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Distinguishing arc-colourings of symmetric digraphs

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Abstract

A symmetric digraph $\overleftrightarrow{G}$ is obtained from an undirected graph G by replacing each edge uv of G by a pair of opposite arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$. An arc-colouring of a digraph is called distinguishing if the only automorphism preserving it is the identity. The least number of colours in a distinguishing arc-colouring, not necessarily proper, of $\overleftrightarrow{G}$ is called the distinguishing index $D'(\overleftrightarrow{G})$. We study bounds for $D'(\overleftrightarrow{G})$. For proper distinguishing arc-colourings, the least number of colours is called the distinguishing chromatic index of $\overleftrightarrow{G}$. There are 15 possible types of proper arc-colourings of a digraph depending on the definition of adjacent arcs. In this paper we investigate distinguishing chromatic indices of $\overleftrightarrow{G}$ for the nine remaining types not considered in our two previous papers. We formulate several conjectures.

Keywords: Symmetry breaking, distinguishing index, distinguishing chromatic index.

Math. Subj. Class.: 05C15, 05C20, 05E18

1 Introduction

We use standard graph theory terminology and notation as in [21]. An edge-colouring of a graph or a digraph is called *general* if it is not necessarily proper. By $[k]$ we denote the set $\{1, \ldots, k\}$ of k smallest positive integers.

We say that a colouring of a graph G *breaks an automorphism* $\varphi \in \mathrm{Aut}(G)$ if φ does not preserve it. A colouring is *distinguishing* if it breaks all non-identity automorphisms. The first paper on distinguishing colourings was published in 1977 by Babai [3]. He studied general distinguishing vertex-colourings of infinite trees. This work was related to his

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investigations of the computational complexity of the graph isomorphism problem the results of which he announced just a few years ago. A wide interest in symmetry breaking by colourings was spawned by the paper [1] of Albertson and Collins in 1996. They introduced the definition of the *distinguishing number* of a graph as the least number of colours in a general distinguishing vertex-colouring. In 2006, Collins and Trenk in [6] initiated investigations of proper distinguishing vertex-colourings. They defined the *distinguishing chromatic number* $\chi_D(G)$ of a graph G as the least number of colours in a proper distinguishing vertex-colouring of G.

Ten years ago, Wilfried Imrich during his stay at our university as a visiting professor introduced the first two authors to the topic of distinguishing colourings. A special issue of ADAM dedicated to him is a good place to thank him for introducing us to the field. Wilfried contributed significantly to this area as a co-author of a dozen of papers. He was also one of the organizers of the BIRS workshop "Symmetry Breaking in Discrete Structures" held in September 2018 in Oaxaca, Mexico.

Consequently in 2015, in paper [12] we introduced the *distinguishing index* $D'(G)$ of a graph G as the least number of colours in a general distinguishing edge-colouring of G. In the same paper, its counterpart for proper colourings called the *chromatic distinguishing index*, denoted by $\chi'_D(G)$, was also defined. Clearly, both invariants are defined for any connected graph except K_2.

The concept of distinguishing edge-colourings of a graph can be naturally extended to arc-colourings of digraphs. By $\overleftrightarrow{G}$ we denote a symmetric digraph obtained from a simple graph G by replacing each edge uv by a pair of opposite arcs $\overrightarrow{uv}, \overrightarrow{vu}$. The definition of proper arc-colouring of a digraph depends on the definition of adjacent arcs. There are four digraphs A_1, A_2, A_3, A_4 with two arcs having at least one vertex in common (see Figure 1):

- 2-*cycle* A_1 with arcs $\overrightarrow{uv}, \overrightarrow{vu}$,
- 2-*path* A_2 with arcs $\overrightarrow{uv}, \overrightarrow{vw}$,
- *source* A_3 with arcs $\overrightarrow{uv}, \overrightarrow{uw}$,
- *sink* A_4 with arcs $\overrightarrow{uv}, \overrightarrow{wv}$.

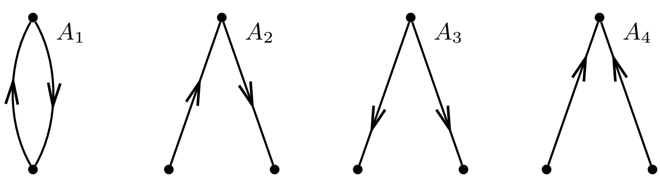

Figure 1: Four weakly connected digraphs with two arcs

Thus, there are 15 possible definitions of a proper colouring of a digraph since there are 15 possible definitions of adjacency of arcs corresponding to non-empty forbidden monochromatic subsets of the set of the four digraphs $A_i, i = 1, 2, 3, 4$. We denote by $\chi'_i(\overleftrightarrow{G})$ the *chromatic index* of a symmetric digraph $\overleftrightarrow{G}$, i.e. the least number of colours in a proper arc-colouring of $\overleftrightarrow{G}$, where "proper" means "without monochromatic digraph A_i". We also use the notation $\chi'_{i,j}(\overleftrightarrow{G}), \chi'_{i,j,k}(\overleftrightarrow{G}), \chi'_{i,j,k,l}(\overleftrightarrow{G})$ if more monochromatic two-arc digraphs are forbidden. A bit surprisingly, only four types of proper arc-colourings of digraphs have been already investigated in literature. These are the types corresponding to $\chi'_{1,2}$ (Behzad [4]), $\chi'_{1,2,3}$ or equivalently $\chi'_{1,2,4}$ (Algor and Alon [2]), and $\chi'_{3,4}$ (West [21]).

Similarly, $\chi'_{D_i}(\overleftrightarrow{G}), \chi'_{D_{i,j}}(\overleftrightarrow{G}), \chi'_{D_{i,j,k}}(\overleftrightarrow{G}), \chi'_{D_{i,j,k,l}}(\overleftrightarrow{G})$ stands for the *distinguishing chromatic index* of $\overleftrightarrow{G}$, i.e. the least number of colours in a distinguishing proper arc-colouring, where the indicated two-arc digraphs cannot be monochromatic.

This paper completes our investigations of distinguishing arc-colourings of symmetric digraphs initiated in our papers [13, 14]. That is, we determine bounds for the distinguishing chromatic indices and discuss their sharpness for all the cases which were not considered in our two previous papers. We restrict our investigations of distinguishing colourings to connected graphs G to avoid dealing with isomorphic components. We also determine bounds for the corresponding chromatic indices if they were not studied yet, what is rather easy for these types of proper colourings. In the last section we summarize the results by listing bounds for both indices for a general colouring and for all 15 types of proper colourings.

2 Useful known facts

By $\widehat{G}$ we denote a *subdivision* of a graph G, i.e. a graph obtained by replacing each edge uv of G by a path uxv of length two.

If c is an arc-colouring of $\overleftrightarrow{G}$, then we define a corresponding edge-colouring $\widehat{c}$ of $\widehat{G}$ by the following rule. If uv is an edge of G and x is a vertex of $\widehat{G}$ adjacent to both u and v, then $\widehat{c}(ux) = c(\overrightarrow{uv})$ and $\widehat{c}(vx) = c(\overrightarrow{vu})$. Sometimes it is more convenient to consider the colouring $\widehat{c}$ of $\widehat{G}$ instead of c, by using the following two facts proved in [13].

Proposition 2.1 ([13]). *For every connected graph G, $\mathrm{Aut}(\overleftrightarrow{G}) \cong \mathrm{Aut}(G)$. Moreover, if G is not a cycle, then $\mathrm{Aut}(\overleftrightarrow{G}) \cong \mathrm{Aut}(\widehat{G}) \cong \mathrm{Aut}(G)$.*

Lemma 2.2 ([13]). *Let G be a connected graph different from a cycle. Then an arc-colouring c of $\overleftrightarrow{G}$ is distinguishing if and only if the corresponding edge-colouring $\widehat{c}$ is a distinguishing edge-colouring of $\widehat{G}$.*

If G is a cycle and the edge-colouring $\widehat{c}$ of $\widehat{G}$ is distinguishing, then so also is the arc-colouring c of $\overleftrightarrow{G}$, but not necessarily conversely.

An edge-colouring of $\widehat{G}$ can be also viewed as a colouring of the two *halfedges* of each edge of the graph G, and distinguishing colourings of them have applications in computer science (cf. [7, 8]). This is an additional motivation for our investigations.

In the next section we use the following sharp upper bound for the distinguishing index $D'(G)$ of a connected graph G.

Theorem 2.3 ([12]). *If G is a connected graph, then*

$$D'(G) \leq \Delta(G)$$

except for three small cycles C_3, C_4, C_5 which need three colours.

This result was strengthened by the second author. A tree is *symmetric* (resp. *bisymmetric*) if it has a central vertex v_c (resp. a central edge e_c), all leaves are of the same distance from v_c (resp. e_c), and every vertex that is not a leaf has the same degree.

Theorem 2.4 ([16]). *If G is a connected graph, then $D'(G) = \Delta(G)$ if and only if G is a cycle C_k of length $k \geq 6$, a symmetric or bisymmetric tree, K_4 or $K_{3,3}$.*

In Section 4 two of our other previous results concerning the distinguishing chromatic index $\chi'(G)$ are used.

Theorem 2.5 ([12]). *The chromatic distinguishing index of every connected graph $G \neq K_2$ with maximum degree Δ satisfies the inequalities*

$$\Delta \leq \chi'_D(G) \leq \Delta + 1$$

except for four small graphs $C_4, K_4, C_6, K_{3,3}$, whose distinguishing index equals $\Delta + 2$.

Theorem 2.6 ([12]). *If T is a tree of order $n \geq 3$, then $\chi'_D(T) = \Delta(T) + 1$ if and only if T is a bisymmetric tree.*

3 General colourings

In this section we consider general arc-colourings of $\overleftrightarrow{G}$. Obviously, any general colouring demands only 1 colour.

It follows from Lemma 2.2 that, for every connected graph G distinct from a cycle, the distinguishing index of the symmetric digraph $\overleftrightarrow{G}$ equals the distinguishing index of the graph $\widehat{G}$, that is, $D'(\overleftrightarrow{G}) = D'(\widehat{G})$. For a cycle C_n, we have $D'(\overleftrightarrow{C_n}) = 2$ since it suffices to use the same colour for all arcs except for one arc $\overrightarrow{uv}$ which gets another colour. Indeed, this is a distinguishing colouring of $\overleftrightarrow{C_n}$ because it fixes two consecutive vertices u, v of $\overleftrightarrow{C_n}$.

First we prove bounds for the distinguishing index of $\overleftrightarrow{G}$ in terms of the maximum degree of the underlying graph G.

Theorem 3.1. *For every connected graph G,*

$$1 \leq D'(\overleftrightarrow{G}) \leq \left\lceil \sqrt{\Delta(G)} \right\rceil.$$

Proof. Let T be a spanning tree of the graph G. We pick a leaf v_1 and view it as a root of T. We construct a distinguishing arc-colouring of $\overleftrightarrow{G}$ with colours from the set $[\lceil \sqrt{\Delta(G)} \rceil]$. First, we colour arcs of the symmetric digraph $\overleftrightarrow{T}$ as follows.

Let v_2 be the only neighbour of v_1 in T. We begin with colouring the pair of arcs $(\overrightarrow{v_1v_2}, \overrightarrow{v_2v_1})$ by the pair $(1, 1)$. Then we do not use the pair $(1, 1)$ for a pair of opposite arcs any more. Next we colour all other pairs of opposite arcs incident to v_2 with distinct ordered pairs of colours, but one of them we necessarily colour with the pair $(1, 2)$, i.e. for some neighbour v_3 of v_2, we colour the arc $\overrightarrow{v_2v_3}$ by 1, and the arc v_3v_2 by 2. Then we proceed recursively according to the BFS ordering $v_1, v_2, \ldots, v_n$ of vertices of the tree T rooted at v_1. Suppose that we have coloured all arcs incident to the vertices $v_1, \ldots, v_{i-1}$ for some $i \geq 3$. We colour all yet uncoloured pairs of opposite arcs incident to v_i with distinct pairs (α, β), where α is the colour of the arc outgoing from v_i. We have enough colours to do this since there are at most $\Delta(G) - 1$ pairs of arcs to be coloured, and we have $\lceil \sqrt{\Delta(G)} \rceil^2 - 1$ distinct pairs of colours at our disposal, as the pair $(1, 1)$ is excluded.

When the colouring of $\overleftrightarrow{T}$ is completed, we colour all uncoloured pairs of opposite arcs of $\overleftrightarrow{G}$ with the pair $(2, 2)$. In our arc-colouring of $\overleftrightarrow{G}$, the vertex v_1 is the only vertex with an incident pair of opposite arcs coloured with $(1, 1)$ and without an incident pair of arcs coloured with $(1, 2)$. Therefore, v_1 is fixed by every automorphism of $\overleftrightarrow{G}$ preserving our colouring.

For $i \geq 0$, let $S_i(v_1)$ denote the *i-th sphere centered at* v_1, i.e. the set of all vertices of T whose distance in T from v_1 equals i. Hence, each sphere $S_i(v_1)$ is fixed setwise because automorphisms preserve distances. But it is also fixed pointwise what can be easily shown by induction on i. Indeed, each pair of opposite arcs between a vertex $v \in S_{i-1}(v_1)$ and its neighbour in $S_i(v_1)$ has a distinct pair of colours (α, β). Consequently, each neighbour of v is also fixed. Therefore, our colouring of $\overleftrightarrow{G}$ is distinguishing. □

By Proposition 2.1, $D'(\overleftrightarrow{G}) = 1$ if and only if G is an asymmetric graph, i.e. $\mathrm{Aut}(G) = \{\mathrm{id}\}$. The following result shows that the upper bound in Theorem 3.1 is tight as well.

Proposition 3.2. *If T is a symmetric tree, then $D'(\overleftrightarrow{T}) = \left\lceil \sqrt{\Delta(T)} \right\rceil$.*

Proof. By Theorem 3.1, we have $D'(\overleftrightarrow{T}) \leq \lceil \sqrt{\Delta(T)} \rceil$. On the other hand, suppose that there exists a distinguishing arc-colouring c' of $\overleftrightarrow{T}$ with less than $\lceil \sqrt{\Delta(T)} \rceil$ colours. Let v_0 be the central vertex of a symmetric tree T. Hence, every edge uv of T, such that u lies on a path from v_0 to v is coloured by an ordered pair (i, j), where $i = c'(\overrightarrow{uv})$ and $j = c'(\overrightarrow{vu})$. This yields a distinguishing edge-colouring of T with less than $\Delta(T)$ colours since the number of such pairs (i, j) is less than $\Delta(T)$. This contradicts the fact that $D'(T) = \Delta(T)$ stated in Theorem 2.4. □

Recall that $D'(G) \leq \Delta(G)$ for connected graphs G with few exceptions, due to Theorem 2.3. We believe that Theorem 3.1 can be strengthened as follows.

Conjecture 3.3. *If G is a connected graph, then*

$$D'(\overleftrightarrow{G}) \leq \left\lceil \sqrt{D'(G)} \right\rceil.$$

We now confirm this conjecture for graphs G with $D'(G) \leq 3$. This condition is fulfilled by a couple of classes of graphs, e.g. regular graphs [15], graphs with a Hamiltonian path [16], 3-connected planar graphs [17], Cartesian powers of connected graphs [9], connected graphs of maximum degree at most four without pendant edges [11], and for claw-free graphs [16].

Proposition 3.4. *If G is a connected graph with $D'(G) \leq 3$, then*

$$D'(\overleftrightarrow{G}) \leq 2.$$

Proof. Let c be a distinguishing edge-colouring of G with three colours $1, 2, 3$. Consider the correspondence $\iota(1) = (1, 1), \iota(2) = (2, 2), \iota(3) = (1, 2)$. Every pair $(\overrightarrow{uv}, \overrightarrow{vu})$ of $\overleftrightarrow{G}$ we colour with the pair $\iota(c(uv))$. This yields a distinguishing arc-colouring of $\overleftrightarrow{G}$ since c is distinguishing. □

Conjecture 3.3 is also true for some bipartite graphs, including trees.

Proposition 3.5. *Let G be a connected bipartite graph with the partition sets X and Y. If G does not admit an automorphism that interchanges X and Y, or $D'(G)$ is not a square of an integer, then*

$$D'(\overleftrightarrow{G}) \leq \left\lceil \sqrt{D'(G)} \right\rceil.$$

Proof. The claim is true for even cycles since they are regular. Then assume G is a connected bipartite graph different from a cycle. Clearly, if α is an automorphism of G, then either $\alpha(X) = X$ and $\alpha(Y) = Y$, or $\alpha(X) = Y$ and $\alpha(Y) = X$.

Suppose that c is a distinguishing edge-colouring of G with the set $[D'(G)]$ of colours. Choose an injective map ι from the set $[D'(G)]$ into the set $[\lceil\sqrt{D'(G)}\rceil]^2$. Let $e = uv$ be any edge of G, where $u \in X, v \in Y$, and let $x \in V(\widehat{G}) \setminus V(G)$ be a vertex subdividing e. If $\iota(c(e)) = (i, j)$, then we define an edge-colouring $\widehat{c}$ of $\widehat{G}$ by putting $\widehat{c}(ux) = i$ and $\widehat{c}(xv) = j$. Then $\widehat{c}$ is a distinguishing colouring unless there exists an automorphism $\widehat{\alpha}$ of $\widehat{G}$ that maps a path of length two between vertices of G coloured with a pair (i, j) onto a path coloured with (j, i). Since c is distinguishing, this would be possible only if the corresponding automorphism α of G interchanged the sets X and Y.

Thus we are left with the case when $\sqrt{D'(G)}$ is not an integer. Then $\lceil\sqrt{D'(G)}\rceil^2 > D'(G)$, and therefore the set $[\lceil\sqrt{D'(G)}\rceil]^2$ has more elements than $[D'(G)]$. Consequently, we can choose an injection ι such that there exists a pair (i, j) with $(i, j) = \iota(k)$ for some $k \in [D'(G)]$, while $(j, i) \neq \iota(l)$ for any $l \in [D'(G)]$. It follows that $\alpha(X) = X$ for every automorphism α preserving the edge-colouring $\widehat{c}$ of $\widehat{G}$ defined above. Hence, $\widehat{c}$ is a distinguishing edge-colouring of $\widehat{G}$.

The conclusion follows from Lemma 2.2. $\square$

Corollary 3.6. *Every tree T satisfies the inequality $D'(\overleftrightarrow{T}) \leq \left\lceil \sqrt{D'(T)} \right\rceil$.*

Proof. If a tree T has a central vertex, then the conclusion follows from Proposition 3.5 because T does not admit an automorphism interchanging the sets of bipartition.

Suppose then that T has a central edge $e = uv$. Let x be a vertex of $\widehat{G}$ subdividing e. For a distinguishing edge-colouring $\widehat{c}$ defined in the proof of Proposition 3.5, it suffices to choose an injection ι such that $\widehat{c}(ux) \neq \widehat{c}(xv)$. Then $\widehat{c}$ breaks every automorphism of T switching u and v. $\square$

In virtue of Proposition 3.2, there are graphs for which the bound in Conjecture 3.3 is achieved. There are also infinitely many graphs G satisfying the inequality $D'(\overleftrightarrow{G}) < \lceil\sqrt{D'(G)}\rceil$. Let F_k be a *windmill with k wings*, that is, a graph obtained by gluing together k triangles in one vertex, say w. If wu and wv are two edges of one of these triangles, then they have to be coloured with two distinct colours in every distinguishing edge-colouring of the windmill F_k. However, in $\overleftrightarrow{F}$ the pairs of arcs connecting w with u and v, respectively, can have the same colouring if the arcs $\overrightarrow{uv}, \overrightarrow{vu}$ have distinct colours. Then it is easy to see that there are infinitely many windmills for which the bound in Conjecture 3.3 is not achieved. The smallest one is F_{25} with $D'(F_{25}) = 5$, while $D'(\overleftrightarrow{F_{25}}) = 2 < \lceil\sqrt{D'(F_{25})}\rceil$.

4 Proper colourings

As it was already mentioned in the Introduction, there are 15 possible definitions of proper arc-colourings depending on a collection of forbidden monochromatic two-arc digraphs. In this section we study nine remaining cases that were not considered in our previous papers [13, 14]. The following observation will be useful in some cases of proper colourings of $\overleftrightarrow{G}$.

Lemma 4.1 ([13]). *Let c be an arc-colouring of $\overleftrightarrow{G}$ without monochromatic sources or sinks. If there exists a vertex which is fixed by every automorphism of $\overleftrightarrow{G}$ preserving c, then c is a distinguishing colouring.*

In each of the next subsections, we use the term *proper arc-colouring* in the sense indicated in the title of the subsection.

Given a set of forbidden monochromatic digraphs containing the source A_3, if we replace it by the sink A_4, then the corresponding invariants in question will be equal, since it suffices to reverse the arcs in the coloured digraph. Hence, we consider both cases in the same subsection.

4.1 Forbidden monochromatic 2-cycles and sources (or sinks)

First assume that monochromatic 2-cycles and sources are forbidden.

Proposition 4.2. *For every graph G of maximum degree $\Delta \geq 2$, we have $\chi'_{1,3}(\overleftrightarrow{G}) = \Delta$.*

Proof. It is easy to observe that an arc-colouring c of $\overleftrightarrow{G}$ has no monochromatic 2-cycles and sources if and only if the corresponding colouring $\widehat{c}$ is a proper edge-colouring of $\widehat{G}$. Thus $\chi'_{1,3}(\overleftrightarrow{G}) = \chi'(\widehat{G})$. By Kőnig's theorem, $\chi'(\widehat{G}) = \Delta(\widehat{G})$ since $\widehat{G}$ is a bipartite graph. □

Now, we determine sharp bounds for the chromatic distinguishing index $\chi'_{D_{1,3}}(\overleftrightarrow{G})$.

Proposition 4.3. *For every connected graph G,*

$$\Delta(G) \leq \chi'_{D_{1,3}}(\overleftrightarrow{G}) \leq \Delta(G) + 1.$$

Proof. Let us first consider a cycle C_n with $n \geq 3$. To obtain a proper arc-colouring of $\overleftrightarrow{C_n}$, we colour all arcs of a directed cycle $\overrightarrow{C_n}$ with colour 1, and all arcs of the opposite directed cycle with colour 2. This is a unique proper colouring of $\overleftrightarrow{C_n}$ up to a permutation of colours. To obtain a distinguishing colouring, it suffices to recolour exactly one arc, say $\overrightarrow{uv}$, with a third colour, thus fixing two consecutive vertices u, v of the cycle.

Suppose now that G is not a cycle. Then, as we have already observed in the last paragraph of the proof of the previous proposition, an arc-colouring c of $\overleftrightarrow{G}$ is proper if and only if the corresponding edge-colouring $\widehat{c}$ of $\widehat{G}$ is proper. Therefore, $\chi'_{D_{1,3}}(\overleftrightarrow{G}) = \chi'_D(\widehat{G})$ in virtue of Lemma 2.2. It follows from Theorem 2.5 applied to $\widehat{G}$ that $\Delta(G) \leq \chi'_D(\widehat{G}) \leq \Delta(G) + 1$ since none of the four exceptional graphs $C_4, K_4, C_6, K_{3,3}$ is a subdivision of a graph different from a cycle. □

In virtue of Theorem 2.6, the lower bound in the above Proposition 4.3 is achieved by each tree T. Indeed, $\chi'_{D_{1,3}}(\overleftrightarrow{T}) = \chi'_D(\widehat{T})$ and the subdivision $\widehat{T}$ cannot be a bisymmetric tree because $\widehat{T}$ has an even number of edges. As we showed, the upper bound is achieved by each cycle C_n. We conjecture that this is the case only for cycles.

Conjecture 4.4. *If $G \neq K_2$ is a connected graph, then $\chi'_{D_{1,3}}(\overleftrightarrow{G}) = \Delta(G)$ unless $G = C_n$.*

Now, assume that monochromatic 2-cycles and sinks are forbidden. By reversing arcs in $\overleftrightarrow{G}$, we immediately infer that $\chi'_{1,4}(\overleftrightarrow{G}) = \chi'_{1,3}(\overleftrightarrow{G})$ and $\chi'_{D_{1,4}}(\overleftrightarrow{G}) = \chi'_{D_{1,3}}(\overleftrightarrow{G})$.

4.2 Forbidden monochromatic 2-paths and sources (or sinks)

By reversing arcs, we have $\chi'_{2,4}(\overleftrightarrow{G}) = \chi'_{2,3}(\overleftrightarrow{G})$ and $\chi'_{D_{2,4}}(\overleftrightarrow{G}) = \chi'_{D_{2,3}}(\overleftrightarrow{G})$. Hence, we describe only the case when 2-paths and sources cannot be monochromatic.

Proposition 4.5. *For every graph G,*

$$\Delta(G) \leq \chi'_{2,3}(\overleftrightarrow{G}) \leq \Delta(G) + 1.$$

Proof. Clearly, $\chi'_{2,3}(\overleftrightarrow{G}) \geq \Delta(G)$ since monochromatic sources are forbidden. On the other hand, let c' be a proper edge-colouring of G. For every edge uv of G, we colour both opposite arcs $\overrightarrow{uv}, \overrightarrow{vu}$ of $\overleftrightarrow{G}$ with the same colour $c'(uv)$. This yields an arc-colouring of $\overleftrightarrow{G}$ without monochromatic 2-paths and sources. Hence, $\chi'_{2,3}(\overleftrightarrow{G}) \leq \Delta(G) + 1$, due to the theorem of Vizing. □

It follows from the proof, that $\chi'_{2,3}(\overleftrightarrow{G}) = \Delta(G)$ for every graph G of Class 1. As monochromatic 2-paths and sources are forbidden, in any arc-colouring of a symmetric digraph $\overleftrightarrow{G}$ with $\Delta(G)$ colours, each pair of opposite arcs incident to a vertex of degree $\Delta(G)$ in G, has to be coloured with the same colour. Consequently, $\chi'_{2,3}(\overleftrightarrow{G}) = \Delta(G) + 1$ for every regular graph of Class 2. This encourages us to formulate the following bold conjecture.

Conjecture 4.6. *Every graph G satisfies the equality $\chi'_{2,3}(\overleftrightarrow{G}) = \chi'(G)$.*

Let us now move on to distinguishing colourings.

Proposition 4.7. *For every connected graph G,*

$$\Delta(G) \leq \chi'_{D_{2,3}}(\overleftrightarrow{G}) \leq \Delta(G) + 1.$$

Proof. The first inequality is obvious. By Theorem 2.5, there exists a distinguishing proper edge-colouring c' of the graph G with at most $\Delta(G) + 1$ colours (it can be easily checked that the claim holds for the four exceptional small graphs). In the symmetric digraph $\overleftrightarrow{G}$, we colour each pair of opposite arcs $\overrightarrow{uv}, \overrightarrow{vu}$ with the same colour $c'(uv)$, thus obtaining a proper distinguishing arc-colouring. □

The lower bound is achieved for every tree T which is not bisymmetric. Indeed, by Theorem 2.6, such a tree admits a distinguishing proper edge-colouring c' with $\Delta(T)$ colours. If uv is an edge of T, then we colour both arcs $\overrightarrow{uv}, \overrightarrow{vu}$ with $c'(uv)$, and thus we get a distinguishing proper arc-colouring of $\overleftrightarrow{T}$.

The upper bound is achieved by every bisymmetric tree T. To see that, take any proper arc-colouring c of $\overleftrightarrow{T}$ with $\Delta(T)$ colours. Then each pair of opposite arcs creates a monochromatic 2-cycle, and such a colouring is unique up to a permutation of colours. Hence, c does not break any automorphism reversing the central edge, so it cannot be distinguishing. The upper bound is also achieved by every Class 2 regular graph G since $\chi'_{D_{2,3}}(\overleftrightarrow{G}) \geq \chi'_{2,3}(\overleftrightarrow{G}) = \Delta(G) + 1$.

4.3 Forbidden monochromatic sources and sinks

This type of proper arc-colouring of digraphs was studied by several authors (cf. [21]). It is well known that $\chi'_{3,4}(\overleftrightarrow{G}) = \Delta(G)$.

Proposition 4.8. *For every connected graph G,*

$$\Delta(G) \leq \chi'_{D_{3,4}}(\overleftrightarrow{G}) \leq \Delta(G) + 1.$$

Proof. Let c be a proper arc-colouring of $\overleftrightarrow{G}$ with $\Delta(G)$ colours. We pick an arc $\overrightarrow{uv}$ and recolour it with an extra colour. Hence, the vertex u is fixed by every automorphism preserving this new colouring since u is the only vertex with an outgoing arc with the extra colour. The claim follows from Proposition 4.1. □

Proposition 4.8 is tight. It is easily seen that $\chi'_{D_{3,4}}(\overleftrightarrow{C_n}) = \Delta(C_n) + 1 = 3$. On the other hand, $\chi'_{D_{3,4}}(\overleftrightarrow{K_n}) = \Delta(K_n)$, except for $n \leq 4$, as we show below.

Proposition 4.9. $\chi'_{D_{3,4}}(\overleftrightarrow{K_n}) = n - 1$ *for* $n \geq 5$.

Proof. Suppose first that $n \geq 7$. To construct a distinguishing proper arc-colouring of $\overleftrightarrow{K_n}$ we take a specific factorization $\mathcal{F}$ of K_n. As an initial factor F_1 we take a Hamiltonian cycle of K_n. The next factor F_2 depends on the parity of n. If n is odd, then F_2 is a union of a triangle and a cycle of length $n - 3 \geq 4$. For even n, the factor F_2 is a perfect matching of K_n that creates exactly one triangle with the edges of the Hamiltonian cycle F_1. Deletion of the arcs of F_1 and F_2 yields a regular graph of even degree which has a 2-factorization due to the classic theorem of Petersen.

The factors of $\mathcal{F}$ are 2-factors, with one possible exception which is a 1-factor. To each 2-factor $F \in$ we assign two colours and choose an orientation of F in such a way that each cycle of F becomes a directed cycle. Each such orientation is coloured with one and the same colour, while the opposite orientation gets the other colour. When F is a 1-factor, then we colour all arcs corresponding to the edges of F with the same colour. Our colouring of the Hamiltonian cycle F_1 breaks all automorphisms of K_n except rotations of F_1. But the colouring of F_2 breaks them. Hence, this is a distinguishing arc-colouring of K_n.

For $n = 6$, let $v_1, \ldots, v_6$ be the vertices of $\overleftrightarrow{K_6}$. We decompose $\overleftrightarrow{K_6}$ into five digraphs (actually oriented graphs except one 2-cycle) $H_1, \ldots, H_5$, and the arcs of each digraph H_i get colour i. Namely, H_1 is a directed Hamiltonian cycle $v_1 \ldots v_6 v_1$, H_2 is a union of the directed 4-cycle $v_3v_2v_6v_5v_3$ and a 2-cycle $v_1v_4v_1$, H_3 is a union of two directed triangles $v_2v_1v_6v_2$ and $v_5v_4v_3v_5$, H_4 is a directed Hamiltonian cycle $v_1v_5v_2v_4v_6v_3v_1$, and H_5 is an opposite orientation of H_4. The colouring of H_1 and H_2 is preserved only by a rotation of H_1 mapping v_1 into v_4, which is broken by the colouring of H_4.

For $n = 5$, we decompose $\overleftrightarrow{K_5}$ into three directed Hamiltonian cycles $H_1 = v_1 \ldots v_5 v_1$, $H_2 = v_1v_3v_2v_5v_4v_1$, $H_3 = v_1v_5v_2v_4v_3v_1$, and a union H_4 of a directed triangle $v_2v_1v_4v_2$ and a 2-cycle $v_3v_5v_3$. We colour each H_i with colour i. The colouring of H_4 breaks each rotation of H_1. □

We conclude this subsection with a conjecture.

Conjecture 4.10. *If G is a connected graph, then* $\chi'_{D_{3,4}}(\overleftrightarrow{G}) = \Delta(G)$ *unless G is a cycle or* K_2 *or* K_4.

4.4 Forbidden monochromatic 2cycles, 2paths and sources (or sinks)

Here again, by reversing arcs, we have $\chi'_{1,2,3}(\overleftrightarrow{G}) = \chi'_{1,2,4}(\overleftrightarrow{G})$ and $\chi'_{D_{1,2,3}}(\overleftrightarrow{G}) = \chi'_{D_{1,2,4}}(\overleftrightarrow{G})$. Therefore we can focus on arc-colourings without monochromatic 2-cycles, 2-paths and sources.

In 1989, Algor and Alon [2] introduced the *directed star arboricity* of a digraph D as the least number of directed star forests into which D can be decomposed, where arcs of each star are directed to its center. Clearly, for a symmetric digraph $\overleftrightarrow{G}$ this invariant coincides with $\chi'_{1,2,3}(\overleftrightarrow{G})$. In 1997, Guiduli [10] showed that the directed star arboricity of a symmetric digraph $\overleftrightarrow{G}$ equals the incidence chromatic number of the graph G, denoted $\chi_{\mathrm{inc}}(G)$, a concept introduced by Brualdi and Quinn Massey [5] in 1993, and then investigated by tens of authors (cf. a constantly updated special webpage [19]). Consequently, $\chi'_{1,2,3}(\overleftrightarrow{G}) = \chi_{\mathrm{inc}}(G)$ for any graph G.

An *incidence* of a graph G is a pair (v, e) where v is a vertex of G and e an edge of G incident with v. Two incidences (v, e) and (w, f) are *adjacent* if either $v = w$, or $e = f$, or $vw \in \{e, f\}$. The *incidence chromatic number* of a graph G is the smallest number of colours in a colouring of the set of incidences of G such that adjacent incidences get distinct colours. It is known [5] that $\Delta(G) + 1 \leq \chi_{\mathrm{inc}}(G)$, but no sharp upper bound for $\chi_{\mathrm{inc}}(G)$ was found. A current best general upper bound proved in [10] is $\chi_{\mathrm{inc}}(G) \leq \Delta(G) + 20 \log \Delta(G) + 84$, and there are infinitely many Paley graphs with $\chi_{\mathrm{inc}}(G) = \Delta(G) + \Omega(\log \Delta(G))$.

For distinguishing colourings, we only have the following easy observation.

Proposition 4.11. *If G is a connected graph, then*

$$\chi_{\mathrm{inc}}(G) \leq \chi'_{D_{1,2,3}}(\overleftrightarrow{G}) = \chi'_{D_{1,2,4}}(\overleftrightarrow{G}) \leq \chi_{\mathrm{inc}}(G) + 1.$$

Proof. The left-hand side inequality is obvious. Let c be a proper arc-colouring of $\overleftrightarrow{G}$ with $\chi_{\mathrm{inc}}(G)$ colours. We recolour one arc of $\overleftrightarrow{G}$, say outgoing from a vertex u, with an extra colour. Thus u is fixed by every automorphism preserving the new colouring because u is the only vertex of $\overleftrightarrow{G}$ with an outgoing arc with the extra colour. By Lemma 4.1, this colouring is distinguishing. □

Proposition 4.11 is tight. For complete graphs we have $\chi_{\mathrm{inc}}(K_n) = \chi'_{D_{1,2,3}}(\overleftrightarrow{K_n}) = n = \Delta(K_n) + 1$ since the arc-colouring where all arcs ingoing to the i-th vertex of $\overleftrightarrow{K_n}$ are coloured with colour i, for $i = 1, \ldots, n$, is proper and distinguishing. On the other hand, it is not difficult to check that $\chi'_{D_{1,2,3}}(\overleftrightarrow{C_{3k}}) = \chi_{\mathrm{inc}}(C_{3k}) + 1 = 4$ for $k \geq 2$. To see this, observe that a proper arc-colouring of $\overleftrightarrow{C_{3k}}$ with $\chi'_{1,2,3}(\overleftrightarrow{C_{3k}}) = 3$ colours is unique up to permutation of colours and is preserved by some rotation of C_{3k}.

4.5 Forbidden monochromatic 2paths, sources and sinks

In this case, we can use the same arguments as in Subsection 4.2. Also, the examples showing the sharpness can be similar. Hence, we omit proofs and discussion of sharpness of the two propositions below.

Proposition 4.12. *For every graph G, $\Delta(G) \leq \chi'_{2,3,4}(\overleftrightarrow{G}) \leq \Delta(G) + 1$.* □

Proposition 4.13. *If G is a connected graph, then $\Delta(G) \leq \chi'_{D_{2,3,4}}(\overleftrightarrow{G}) \leq \Delta(G)+1$.* □

4.6 Forbidden all four weakly connected monochromatic two-arc digraphs

In this subsection, an arc-colouring c of $\overleftrightarrow{G}$ is proper if and only if any two arcs with the same colour do not have a vertex in common. Thus every colour class with respect to c corresponds to a matching in G.

Proposition 4.14. *For every graph G,*

$$2\Delta(G) \leq \chi'_{1,2,3,4}(\overleftrightarrow{G}) \leq 2\Delta(G)+2.$$

Proof. Let G' be a multigraph obtained from the symmetric digraph $\overleftrightarrow{G}$ by ignoring orientations of arcs. In other words, G' arises from the graph G by substituting all edges by double edges. Clearly, $\chi'_{1,2,3,4}(\overleftrightarrow{G}) = \chi'(G')$, and by the well-known theorem of Vizing [20] about the chromatic index of a multigraph $\Delta(G') \leq \chi(G') \leq \Delta(G')+2$, where $\Delta(G') = 2\Delta(G)$. □

Theorem 4.15. *For every connected graph G,*

$$\chi'_{D_{1,2,3,4}}(\overleftrightarrow{G}) = \chi'_{1,2,3,4}(\overleftrightarrow{G}).$$

Whence, $2\Delta(G) \leq \chi'_{D_{1,2,3,4}}(\overleftrightarrow{G}) \leq 2\Delta(G)+2$*.*

Proof. Let $c'\colon E(G') \to [\chi'_{1,2,3,4}(\overleftrightarrow{G})]$ be a proper edge-colouring of the multigraph G' defined in the previous proof. We pick a vertex w of degree $\Delta(G)$ in G. We define an arc-colouring c of $\overleftrightarrow{G}$ based on the colouring c' as follows. Without loss of generality we may assume that $[\Delta(G)]$ is the set of colours of arcs outgoing from w and $[2\Delta(G)] \setminus [\Delta(G)]$ is the set of colours of the ingoing arcs. Consider w as a root of a BFS spanning tree T of the graph G. For each pair of edges $uv \in E(G')$, we orient them in such a way that $c(\overrightarrow{uv}) < c(\overrightarrow{vu})$ if u precedes v in the BFS ordering of T.

Suppose now that $\varphi \in \mathrm{Aut}(\overleftrightarrow{G})$ preserves the colouring c, and $\varphi(w) = v$. If $v \neq w$, then there is an in-neighbour u of v that precedes v in the BFS ordering. Then $c(\overrightarrow{uv}) \in [2\Delta(G)] \setminus [\Delta(G)]$ as $\overrightarrow{uv}$ is an arc ingoing to $v = \varphi(w)$. Hence, $c(\overrightarrow{vu}) > \Delta(G)$ by our definition of c. However, the colours of all arcs outgoing from w, and hence from $\varphi(w)$, belong to the set $[\Delta(G)]$, a contradiction. Therefore, φ is the identity, by Lemma 4.1. □

Each of the three possible values of $\chi'_{1,2,3,4}(\overleftrightarrow{G})$, and thus of $\chi'_{D_{1,2,3,4}}(\overleftrightarrow{G})$, are attained. If G is a Class 1 graph, then $\chi'_{1,2,3,4}(\overleftrightarrow{G}) = 2\Delta(G)$. To see this, it suffices to consider two proper edge-colourings of G with disjoint sets of $\Delta(G)$ colours, and use them for a proper arc-colouring of $\overleftrightarrow{G}$.

We have $\chi'_{1,2,3,4}(\overleftrightarrow{C_{2k+1}}) = 2\Delta(C_{2k+1})+1 = 5$ for every odd cycle C_{2k+1} with $k \geq 2$. Indeed, a maximal matching of C_{2k+1} contains k edges. Hence, $\chi'_{D_{1,2,3,4}}(\overleftrightarrow{C_{2k+1}}) > 4$ since $4k$ is smaller than the numbers $2(2k+1)$ of arcs of $\overleftrightarrow{C_{2k+1}}$. But it is not difficult to decompose the multigraph C'_{2k+1}, i.e. the cycle C_{2k+1} with doubled edges, into four matchings $M_i, i = 1,2,3,4$, of size k and a matching M_5 of size two, and colour the arcs of M_i with colour i, for $i = 1,\dots,5$. Namely, let $v_1,\dots,v_{2k+1}$ be consecutive vertices of

C_{2k+1}. In our construction, the set of colours of the arcs incident to v_i equals $[5] \setminus \{i\}$, for $i = 1, 2, 3, 4$, and $[4]$ for $i = 5, \dots, 2k+1$.

For complete graphs of odd order we have, $\chi'_{1,2,3,4}(\overleftrightarrow{K_{2k+1}}) = 4k+2 = 2\Delta(K_{2k+1})+2$, since a maximal matching of K_{2k+1} is of size k. With $4k+1$ colours we could properly colour at most $(4k+1)k$ arcs, less than $2k(2k+1)$, the number of arcs of $\overleftrightarrow{K_{2k+1}}$.

5 Summary of all cases

Table 1 presents lower and upper bounds for chromatic indices and distinguishing chromatic indices of symmetric digraphs $\overleftrightarrow{G}$ in terms of $\Delta = \Delta(G)$, according to definitions of proper arc-colourings, i.e. forbidden sets of monochromatic two-arc digraphs $A_i, i = 1, 2, 3, 4$. These bounds are tight except for items 12 and 13 where only an asymptotic bound $\Delta + \Theta(\log \Delta)$, indicated in Table 1 by $\star$, is known. References for the results not proved in this paper are provided (obviously, $\chi'_1(\overleftrightarrow{G}) = 2$ since each 2-cycle can be properly coloured by 2 colours).

For convenience, let us recall that A_1 is a 2-cycle, A_2 is a 2-path, A_3 is a source and A_4 is a sink.

item	chromatic index		distinguishing chrom. index	
none	1		$1 \le D'(\overleftrightarrow{G}) \le \lceil\sqrt{\Delta}\rceil$	
A_1	$\chi'_1(\overleftrightarrow{G}) = 2$		$2 \le \chi'_{D_1}(\overleftrightarrow{G}) \le \lceil\sqrt{\Delta}\rceil + 1$	[13]
A_2	$2 \le \chi'_2(\overleftrightarrow{G}) \le \lceil\sqrt{\Delta}\rceil + 1$	[14]	$2 \le \chi'_{D_2}(\overleftrightarrow{G}) \le \lceil 2\sqrt{\Delta}\rceil$	[14]
A_3	$\chi'_3(\overleftrightarrow{G}) = \Delta$	[13]	$\chi'_{D_3}(\overleftrightarrow{G}) = \Delta$	[13]
A_4	$\chi'_4(\overleftrightarrow{G}) = \Delta$	[13]	$\chi'_{D_4}(\overleftrightarrow{G}) = \Delta$	[13]
A_1, A_2	$2 \le \chi'_{1,2}(\overleftrightarrow{G}) \le \lceil\sqrt{\Delta}\rceil + 1$	[18]	$2 \le \chi'_{D_{1,2}}(\overleftrightarrow{G}) \le \lceil 2\sqrt{\Delta}\rceil$	[14]
A_1, A_3	$\chi'_{1,3}(\overleftrightarrow{G}) = \Delta$		$\Delta \le \chi'_{D_{1,3}}(\overleftrightarrow{G}) \le \Delta + 1$	
A_1, A_4	$\chi'_{1,4}(\overleftrightarrow{G}) = \Delta$		$\Delta \le \chi'_{D_{1,4}}(\overleftrightarrow{G}) \le \Delta + 1$	
A_2, A_3	$\Delta \le \chi'_{2,3}(\overleftrightarrow{G}) \le \Delta + 1$		$\Delta \le \chi'_{D_{2,3}}(\overleftrightarrow{G}) \le \Delta + 1$	
A_2, A_4	$\Delta \le \chi'_{2,4}(\overleftrightarrow{G}) \le \Delta + 1$		$\Delta \le \chi'_{D_{2,4}}(\overleftrightarrow{G}) \le \Delta + 1$	
A_3, A_4	$\chi'_{3,4}(\overleftrightarrow{G}) = \Delta$	[21]	$\Delta \le \chi'_{D_{3,4}}(\overleftrightarrow{G}) \le \Delta + 1$	
A_1, A_2, A_3	$\Delta + 1 \le \chi'_{1,2,3}(\overleftrightarrow{G}) \le \star$	[10]	$\Delta + 1 \le \chi'_{D_{1,2,3}}(\overleftrightarrow{G}) \le \star$	
A_1, A_2, A_4	$\Delta + 1 \le \chi'_{1,2,4}(\overleftrightarrow{G}) \le \star$	[10]	$\Delta + 1 \le \chi'_{D_{1,2,4}}(\overleftrightarrow{G}) \le \star$	
A_1, A_3, A_4	$\Delta \le \chi'_{1,3,4}(\overleftrightarrow{G}) \le \Delta + 1$	[13]	$\Delta \le \chi'_{D_{1,3,4}}(\overleftrightarrow{G}) \le \Delta + 1$	[13]
A_2, A_3, A_4	$\Delta \le \chi'_{2,3,4}(\overleftrightarrow{G}) \le \Delta + 1$		$\Delta \le \chi'_{D_{2,3,4}}(\overleftrightarrow{G}) \le \Delta + 1$	
all four	$2\Delta \le \chi'_{1,2,3,4}(\overleftrightarrow{G}) \le 2\Delta + 2$		$2\Delta \le \chi'_{D_{1,2,3,4}}(\overleftrightarrow{G}) \le 2\Delta + 2$	

Table 1: Bounds for chromatic indices and distinguishing chromatic indices for all types of arc-colourings of symmetric digraphs. The second column contains forbidden monochromatic two-arc digraphs.

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References

[1] M. O. Albertson and K. L. Collins, Symmetry breaking in graphs, *Electron. J. Comb.* **3** (1996), Research Paper 18, approx. 17, doi:10.37236/1242, https://doi.org/10.37236/1242.

[2] I. Algor and N. Alon, The star arboricity of graphs, *Discrete Math.* **75** (1989), 11–22, doi:10.1016/0012-365X(89)90073-3, https://doi.org/10.1016/0012-365X(89)90073-3.

[3] L. Babai, Asymmetric trees with two prescribed degrees, *Acta Math. Acad. Sci. Hungar.* **29** (1977), 193–200, doi:10.1007/bf01896481, https://doi.org/10.1007/bf01896481.

[4] M. Behzad, *Graphs and their chromatic numbers*, Ph.D. thesis, Michigan State University, 1965.

[5] R. A. Brualdi and J. J. Q. Massey, Incidence and strong edge colorings of graphs, *Discrete Math.* **122** (1993), 51–58, doi:10.1016/0012-365X(93)90286-3, https://doi.org/10.1016/0012-365X(93)90286-3.

[6] K. L. Collins and A. N. Trenk, The distinguishing chromatic number, *Electron. J. Comb.* **13** (2006), Research Paper 16, 19, doi:10.37236/1042, https://doi.org/10.37236/1042.

[7] D. Dereniowski, A. Kosowski and D. Pająk, Distinguishing views in symmetric networks: a tight lower bound, *Theoret. Comput. Sci.* **582** (2015), 27–34, doi:10.1016/j.tcs.2015.03.018, https://doi.org/10.1016/j.tcs.2015.03.018.

[8] M. Fernández, H. Kirchner and B. Pinaud, Labelled port graph—a formal structure for models and computations, in: *The 12th Workshop on Logical and Semantic Frameworks, with Applications*, Elsevier Sci. B. V., Amsterdam, volume 338 of *Electron. Notes Theor. Comput. Sci.*, pp. 3–21, 2018, doi:10.1016/j.entcs.2018.10.002, https://doi.org/10.1016/j.entcs.2018.10.002.

[9] A. Gorzkowska, R. Kalinowski and M. Pilśniak, The distinguishing index of the Cartesian product of finite graphs, *Ars Math. Contemp.* **12** (2017), 77–87, doi:10.26493/1855-3974.909.0e1, https://doi.org/10.26493/1855-3974.909.0e1.

[10] B. Guiduli, On incidence coloring and star arboricity of graphs, *Discrete Math.* **163** (1997), 275–278, doi:10.1016/0012-365X(95)00342-T, https://doi.org/10.1016/0012-365X(95)00342-T.

[11] W. Imrich, R. Kalinowski, M. Pilśniak and M. Woźniak, The distinguishing index of connected graphs without pendant edges, *Ars Math. Contemp.* **18** (2020), 117–126, doi:10.26493/1855-3974.1852.4f7, https://doi.org/10.26493/1855-3974.1852.4f7.

[12] R. Kalinowski and M. Pilśniak, Distinguishing graphs by edge-colourings, *European J. Comb.* **45** (2015), 124–131, doi:10.1016/j.ejc.2014.11.003, https://doi.org/10.1016/j.ejc.2014.11.003.

[13] R. Kalinowski and M. Pilśniak, Proper distinguishing arc-colourings of symmetric digraphs, *Appl. Math. Comput.* **421** (2022), Paper No. 126939, 6, doi:10.1016/j.amc.2022.126939, https://doi.org/10.1016/j.amc.2022.126939.

[14] R. Kalinowski, M. Pilśniak and M. Prorok, Distinguishing symmetric digraphs by proper arc-colourings of type I, manuscript.

[15] F. Lehner, M. Pilśniak and M. Stawiski, A bound for the distinguishing index of regular graphs, *Eur. J. Comb.* **89** (2020), 103145, 9, doi:10.1016/j.ejc.2020.103145, https://doi.org/10.1016/j.ejc.2020.103145.

[16] M. Pilśniak, Improving upper bounds for the distinguishing index, *Ars Math. Contemp.* **13** (2017), 259–274, doi:10.26493/1855-3974.981.ff0, https://doi.org/10.26493/1855-3974.981.ff0.

[17] M. Pilśniak and T. Tucker, Distinguishing index of maps, *Eur. J. Comb.* **84** (2020), 103034, 12, doi:10.1016/j.ejc.2019.103034, https://doi.org/10.1016/j.ejc.2019.103034.

[18] S. Poljak and V. Rödl, On the arc-chromatic number of a digraph, *J. Comb. Theory Ser. B* **31** (1981), 190–198, doi:10.1016/s0095-8956(81)80024-x, https://doi.org/10.1016/s0095-8956(81)80024-x.

[19] E. Sopena, *The Incidence Coloring Page*, https://www.labri.fr/perso/sopena/pmwiki/index.php?n=TheIncidenceColoringPage.

[20] V. G. Vizing, The chromatic class of a multigraph, *Kibernetika (Kiev)* **1965** (1965), 29–39, doi:10.1007/BF01885700, https://doi.org/10.1007/BF01885700.

[21] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 2001.