark, though, that possibly $x_2 \in S$. Recall that by our earlier observations, $S = S'$.

Let $S'' = S \setminus \{x_2\}$. Thus, if $x_2 \notin S$, then $S'' = S$, while if $x_2 \in S$, then $S'' = S \setminus \{x_2\}$. The set S'' is a non-isolating set of T' such that $|S''| \le |S| \le 2\Delta - 1$. As observed earlier, $y_2 \notin Q'$ and $x_2 \notin Q'$. The set $Q' \cup \{y_2, x_2\}$ is a PD-set of $T' - S''$, implying that $|Q'| + 2 \ge \gamma_{\rm pr}(T' - S'') \ge \gamma_{\rm pr}(T') = 2(k-1)$. Hence, $|Q'| \ge 2k - 4$, and so $|Q| = |Q_1| + |Q'| \ge 4 + (2k-4) = 2k$, contradicting the fact that $|Q| \le 2k - 2$. □

Proof of Claim 6.2*, continued:* By Claim 6.2.3, $w_1 \notin Q_1$. This implies that $Q_1 = \{x_1, y_1\}$. The set S_1 therefore consists of the $\Delta - 1$ leaf neighbors of w_1, and so $|S_1| = \Delta - 1$. This is true for every leaf in the tree U. Hence, if u_i is a leaf in U for some $i \in [k]$, then in the corresponding double star H_i of T we have $Q_i = \{x_i, y_i\}$ and $|S_i| = \Delta - 1$. Further, the set S_i consists of the $\Delta - 1$ leaf neighbors of w_i. In particular, $|Q_1| = 2$ and $|S_1| = \Delta - 1$. Since the underlying tree U of T has order $k \ge 3$, there are at least two leaves in U. Thus, u_p is a leaf in U for some $p \in [k] \setminus \{1\}$, implying that $|Q_p| = 2$ and $|S_p| = \Delta - 1$.

If $|Q_i| \ge 2$ for all $i \in [k]$, then $|Q| \ge 2k$, a contradiction. Hence, $|Q_q| \le 1$ for some $q \in [k]$. By our earlier observations, u_q is not a leaf in the tree U, and so $q \notin \{1, p\}$. If $|Q_q| = 0$, then $\{w_q, x_q\} \subseteq S_q$, and so $|S_q| \ge 2$ (in fact, $|S_q| \ge 2\Delta - 1$) and $|S| \ge |S_1| + |S_p| + |S_q| \ge (\Delta - 1) + (\Delta - 1) + 2 = 2\Delta$, a contradiction. Hence, $|Q_q| = 1$, implying that $Q_q = \{y_q\}$ and $w_q \in S_q$, and so $|S_q| \ge 1$. Since the paired dominating number is an even integer and $|Q| \le 2k$, there exists $r \in [k] \setminus \{1, p, q\}$ such that $|Q_r| = 1$. Therefore, $Q_r = \{y_r\}$ and $|S_r| \ge 1$. Hence, $|S| \ge |S_1| + |S_p| + |S_q| + |S_r| \ge (\Delta - 1) + (\Delta - 1) + 1 + 1 = 2\Delta$, a contradiction. This completes the proof of Claim 6.2. □

Proof of Lemma 6.1*, continued:* By Claim 6.2, we have $|S| \ge 2\Delta$. By our choice of the set S, this implies that ${\rm st}^-_{\gamma_{\rm pr}}(T) = |S| \ge 2\Delta$. Conversely, if we consider the set $S = V(H_1)$, then $S \in {\rm NI}(T)$ satisfies $|S| = 2\Delta$ and $\gamma_{\rm pr}(T - S) = \gamma_{\rm pr}(T') = 2k - 2 < \gamma_{\rm pr}(T)$, and so ${\rm st}^-_{\gamma_{\rm pr}}(T) \le 2\Delta$. Consequently, ${\rm st}^-_{\gamma_{\rm pr}}(T) = 2\Delta$. This completes the proof of Lemma 6.1. □

We determine next the $\gamma^+_{\rm pr}$-stability of a tree in the family $\mathcal{F}_\Delta$.

Lemma 6.3. *For $\Delta \ge 2$, if $T \in \mathcal{F}_\Delta$, then ${\rm st}^+_{\gamma_{\rm pr}}(T) \le \Delta - 1$.*

Proof. Let T be an arbitrary tree in the family $\mathcal{F}_{k,\Delta}$ for some $k \ge 2$ and $\Delta \ge 2$. We use the same notation as in the proof of Lemma 6.1. In particular, $\gamma_{\rm pr}(T) = 2k$ and H_1 corresponds to a leaf u_1 in the underlying tree U of T. Moreover, $y_1 y_2$ is the edge joining H_1 and H_2 in T. Also, w_i and x_i are the support vertices in the double star H_i and $w_i x_i y_i$ is a path in H_i for $i \in [k]$. Let L be the set of $\Delta - 2$ leaf neighbors of x_1 in T, and let $S = L \cup \{x_1\}$. We resulting set $S \in NI(T)$ and the forest $T - S$ has two components, say F_1 and F_2 where $w_1 \in V(F_1)$ and $y_1 \in V(F_2)$. Moreover, $\gamma_{\rm pr}(T - S) = \gamma_{\rm pr}(F_1) + \gamma_{\rm pr}(F_2) = 2 + 2k > \gamma_{\rm pr}(T)$. Therefore, ${\rm st}^+_{\gamma_{\rm pr}}(T) \le |S| = \Delta - 1$. □

Recall that by Proposition 3.1, for $\Delta \geq 3$, if $T \in \mathcal{H}_\Delta$, then $\mathrm{st}_{\gamma_{\mathrm{pr}}}(T) = 2\Delta - 1$. Further we remark that $\mathrm{st}^+_{\gamma_{\mathrm{pr}}}(T) = \infty$. We next define another family of trees T with maximum degree Δ such that $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = 2\Delta - 1$. For integers $\Delta \geq 3$ and $\Delta - 1 \geq k \geq 3$, let $E_{k,\Delta}$ be a graph obtained from the path P_2 with vertices u and v and the disjoint union of $2k$ double stars $S(\Delta - 1, \Delta - 1)$ by selecting one leaf from each double star and identifying half of the selected leaves with the vertex v and the other half of the selected leaves with the vertex u (see Figure 4). Let

$$\mathcal{E}_\Delta = \bigcup_{k \geq 3} E_{k,\Delta}.$$

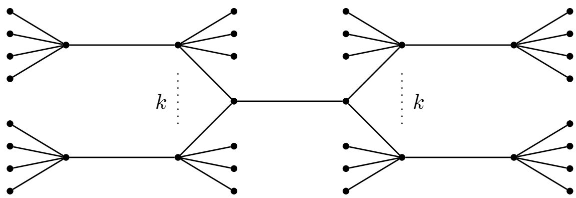

Figure 4: A tree $E_{k,5}$ from the family $\mathcal{E}_5$.

If T is a tree from the family $\mathcal{E}_\Delta$, then $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) = 2\Delta - 1$. Moreover, if T is isomorphic to the graph $E_{k,\Delta}$, then $\mathrm{st}^+_{\gamma_{\mathrm{pr}}}(T) = k(\Delta - 1)$. In contrast to the family $\mathcal{H}_\Delta$, the trees from the family $\mathcal{E}_\Delta$ have finite γ^+_{pr}-stability.

7 Proof of Theorem 2.2

In this section we present a proof of Theorem 2.2, which we restate below.

Theorem 2.2. *If T is a tree with maximum degree Δ satisfying $\gamma_{\mathrm{pr}}(T) \geq 4$, then the following hold.*

(a) $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \leq 2\Delta$, *with equality if and only if $T \in \mathcal{F}_\Delta$.*

(b) $\mathrm{st}_{\gamma_{\mathrm{pr}}}(T) \leq 2\Delta - 1$, *and this bound is sharp for all $\Delta \geq 2$.*

Proof. We first prove the statement given in part (a). Since $\gamma_{\mathrm{pr}}(T) \geq 4$, we have $\Delta \geq 2$. If $\Delta = 2$, then G is a path P_n of order $n \geq 5$. In this case, the family $\mathcal{F}_{k,\Delta} = \{P_n : n \equiv 0 \pmod 4$ and $n \geq 8\}$, and Theorem 5.1 and Lemma 6.1 imply the desired result. Suppose, therefore, that $\Delta \geq 3$. The sufficiency of part (a) follows from Lemma 6.1. To prove the necessity, let T be a tree with maximum degree $\Delta \geq 3$ satisfying $\gamma_{\mathrm{pr}}(T) \geq 4$. Let $d = \mathrm{diam}(T)$, and so $d \geq 4$. Let $P\colon v_0 v_1 \ldots v_d$ be a diametral path in G. Thus, v_0 and v_d are leaves in T and $d(v_0, v_d) = \mathrm{diam}(G)$. We now consider the tree T rooted at the vertex v_d. Let D be a γ_{pr}-set of T.

Suppose that there is a child u_1 of v_2 that is a support vertex in T where $u_1 \neq v_1$. Let u_0 be a leaf neighbor of u_1. Since every PD-set of T contains all support vertices, we have $\{v_1, u_1\} \subset D$. Renaming vertices if necessary, we may assume that u_0 and u_1 are paired in D. Thus, if S consists of the vertex u_1 and all leaf neighbors of u_1, then $S \in \mathrm{NI}(T)$ and $\gamma_{\mathrm{pr}}(T - S) \leq |D| - 2 = \gamma_{\mathrm{pr}}(T) - 2$. Hence, $\mathrm{st}^-_{\gamma_{\mathrm{pr}}}(T) \leq |S| \leq \Delta < 2\Delta - 1$, and the desired result follows. Assume, therefore, that every child of v_2 different from v_1 is a leaf.

Suppose that there is a $\gamma_{\rm pr}$-set, $D_{2,3}$, of T such that v_2 and v_3 are paired in $D_{2,3}$. Necessarily, $v_1 \in D_{2,3}$ and v_1 is paired in $D_{2,3}$ with one of its leaf neighbors. Let S consist of the vertex v_1 and all of its leaf neighbors. Thus, $S \in {\rm NI}(T)$ and $\gamma_{\rm pr}(T-S) \le |D_{2,3}| - 2 = \gamma_{\rm pr}(T) - 2 < \gamma_{\rm pr}(T)$, implying that ${\rm st}^-_{\gamma_{\rm pr}}(T) \le |S| \le \Delta < 2\Delta - 1$, once again implying the desired result. Therefore, we may assume that in every $\gamma_{\rm pr}$-set of T the vertices v_2 and v_3 are not paired.

Suppose that there is a $\gamma_{\rm pr}$-set, D_3, of T which contains a neighbor of v_3 different from v_2. In this case, if S consists of the vertex v_2 and all its descendants, then $|S| \le 2\Delta - 1$, $S \in {\rm NI}(T)$ and $\gamma_{\rm pr}(T-S) \le |D_3| - 2 = \gamma_{\rm pr}(T) - 2 < \gamma_{\rm pr}(T)$, noting that the set $D_3 \setminus S$ is a PD-set of $T - S$ and, by the minimality of D_3 we have $|D_3 \cap S| = 2$. Thus, ${\rm st}^-_{\gamma_{\rm pr}}(T) \le |S| \le 2\Delta - 1$, and the desired result follows. Hence, we may assume that every $\gamma_{\rm pr}$-set of T contains the vertex v_2 but no other vertex in $N[v_3]$. In particular, $N[v_3] \cap D = \{v_2\}$.

Suppose that $d_T(v_1) < \Delta$ or $d_T(v_2) < \Delta$. Thus, $d_T(v_1) + d_T(v_2) \le 2\Delta - 1$. In order to dominate the vertex v_0, we have $v_1 \in D$. By our earlier assumptions, $v_2 \in D$ and every child of v_2 different from v_1 is a leaf. Thus by the minimality of the set D, the vertex v_1 is the only descendant of v_2 that belongs to the set D, and the vertices v_1 and v_2 are paired in D. Hence, if $S = N[v_2] \cup N[v_1]$, then $S \in {\rm NI}(T)$ and $|S| = d_T(v_1) + d_T(v_2) \le 2\Delta - 1$. Further, $D \setminus \{v_1, v_2\}$ is a PD-set of $T - S$, and so $\gamma_{\rm pr}(T-S) \le |D| - 2 = \gamma_{\rm pr}(T) - 2$, implying that ${\rm st}^-_{\gamma_{\rm pr}}(T) \le |S| \le 2\Delta - 1$, yielding the desired result. Hence, we may assume that $d_T(v_1) = d_T(v_2) = \Delta$.

Suppose that $d_T(v_3) \ge 3$, and let u_2 be a child of v_3 different from v_2. If u_2 is a leaf, then v_3 belongs to every $\gamma_{\rm pr}$-set of T, while if u_2 is not a leaf, then from the structure of the rooted tree T the vertex u_2 can be chosen to belong to some $\gamma_{\rm pr}$-set of T. In both cases, we contradict our earlier assumption that every $\gamma_{\rm pr}$-set of T contains the vertex v_2 but no other vertex in $N[v_3]$. Hence, $d_T(v_3) = 2$. We now let $S = N[v_1] \cup N[v_2]$, and so $S \in {\rm NI}(T)$ and $|S| = d_T(v_1) + d_T(v_2)$. By our earlier observations, $|S| = 2\Delta$ and $\gamma_{\rm pr}(T-S) = \gamma_{\rm pr}(T) - 2 < \gamma_{\rm pr}(T)$, implying that ${\rm st}^-_{\gamma_{\rm pr}}(T) \le |S| = 2\Delta$. This proves the desired upper bound.

We show next that if we have equality in the upper bound in part (a), then $T \in \mathcal{F}_\Delta$. Let ${\rm st}^-_{\gamma_{\rm pr}}(T) = 2\Delta$. By our earlier observations, we have that every child of v_2 different from v_1 is a leaf. Further, $d_T(v_1) = d_T(v_2) = \Delta$ and $d_T(v_3) = 2$. We now re-root the tree T at the vertex v_0, thereby interchanging the roles of v_0 and v_d. Identical arguments as before show that every child of v_{d-2} different from v_{d-1} is a leaf. Further, $d_T(v_{d-1}) = d_T(v_{d-2}) = \Delta$ and $d_T(v_{d-3}) = 2$. In particular, $d \ge 6$.

Suppose that $d = 6$, and so $v_{d-3} = v_3$. In this case, the tree T is determined and $\gamma_{\rm pr}(T) = 4$. Letting $S = (N[v_1] \cup N[v_2]) \setminus \{v_3\}$, we have $S \in {\rm NI}(T)$ and $|S| = d_T(v_1) + d_T(v_2) - 1 = 2\Delta - 1$. Further, $\gamma_{\rm pr}(T-S) = 2 < \gamma_{\rm pr}(T)$. Therefore, ${\rm st}^-_{\gamma_{\rm pr}}(T) \le |S| = 2\Delta - 1$, a contradiction. Hence, $d \ge 7$, and so $v_{d-3} \ne v_3$.

We now consider the tree $T' = T - (N[v_1] \cup N[v_2])$. If $\gamma_{\rm pr}(T') = 2$, then by our earlier observations, we have $d = 7$ and $T' \cong S(\Delta - 1, \Delta - 1)$ where v_{d-1} and v_{d-2} are the two (adjacent) vertices in T' that are not leaves. Therefore, $T \in T_{2,\Delta}$, and so $T \in T_\Delta$. Hence, we may assume that $\gamma_{\rm pr}(T') \ge 4$, for otherwise the desired characterization follows. In particular, $d \ge 8$. As observed earlier, $d_T(v_{d-1}) = d_T(v_{d-2}) = \Delta$, implying that $\Delta(T') = \Delta$ and ${\rm st}^-_{\gamma_{\rm pr}}(T') \le 2\Delta$.

Let D be a $\gamma_{\rm pr}$-set of T. Since every PD-set of T contains the set of support vertices, we note that $v_1, v_2 \in D$. By the minimality of D, no leaf-neighbor of v_1 or v_2 belongs to

D. If $v_3 \in D$, then $v_4 \in D$ (with v_3 and v_4 paired in D). However in this case, we can replace v_3 in D with an arbitrary neighbor of v_4 that does not belong to D. Hence, we can choose the $\gamma_{\rm pr}$-set D of T so that $v_3 \notin D$. The resulting set D when restricted to $V(T')$ is a PD-set of T', implying that $\gamma_{\rm pr}(T') \le |D| - 2 = \gamma_{\rm pr}(T) - 2$. Conversely, every PD-set of T' can be extended to a PD-set of T by adding to it the vertices v_1 and v_2 (with v_1 and v_2 paired), and so $\gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T') + 2$. Consequently, $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(T') + 2$.

Suppose that $\mathrm{st}^-_{\gamma_{\rm pr}}(T') < 2\Delta$. Let S' be a $\mathrm{st}^-_{\gamma_{\rm pr}}$-set of T'. Thus, S is a set in $\mathrm{NI}(T')$ with $|S'| = \mathrm{st}^-_{\gamma_{\rm pr}}(T') < 2\Delta$ such that $\gamma_{\rm pr}(T - S') < \gamma_{\rm pr}(T')$. If D' is a $\gamma_{\rm pr}$-set of $T' - S'$, then $D' \cup \{v_1, v_2\}$ is a PD-set of $T - S$, and so $\gamma_{\rm pr}(T - S') \le |D'| + 2 = \gamma_{\rm pr}(T - S') + 2 < \gamma_{\rm pr}(T') + 2 = \gamma_{\rm pr}(T)$. Hence, $S' \in \mathrm{NI}(T)$ and $\gamma_{\rm pr}(T - S') < \gamma_{\rm pr}(T')$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \le |S'| = \mathrm{st}^-_{\gamma_{\rm pr}}(T') < 2\Delta$, a contradiction. Therefore, $\mathrm{st}^-_{\gamma_{\rm pr}}(T') = 2\Delta$.

Hence, the tree T' satisfies $\Delta(T') = \Delta$, $\gamma_{\rm pr}(T') \ge 4$ and $\mathrm{st}^-_{\gamma_{\rm pr}}(T') = 2\Delta$. Proceeding by induction, we have $T' \in \mathcal{F}_\Delta$. Thus, T' is constructed from the disjoint union of k' double stars each isomorphic to $S(\Delta - 1, \Delta - 1)$, by selecting one leaf from each double star and adding $k' - 1$ edges between these selected leaves to produce a tree with maximum degree Δ. The resulting tree T' satisfies $\gamma_{\rm pr}(T') = 2k'$ with the $2k'$ support vertices forming a $\gamma_{\rm pr}$-set of T'.

By construction of T', the tree T' contains the vertex v_4 but not the vertex v_3. Suppose that v_4 is a support vertex in T', implying by construction of T' that v_4 is a vertex of degree Δ in T'. Let $S = (N[v_1] \cup N[v_2]) \setminus \{v_3\}$. We note that $S \in \mathrm{NI}(T)$ and $|S| = 2\Delta - 1$. Let D' be the (unique) $\gamma_{\rm pr}$-set of T', and so D' is the set of $2k'$ support vertices in T'. In particular, we note that $v_4 \in D'$. The set D' is a PD-set of $T - S$, and so $\gamma_{\rm pr}(T - S) \le |D'| = \gamma_{\rm pr}(T') = \gamma_{\rm pr}(T) - 2$. Therefore, $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \le |S| = 2\Delta - 1$, a contradiction. Hence, v_4 is a leaf of T', and so v_4 is a leaf in one of the k' double stars in the construction of T'. Selecting the leaf v_4 from this double star and selecting the leaf v_3 from the double star induced by $N[v_1] \cup N[v_2]$, which is isomorphic to $S(\Delta - 1, \Delta - 1)$, and adding back the edge v_3v_4 we re-construct the tree T, showing that $T \in \mathcal{F}_\Delta$. This completes the proof of part (a).

Part (b) now follows readily from part (a). If $T \in \mathcal{F}_\Delta$ for some $\Delta \ge 2$, then by Lemmas 6.1 and 6.3, we have $\mathrm{st}_{\gamma_{\rm pr}}(T) \le \Delta - 1$. Hence, we may assume that $T \notin \mathcal{F}_\Delta$ for any $\Delta \ge 2$, for otherwise the bound in part (b) is immediate. With this assumption, the upper bound in part (b) follows immediately from part (a) noting that $\mathrm{st}_{\gamma_{\rm pr}}(T) \le \mathrm{st}^-_{\gamma_{\rm pr}}(T) \le 2\Delta - 1$. That the bound is tight for all $\Delta \ge 2$ follows from Proposition 3.1. □

8 Proof of Theorem 2.3

In this section we present a proof of Theorem 2.3, which we restate below.

Theorem 2.3. *If G is a connected graph with $\gamma_{\rm pr}(G) \ge 4$, then $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \le 2\Delta(G)$, and this bound is sharp.*

Proof. Let G be a connected graph with $\gamma_{\rm pr}(G) \ge 4$ and let $\Delta = \Delta(G)$. Since $\gamma_{\rm pr}(G) \ge 4$, we have $\Delta \ge 2$. If $\Delta = 2$, then G is a path P_n or a cycle C_n, and by Theorem 5.1, we have $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \le 2\Delta$, with equality if and only if $n \equiv 0 \pmod 4$. Assume, therefore, that $\Delta \ge 3$.

Let T be a spanning tree of G such that $\gamma_{\rm pr}(T) = \gamma_{\rm pr}(G)$. We note that such a tree exists by Lemma 4.1. Let S be a $\mathrm{st}^-_{\gamma_{\rm pr}}$-set of T. Thus, S is a set in $\mathrm{NI}(T)$ with $|S| = \mathrm{st}^-_{\gamma_{\rm pr}}(T)$ such that $\gamma_{\rm pr}(T - S) < \gamma_{\rm pr}(T)$. By Observation 4.2, we have

$|S| = \mathrm{st}^-_{\gamma_{\rm pr}}(T) \le n-2$. Since $S \in \mathrm{NI}(T)$, every vertex in $T-S$, and therefore in the supergraph $G-S$, has degree at least 1. Hence, $S \in \mathrm{NI}(G)$ and since $\gamma_{\rm pr}(G-S) \le \gamma_{\rm pr}(T-S)$, we have $\gamma_{\rm pr}(G-S) < \gamma_{\rm pr}(G)$. Thus, $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \le |S| = \mathrm{st}^-_{\gamma_{\rm pr}}(T)$. By Theorem 2.2, we have $\mathrm{st}^-_{\gamma_{\rm pr}}(T) \le 2\Delta(T)$. Noting that $\Delta(T) \le \Delta(G)$, we therefore have that $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \le \mathrm{st}^-_{\gamma_{\rm pr}}(T) \le 2\Delta(T) \le 2\Delta(G) = 2\Delta$.

To show that the upper bound in Theorem 2.3 is tight, we present a family of graphs with maximum degree Δ and $\gamma_{\rm pr}(G) \ge 4$ satisfying $\mathrm{st}^-_{\gamma_{\rm pr}}(G) = 2\Delta$. Our first family, $\mathcal{G}_\Delta$, is constructed as follows. For $k \ge 2$ and $\Delta \ge 2$, let $G_{k,\Delta}$ be a graph obtained from k double stars $S(\Delta-1, \Delta-1)$ by choosing two leaves at distance 3 apart in each double star and adding k edges between the chosen leaves in such a way, that every chosen vertex has degree 2 in the resulting graph. Let $\mathcal{G}_\Delta$ be the family of all such graphs $G_{k,\Delta}$ for all $k \ge 2$. The graph $G_{2,6} \in \mathcal{G}_6$, for example, is illustrated in Figure 5. We note that $\gamma_{\rm pr}(G_{k,\Delta}) = 2k$ and that set of $2k$ vertices of degree Δ is the unique $\gamma_{\rm pr}$-set of $G_{k,\Delta}$. Furthermore, $\mathrm{st}^-_{\gamma_{\rm pr}}(G_{k,\Delta}) = 2\Delta$. □

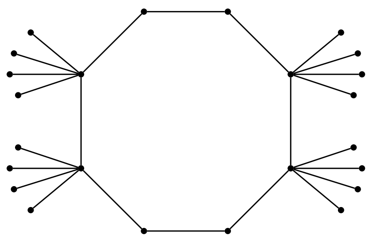

Figure 5: The graph $G_{2,6}$ from a class of graphs $G_{k,\Delta}$.

Recall that by definition we have $\mathrm{st}_{\gamma_{\rm pr}}(G) \le \mathrm{st}^-_{\gamma_{\rm pr}}(G)$ for every graph G. Hence, as an immediate consequence of Theorem 2.3 we have Corollary 2.4. Recall its statement.

Corrolary 2.4. *If G is a connected graph with $\gamma_{\rm pr}(G) \ge 4$, then $\mathrm{st}_{\gamma_{\rm pr}}(G) \le 2\Delta(G)$.*

It remains an open problem, however, to determine if the upper bound of Corollary 2.4 is best achievable for all values of possible value of $\Delta(G) = \Delta \ge 2$. If $\Delta = 2$ and G is a path, then $G \cong P_n$ where $n \ge 5$, and $\mathrm{st}_{\gamma_{\rm pr}}(G) \le 2\Delta - 2$ by Corollary 5.3. If $\Delta = 2$ and G is a cycle, then $G \cong C_n$ where $n \ge 5$, and $\mathrm{st}_{\gamma_{\rm pr}}(G) \le 2\Delta$ by Corollary 5.5, with equality if and only if $G = C_8$. Hence, the only connected graph G with maximum degree $\Delta = 2$ satisfying $\gamma_{\rm pr}(G) \ge 4$ and $\mathrm{st}_{\gamma_{\rm pr}}(G) = 2\Delta$ is the 8-cycle, namely $G = C_8$. For $\Delta \ge 3$, we do not know of a connected graph G with maximum degree Δ satisfying $\gamma_{\rm pr}(G) \ge 4$ and $\mathrm{st}_{\gamma_{\rm pr}}(G) = 2\Delta$.

By Corollary 5.5 and Proposition 3.1, for any given $\Delta \ge 2$, there do exists infinite families of connected graphs G with maximum degree Δ satisfying $\mathrm{st}_{\gamma_{\rm pr}}(G) = 2\Delta - 1$. Thus, if the upper bound of Corollary 2.4 can be improved to $\mathrm{st}_{\gamma_{\rm pr}}(G) \le 2\Delta - 1$ in the case when $\Delta \ge 3$, then this bound would be tight.

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References

[1] M. Amraee, N. Jafari Rad and M. Maghasedi, Roman domination stability in graphs, *Math. Rep. (Bucur.)* **21** (2019), 193–204, http://imar.ro/journals/Mathematical_Reports/php/2019/Mrc19_2.php.

[2] A. Aytaç and B. Atay Atakul, Exponential domination critical and stability in some graphs, *Int. J. Found. Comput. Sci.* **30** (2019), 781–791, doi:10.1142/s0129054119500217.

[3] D. Bauer, F. Harary, J. Nieminen and C. L. Suffel, Domination alteration sets in graphs, *Discrete Math.* **47** (1983), 153–161, doi:10.1016/0012-365x(83)90085-7.

[4] R. C. Brigham, P. Z. Chinn and R. D. Dutton, Vertex domination-critical graphs, *Networks* **18** (1988), 173–179, doi:10.1002/net.3230180304.

[5] R. C. Brigham, T. W. Haynes, M. A. Henning and D. F. Rall, Bicritical domination, *Discrete Math.* **305** (2005), 18–32, doi:10.1016/j.disc.2005.09.013.

[6] T. Burton and D. P. Sumner, Domination dot-critical graphs, *Discrete Math.* **306** (2006), 11–18, doi:10.1016/j.disc.2005.06.029.

[7] W. J. Desormeaux, T. W. Haynes and M. A. Henning, Total domination changing and stable graphs upon vertex removal, *Discrete Appl. Math.* **159** (2011), 1548–1554, doi:10.1016/j.dam.2011.06.006.

[8] W. J. Desormeaux, T. W. Haynes and M. A. Henning, Paired domination in graphs, in: T. W. Haynes, S. T. Hedetniemi and M. A. Henning (eds.), *Topics in Domination in Graphs*, Springer, Cham, volume 64 of *Dev. Math.*, pp. 31–77, 2020, doi:10.1007/978-3-030-51117-3_3.

[9] O. Favaron, D. P. Sumner and E. Wojcicka, The diameter of domination k-critical graphs, *J. Graph Theory* **18** (1994), 723–734, doi:10.1002/jgt.3190180708.

[10] T. W. Haynes, S. T. Hedetniemi and M. A. Henning (eds.), *Topics in Domination in Graphs*, volume 64 of *Developments in Mathematics*, Springer, Cham, 2020, doi:10.1007/978-3-030-51117-3.

[11] T. W. Haynes and P. J. Slater, Paired-domination in graphs, *Networks* **32** (1998), 199–206, doi:10.1002/(sici)1097-0037(199810)32:3<199::aid-net4>3.0.co;2-f.

[12] M. A. Henning and M. Krzywkowski, Total domination stability in graphs, *Discrete Appl. Math.* **236** (2018), 246–255, doi:10.1016/j.dam.2017.07.022.

[13] M. A. Henning and N. J. Rad, On total domination vertex critical graphs of high connectivity, *Discrete Appl. Math.* **157** (2009), 1969–1973, doi:10.1016/j.dam.2008.12.009.

[14] M. A. Henning and A. Yeo, *Total Domination in Graphs*, Springer Monographs in Mathematics, Springer, New York, 2013, doi:10.1007/978-1-4614-6525-6.

[15] N. Jafari Rad, E. Sharifi and M. Krzywkowski, Domination stability in graphs, *Discrete Math.* **339** (2016), 1909–1914, doi:10.1016/j.disc.2015.12.026.

[16] Z. Li, Z. Shao and S.-j. Xu, 2-rainbow domination stability of graphs, *J. Comb. Optim.* **38** (2019), 836–845, doi:10.1007/s10878-019-00414-0.

[17] D. P. Sumner, Critical concepts in domination, *Discrete Math.* **86** (1990), 33–46, doi:10.1016/0012-365x(90)90347-k.

[18] D. P. Sumner and P. Blitch, Domination critical graphs, *J. Comb. Theory Ser. B* **34** (1983), 65–76, doi:10.1016/0095-8956(83)90007-2.