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A note on decompositions of transitive tournaments

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Abstract

For any positive integer n, we determine all connected digraphs G of size at most four, such that a transitive tournament of order n is G-decomposable. Among others, these results disprove a generalization of a theorem of Sali and Simonyi [Orientations of self-complementary graphs and the relation of Sperner and Shannon capacities, European J. Combin. 20 (1999), 93–99].

MSC: 05C70; 05C35

Keywords: Transitive tournaments; Decomposition

1. Introduction

Let G be a digraph of order n with the vertex set $V(G)$ and the arc set $E(G)$. The *outdegree* of a vertex $v \in V(G)$ is denoted by $d^+(v)$, and its *indegree* by $d^-(v)$. The *degree* of a vertex v is the sum $d(v) = d^-(v) + d^+(v)$. A *reverse of a digraph* G is the digraph $\overleftarrow{G}$ obtained from G by converting each arc $(u, v) \in E(G)$ into (v, u). An *oriented graph* is a digraph without directed cycles of length two. Replacing of every arc (u, v) in an oriented graph G by an edge uv yields the *underlying graph* Γ of G, and then G is called an *orientation* of Γ.

A *tournament* is an orientation of a complete graph. A digraph G is called *transitive* when it satisfies the condition of transitivity: if (u, v) and (v, w) are two arcs of G then (u, w) is an arc, too. A transitive tournament of order n will be denoted by TT_n. Since TT_n is unique up to isomorphism, throughout the paper we will view it as shown in Fig. 1. Namely, $V(TT_n) = \{1, \dots, n\}$ and $E(TT_n) = \{(i, j) : 1 \leqslant i < j \leqslant n\}$. The vertices 1, 2 and n will be called the *first*, the *second* and the *last vertex* of TT_n, respectively. We define the *length of an arc* (i, j) as the difference $j - i$.

Let G and H be two digraphs. We say that *H can be decomposed into G* (or *H is G-decomposable*, for short), if there exists a partition of the arc set $E(H)$ into pairwise disjoint subsets each of which creates a subgraph isomorphic to G. An obvious necessary condition for the existence of a G-decomposition of H is the divisibility of $|E(H)|$ by $|E(G)|$.

This paper has been inspired by a theorem of Sali and Simonyi [3] (for a nice, short proof consult Gyárfás [2]). Its slightly weaker version can be formulated as follows.

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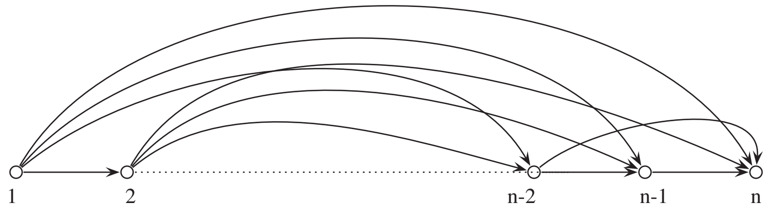

Fig. 1. Transitive tournament TT_n.

Theorem (*Sali and Simonyi [3]*). *If Γ is a self-complementary graph of order n, then there exists an orientation G of Γ such that a transitive tournament TT_n is G-decomposable.*

One can try to generalize this result and pose a more general question. Suppose a complete graph K_n can be decomposed into k copies of a graph Γ. Does there always exist an orientation G of Γ such that TT_n is G-decomposable? The above theorem of Sali and Simonyi answers this question in affirmative for $k = 2$. We will show that, in general, the answer is negative.

2. Some lemmas

We start with an immediate consequence of the fact that the transitive tournament TT_n is isomorphic to its reverse $\overleftarrow{TT_n}$.

Lemma 1. *A transitive tournament TT_n is G-decomposable if and only if it is $\overleftarrow{G}$-decomposable.*

The subsequent lemmas will be useful in disproving the existence of some decompositions of TT_n.

Lemma 2. *Assume that every subgraph of TT_n isomorphic to G has at most two arcs in the set*

$$F = \left\{(i, j) \in E(TT_n) : i \leqslant \frac{n}{2} < j\right\}.$$

If TT_n can be decomposed into G, then the number of copies of G in a decomposition cannot be smaller than $\lceil (n^2 - 1)/8 \rceil$.

Proof. Observe that $|F| = \lceil (n^2 - 1)/4 \rceil$. □

Lemma 3. *Let G be a digraph of order at least three such that G has exactly one vertex x of indegree zero. Then TT_n cannot be G-decomposed in each of the following three cases*:

(A) *the underlying graph of G is a star with a center x,*
(B) *every vertex of G has outdegree zero, except for two vertices x and y with $d^+(x) = d^+(y) = 2$,*
(C) *$d^+(x) = 3$, and every other vertex of G has outdegree less than two.*

Proof. Let d denote the degree of the vertex x in G. Thus $d \geqslant 2$ in case (A), $d = 2$ in case (B), and $d = 3$ in case (C). Suppose there exists a decomposition of TT_n into G. If the first vertex of TT_n belongs to a copy of G, then it has to be the vertex x. Hence, d divides the degree $n - 1$ of a vertex in TT_n.

In the decomposition, there is a unique copy of G that contains the arc $(1, 2)$ of TT_n. In all other copies of G, the second vertex of TT_n cannot be different from x. Therefore d has to divide $n - 1 - c$, where $c = 1$ in case (A), $c = 3$ in case (B), and $c \in \{1, 2\}$ in case (C). In each case, this contradicts the divisibility of $n - 1$ by d. □

Lemma 4. *If for every arc $(u, v) \in E(G)$, at least one of its vertices u, v has both the outdegree and the indegree positive, then TT_n is not G-decomposable.*

Proof. Clearly, the longest arc $(1, n)$ of TT_n cannot belong to any copy of G. □

3. Decomposition into connected digraphs of size at most three

In this section we determine all cases when a transitive tournament TT_n can be decomposed into a connected digraph G of size at most three. Naturally, the size of G has to divide the size of TT_n, and G must not contain a directed cycle. It is clear that TT_n can be decomposed into single arcs.

Theorem 5. *There does not exist a decomposition of TT_n into any connected digraph of size two.*

Proof. By Lemma 1, it suffices to consider only two digraphs $A1$ and $A2$ shown in Fig. 2. It is easy to see that, by Lemma 4, any TT_n cannot be decomposed into copies of $A1$. By Lemma 3(A), the same is true for $A2$. □

It is well known (cp. [1]) that a complete graph K_n can be decomposed into a path P_3 of length two if and only if the size of K_n is even, i.e. $n \equiv 0$ or $1 \pmod 4$. We have thus shown that there does not exist an orientation of P_3 that would decompose any TT_n. This gives a negative answer to the question formulated at the end of the Introduction. Other counterexamples follow from Theorems 6 and 7.

Theorem 6. *Let G be a connected digraph of size three. There exists a decomposition of TT_n into G if and only if G is isomorphic to one of the following digraphs* (*see Fig.* 3):

1. $B1$, *and* $n \equiv 1$ *or* $3 \pmod 6$, $n \geqslant 3$,
2. $B5$ *or* $\overleftarrow{B5}$, *and* $n \equiv 1$ *or* $3 \pmod 6$, $n \geqslant 7$,
3. $B6$, *and* $n \equiv 0, 1, 3$ *or* $4 \pmod 6$, $n \geqslant 4$.

Proof. Since $\binom{n}{2}$ has to be divisible by three, we exclude $n \equiv 2 \pmod 6$ and $n \equiv 5 \pmod 6$. Due to Lemma 1, we consider only six subgraphs $B1$, $B2$, $B3$, $B4$, $B5$ and $B6$ of TT_n (see Fig. 3). We immediately observe that, by Lemma 3(A), TT_n cannot be decomposed into $B2$, and by Lemma 4, neither into $B3$ nor into $B4$.

The degree of every vertex of $B1$ equals two, hence vertices of TT_n have to be of even degree, so either $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$. As it is well known (cp. [1]), for all such $n \geqslant 3$, there exist Steiner triple systems that give decompositions of a complete graph K_n into triangles. If we replace all edges of K_n by arcs to obtain a transitive tournament, the resulting oriented triangles will always be isomorphic to $B1$.

Let us label the vertices of $B5$ with x, y, z, t, so that the arcs of $B5$ are: (x, y), (y, z) and (t, z). The vertices x, y and t have positive outdegree, while $d(z) = d^-(z) = 2$. Thus, in any decomposition of TT_n into $B5$, the last vertex of TT_n has to be the vertex z in any copy of $B5$ it belongs to. As previously, this implies that either $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$, with $n \geqslant 7$ (since TT_3 is not $B5$-decomposable).

From now on, to describe the required decompositions, we will use the following notation. Every copy of $B5$ in TT_n will be represented by a sequence of four integers $(abcd)$ that indicate vertices of TT_n corresponding to x, y, z and t, respectively. Consequently, a decomposition of TT_7 into $B5$ can be given by a set of seven sequences: (1273), (1364), (2341), (2451), (2561), (2674), (3571).

Next, take any n of the form $n = 6k + 1$ with $k \geqslant 2$. Partition the set $V(TT_n) \backslash \{n\} = \{1, \ldots, n-1\}$ into six-element sets $V_1, \ldots, V_k$, where

$$V_i = \{i, 2k - i + 1, 4k - 2i + 1, 4k - 2i + 2, 5k - i + 1, 6k - i + 1\}, \quad i = 1, \ldots, k.$$

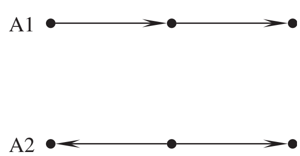

Fig. 2. Directed subgraphs $A1$ and $A2$ of TT_n.

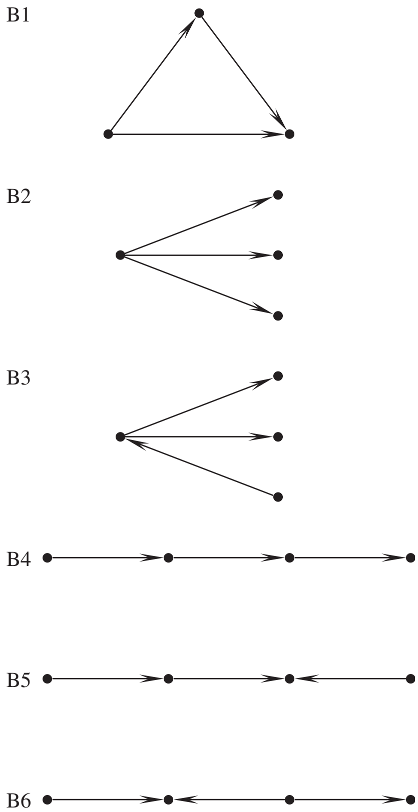

Fig. 3. Oriented graphs of size three to be considered in the proof of Theorem 6.

For $i < j$, let S_{ij} denote the set of all arcs between V_i and V_j in TT_n, i.e.

$$S_{ij} = E(TT_n) \cap (V_i \times V_j \cup V_j \times V_i).$$

Every arc of TT_n belongs either to exactly one subgraph induced by $V_i \cup \{n\}$, for some i, or to exactly one subgraph induced by S_{ij} for some $i < j$. For every i, the vertices of $V_i \cup \{n\}$ induce in TT_n a transitive tournament of order seven, and its decomposition into $B5$ has been already shown. Therefore, it suffices to find a $B5$-decomposition of the subgraph induced by the set of arcs S_{ij} for all $i < j$. To make it more readable, for fixed i and j, we denote the elements of V_i and V_j as

$$V_i = \{a, b, c, d, e, f\}, \quad V_j = \{\breve{a}, \breve{b}, \breve{c}, \breve{d}, \breve{e}, \breve{f}\},$$

assuming that they are listed increasingly. Observe that the ordering of these integers is the following one:

$$a < \breve{a} < \breve{b} < b < \breve{c} < \breve{d} < c < d < \breve{e} < e < \breve{f} < f,$$

since $i < j$. The desired decomposition follows: $(a\breve{a}b\breve{b})$, $(a\breve{b}c\breve{a})$, $(a\breve{c}c\breve{d})$, $(a\breve{d}e\breve{b})$, $(\breve{b}d\breve{f}c)$, $(b\breve{c}d\breve{a})$, $(b\breve{d}f\breve{c})$, $(b\breve{e}f\breve{b})$, $(b\breve{f}f\breve{a})$, $(\breve{c}e\breve{f}a)$, $(\breve{d}d\breve{e}a)$, $(c\breve{e}e\breve{a})$. Thus, we have proved that TT_ncan be decomposed into $B5$ (as well as into $\overleftarrow{B5}$, by Lemma 1) for $n \equiv 1 \pmod 6$, $n \geqslant 7$.

A $B5$-decomposition of TT_9 can be given by the following set of 12 sequences: (1243), (1385), (1453), (1574), (1684), (1892), (2395), (2564), (2691), (2794), (3671), (3782). Let $n = 6k + 3$, $k \geqslant 2$. Consider the partitioning of $V(TT_n) \backslash \{n\}$ into subsets $V_1, \ldots, V_{k-1}$ and W, where

$$W = \{2k - 1, 2k, 2k + 1, 2k + 2, 4k + 1, 4k + 2, 4k + 3, 4k + 4\}$$

and

$$V_i = \{i, 2k-i-1, 4k-2i+1, 4k-2i+2, 5k-i+4, 6k-i+3\}, \quad i=1,\ldots,k-1.$$

Each set $V_i \cup \{n\}$ induces TT_7, and $W \cup \{n\}$ induces TT_9. Observe that the ordering of the integers of V_i and V_j with $i<j$ is the same as for the previous case $n=6k+1$. Therefore, it suffices to decompose a subgraph H_i formed by the set of arcs

$$E(H_i) = E(TT_n) \cap (V_i \times W \cup W \times V_i),$$

for each $i=1,\ldots,k-1$. If we denote $W=\{a,b,c,d,e,f,g,h\}$ and $V_i=\{\breve{a},\breve{b},\breve{c},\breve{d},\breve{e},\breve{f}\}$, then the sequence of 14 integers $(\breve{a}\breve{b}abcd\breve{c}\breve{d}efgh\breve{e}\breve{f})$ is strictly increasing. It is not difficult to derive that the digraph H_i is an edge-disjoint union of eight digraphs, each of which is isomorphic to a subgraph of TT_n induced by the set of five vertices $\{\breve{a}, a, \breve{c}, e, \breve{e}\}$. This is an oriented graph with an underlying graph $K_{3,2}$ and it has the following $B5$-decomposition: $(\breve{a}a\breve{e}e)$, $(a\breve{c}e\breve{a})$.This completes the proof for the case when G is isomorphic to $B5$, and by Lemma 1, to $\overleftarrow{B5}$.

TT_n can be decomposed into $B6$ only if the size of TT_n is divisible by three, hence $n \equiv 0, 1, 3$ or $(4 \bmod 6)$. We will show that this necessary condition is also sufficient in this case.

Let us label the vertices of $B6$ with x, y, z, t so that its arcs are: (x, y), (z, y), (z, t). First, we observe that the oriented graph $\vec{K}_{3,3}$ with the vertex set $\{v_1,\ldots,v_6\}$ and the arc set $\{v_1, v_2, v_3\} \times \{v_4, v_5, v_6\}$, is $B6$-decomposable. Using the same convention as before, we represent a decomposition of $\vec{K}_{3,3}$ into $B6$ by the set of sequences:

$$(v_1v_5v_2v_4), (v_2v_6v_3v_5), (v_3v_4v_1v_6).$$

Let $n=6k$. TT_6 has the following $B6$-decomposition: (1524), (3412), (4536), (4613), (5623). Next, for fixed $k \geqslant 2$ we partition the vertex set of TT_{6k} into $2k$ triples $T_1,\ldots,T_{2k}$, where $T_i=\{3i-2, 3i-1, 3i\}$, $i=1,\ldots,2k$. The sum of two consecutive triples $T_{2j-1} \cup T_{2j}$, with $1 \leqslant j \leqslant k$, induces a transitive tournament of order six. For any other pair of triples T_p and T_q with $p<q$, the subgraph induced by the set of arcs $T_p \times T_q$ is isomorphic to $\vec{K}_{3,3}$. It easily follows that TT_{6k} is $B6$-decomposable.

In case $n=6k+3$, we argue in a similar way. For $k=1$, there exists a decomposition of TT_9 into $B6$: (1523), (1634), (2413), (2659), (2718), (3546), (3829), (4739), (4879), (4967), (6857), (8912). If $k \geqslant 2$, we partition the set $V(TT_n)$ into $2k+1$ triples as above: $T_i=\{3i-2, 3i-1, 3i\}$, $i=1,\ldots,2k+1$. For each $j=1,\ldots,k-1$, the sum $T_{2j-1} \cup T_{2j}$ induces TT_6. The sum of three last triples $T_{2k-1} \cup T_{2k} \cup T_{2k+1}$ induces TT_9. The arcs between any other pair of triples create in TT_n an oriented graph isomorphic to $\vec{K}_{3,3}$.

At last, let $n \equiv 1$ or $4 \pmod 6$, i.e. $n=3k+1$ with $k \geqslant 1$. The transitive tournament TT_4 has the following $B6$-decomposition: (1324), (3412). If $k>1$, we partition the set $V(TT_{3k+1}) \backslash \{n\}$ into $3k$ triples $T_i=\{3i-2, 3i-1, 3i\}$, $i=1,\ldots,k$. For every i, the set $T_i \cup \{n\}$ induces TT_4. The set of arcs $T_p \times T_q$ creates $\vec{K}_{3,3}$, whenever $1 \leqslant p < q \leqslant 3k$. Thus TT_n is $B6$-decomposable for every $n=1$ or $4 \pmod 6$. □

4. Decomposition into connected digraphs of size four

Theorem 7. *Let G be a connected digraph of size four. There exists a decomposition of TT_n into G if and only if $n \equiv 0$ or $1 \pmod 8$, and either G or its reverse $\overleftarrow{G}$ is isomorphic to one of four digraphs*: $C43$, $C46$, $C54$ *and* $C56$ *depicted in Fig. 4.*

Proof. Up to isomorphism, there are five connected graphs of size four. These are graphs $C1, C2, C3, C4, C5$ presented in Fig. 5. We shall investigate all their orientations that are subgraphs of a transitive tournament. Due to Lemma 1, we need not examine reverse orientations. By the necessary condition of decomposibility, we may assume that $n \equiv 0$ or $1 \pmod 8$.

Case $C1$: There are three such orientations of $C1$ (see Fig. 6).

Each vertex of $C11$ is of degree two, therefore a transitive tournament TT_n could be decomposed into $C11$, only if n were odd. Hence $n=8k+1$, for $k \in \mathbb{N}$, and the number of copies of $C11$ in any decomposition would equal $k(8k+1)=n^2-n/8$.

On the other hand, $C11$ satisfies the assumption of Lemma 2. This leads to a contradiction.

It is easy to see that TT_n is not decomposable neither into $C12$, by Lemma 4, nor into $C13$, by Lemma 3(B).

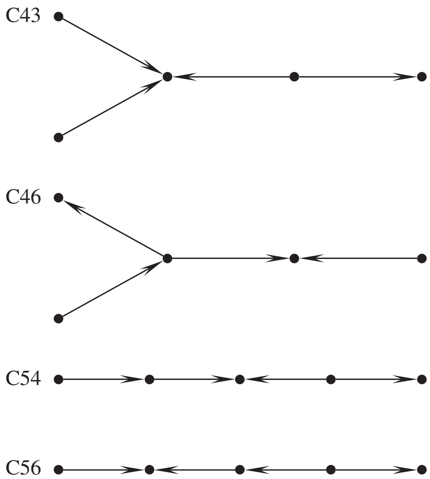

Fig. 4. Four of five digraphs of size four that decompose TT_n.

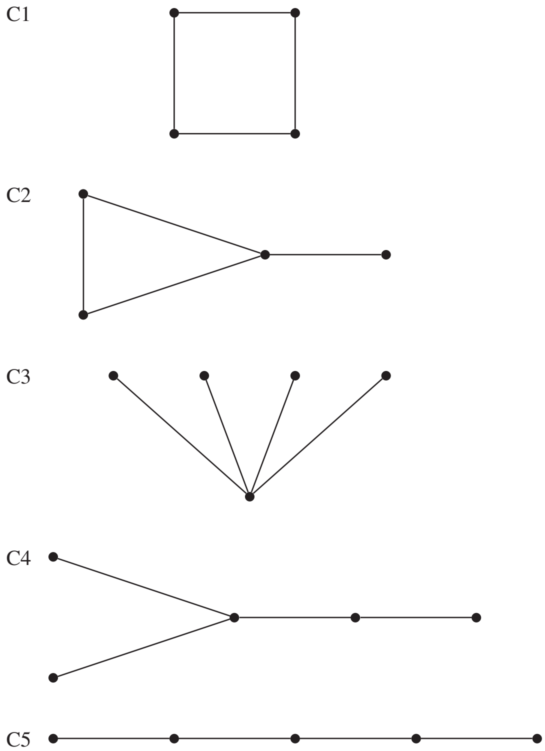

Fig. 5. All connected graphs of size four.

Case $C2$: Fig. 7 presents all orientations of $C2$ in question.

Lemmas 4, 3(C) and 3(B) imply that TT_n cannot be decomposed into $C21$, $C22$ and $C23$, respectively.

Case $C3$: Due to Lemma 1, we consider only three orientations of $C3$ (see Fig. 8).

By Lemma 3(A), the digraph $C31$ does not decompose TT_n, and by Lemma 4, the same is true for $C32$ and $C33$.

Case $C4$: All orientations of $C4$, we have to consider, are depicted in Fig. 9.

Lemma 4 immediately excludes the digraphs $C41$ and $C45$. Further, Lemma 3(B) and (C) excludes $C42$ and $C44$, too.

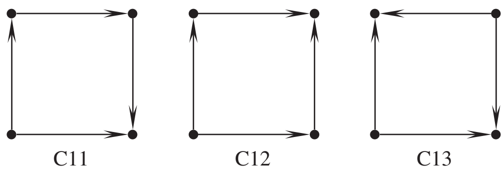

Fig. 6. Orientations of $C1$.

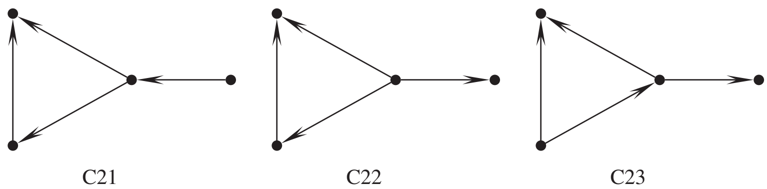

Fig. 7. Orientations of $C2$.

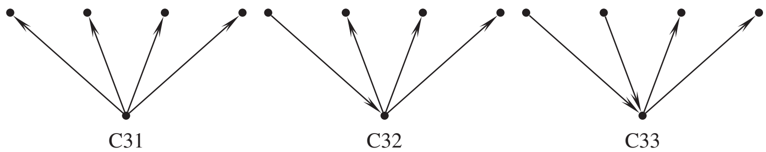

Fig. 8. Orientations of $C3$.

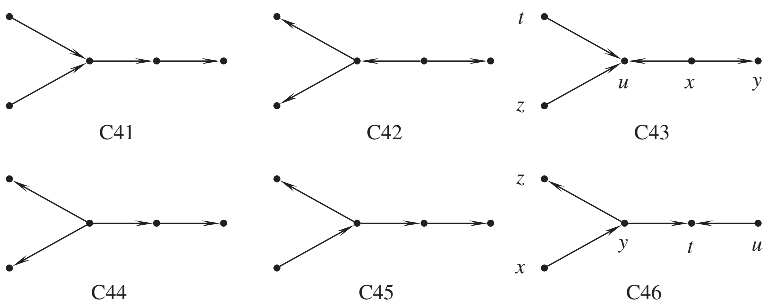

Fig. 9. Orientations of $C4$.

Let us label the vertices of $C43$ by x, y, z, t, u, so that its arcs are (x, y), (x, u), (z, u) and (t, u). Using the convention introduced in the proof of Theorem 6, a decomposition of TT_8 into $C43$ can be represented as a set of sequences: (12345), (13246), (18234), (23167), (25368), (36457), (56478). If $n = 8k$, for some $k \geqslant 2$, then partition the vertex set of TT_n into sets $W_1, \ldots, W_k$, where W_i is the following set of eight consecutive integers:

$$W_i = \{8i - 7, \ldots, 8i\}, \quad i = 1, \ldots, k.$$

Each set W_i induces a $C43$-decomposable tournament TT_8. For $i < j$, let $D_{i,j}$ indicate an oriented graph with the vertex set $V(D_{i,j}) = W_i \cup W_j$ and the edge set $E(D_{i,j}) = W_i \times W_j$. If we show that $D_{i,j}$ can be decomposed into $C43$, then we will prove that TT_{8k} is $C43$-decomposible. To do this, it suffices to generalize in an obvious way the following $C43$-decomposition (3,10,1,2,9), (4,9,1,2,10) of the subgraph D' of TT_n created by all eight arcs from the set $\{1, 2, 3, 4\}$ to the set $\{9, 10\}$. Indeed, each D_{ij} is an arc-disjoint union of eight oriented graphs isomorphic to D'.

There exists a decomposition of TT_9 into $C43$: (12345), (13246), (18234), (19237), (23468), (25349), (36578), (56789), (69457). Let $n = 8k + 1$, $k \geqslant 2$. Partition $V(TT_n)\backslash\{n\}$ into sets $W_1, \ldots, W_k$ as above, and note that $W_i \cup \{n\}$ induces TT_9 and each $D_{i,j}$ is $C43$-decomposible.

Now, consider the oriented graph $C46$ and denote its vertices by x, y, z, t, u so that the arcs of $C46$ are: (x, y), (y, z), (y, t), (u, t). A decomposition of TT_8 into $C46$ is represented by the following set of sequences: (12853), (13675), (14687), (15862), (16872), (23481), (24571). For $k \geqslant 2$, partition the vertex set of TT_{8k} into k subsets $\Upsilon_1, \ldots, \Upsilon_k$, where

$$\Upsilon_i = \{i, 2k - i + 1, 2k + 2i - 1, 2k + 2i, 8k - 4i + 1, 8k - 4i + 2, 8k - 4i + 3, 8k - 4i + 4\},$$

for $i = 1, \ldots, k$. In TT_{8k}, each set Υ_i induces TT_8. Hence, to show that TT_{8k} is decomposable into $C46$, it suffices to find a $C46$-decomposition of the oriented subgraph D^*_{ij} induced by the set of arcs

$$E(TT_{8k}) \cap (\Upsilon_i \times \Upsilon_j \cup \Upsilon_j \times \Upsilon_i)$$

for every $i < j$. Arrange the elements of both sets Υ_i and Υ_j in increasing order and denote them as

$$\Upsilon_i = \{a, b, c, d, e, f, g, h\}, \quad \Upsilon_j = \{\breve{a}, \breve{b}, \breve{c}, \breve{d}, \breve{e}, \breve{f}, \breve{g}, \breve{h}\}.$$

Note that the sequence of 16 integers $(a\breve{a}\breve{b}bcd\breve{c}\breve{d}\breve{e}\breve{f}\breve{g}\breve{h}efgh)$ is increasing. The requested decomposition of D^*_{ij} follows:

$$(a\breve{h}gh\breve{a}), (b\breve{h}ef\breve{a}), (a\breve{g}gh\breve{c}), (b\breve{g}ef\breve{b}), (a\breve{f}hg\breve{c}), (d\breve{f}fe\breve{d}), (a\breve{e}hg\breve{a}), (d\breve{e}fe\breve{c}),$$

$$(a\breve{a}de\breve{b}), (a\breve{b}gh\breve{d}), (\breve{a}b\breve{d}\breve{c}a), (\breve{b}b\breve{f}\breve{e}c), (\breve{a}c\breve{f}\breve{g}d), (\breve{b}c\breve{d}\breve{h}d), (\breve{b}d\breve{d}\breve{c}c), (a\breve{d}gf\breve{c}).$$

TT_9 has the following decomposition into $C46$: (12963), (13582), (14672), (15894), (16798), (23491), (24581), (25671), (37986). Let $n = 8k + 1$, $k \geqslant 2$. Partition the set $V(TT_n)\backslash\{n\} = \{1, \ldots, 8k\}$ into the same eight-element subsets $\Upsilon_1, \ldots, \Upsilon_k$ as above. To see that TT_n is decomposable into $C46$, observe that $\Upsilon_i \cup \{n\}$ induces a transitive tournament of order 9, for each i, and the set of all other arcs is a disjoint union of arc sets of oriented graphs D^*_{ij} with $1 \leqslant i < j \leqslant k$. They are all $C46$-decomposable.

Case $C5$: By Lemma 1, we consider only six orientations of $C5$ (see Fig. 10).

Lemma 4 implies that any transitive tournament cannot be decomposed into $C51$.

Let G be one of the digraphs $C52$ and $C53$. Observe that G has exactly one vertex x with $d^-(x) = 2$. Moreover $d^+(x) = 0$, so the last vertex of TT_n must coincide with x in every copy of G it belongs to. Hence the degree of any vertex in TT_n must be even. It follows that $n = 8k + 1$, for $k \in \mathbb{N}$, and the number of copies of G in any decomposition equals $k(8k + 1) = n(n - 1)/8$. By Lemma 2, there does not exist a decomposition of TT_n into G.

Vertices of the digraph $C54$ can be labeled in such a way that its arc set consists of (x, y), (y, z), (t, z) and (t, u). TT_8 has a $C54$-decomposition: (12435), (13625), (14756), (15823), (16738), (27845), (46817). For $n = 8k$ with $k \geqslant 2$,

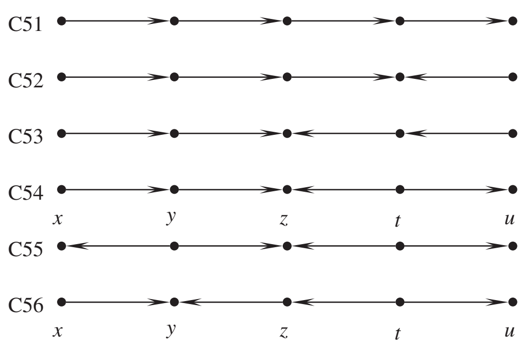

Fig. 10. Orientations of $C5$.

consider the following partitioning of $V(TT_n)$ into eight-element sets

$$V_i = \{i, 2k - i + 1, 2k + 2i - 1, 2k + 2i, 5k - i + 1, 5k + i, 8k - 2i + 1, 8k - 2i + 2\},$$

$i = 1, \dots, k$. Each set V_i induces TT_8, therefore, it suffices to decompose into $C54$ a digraph $D'_{i,j}$ with the vertex set $V_i \cup V_j$ and the arc set

$$E(TT_n) \cap (V_i \times V_j \cup V_j \times V_i),$$

for all $i < j$. To do this, denote

$$V_i = \{a, b, c, d, e, f, g, h\} \quad \text{and} \quad V_j = \{\breve{a}, \breve{b}, \breve{c}, \breve{d}, \breve{e}, \breve{f}, \breve{g}, \breve{h}\},$$

assuming that the elements of each set are listed in increasing order, and observe that the sequence of 16 integers $(a\breve{a}\breve{b}bcd\breve{c}\breve{d}\breve{e}ef\breve{f}\breve{g}\breve{h}gh)$ is increasing. A decomposition of D'_{ij} follows:

$$(a\breve{a}f\breve{e}h), (a\breve{b}h\breve{d}f), (a\breve{c}h\breve{g}g), (c\breve{c}g\breve{b}e), (c\breve{e}e\breve{c}f), (a\breve{f}h\breve{a}g), (\breve{b}d\breve{e}a\breve{g}), (\breve{a}b\breve{c}d\breve{g}),$$

$$(\breve{a}c\breve{h}e\breve{f}), (\breve{a}d\breve{f}c\breve{g}), (\breve{a}e\breve{g}f\breve{f}), (\breve{b}b\breve{d}a\breve{h}), (\breve{b}c\breve{d}d\breve{h}), (b\breve{f}g\breve{h}h), (b\breve{e}g\breve{d}e), (\breve{b}f\breve{h}b\breve{g}).$$

A decomposition of TT_9 into $C54$ is given by a set of sequences: (12436), (13526), (14657), (15823), (16739), (17849), (38927), (45918), (47968). In TT_{8k+1} with $k \geqslant 2$, each set $V_j \cup \{n\}$ induces TT_9, and the digraph $D'_{i,j}$ is $C54$-decomposible, as shown before. Thus, TT_n can be decomposed into $C54$ for every $n = 0, 1 \pmod 8$, $n \geqslant 8$.

By Lemma 3(B), TT_n cannot be decomposed into $C55$.

Let $\{x, y, z, t, u\}$ be the vertex set of the digraph $C56$, so that (x, y), (z, y), (t, z) and (t, u) are the arcs of it. Then the set of sequences (14325), (15426), (16538), (28637), (47218), (57648), (58713) represents a $C56$-decomposition of TT_8, and (14325), (15426), (16537), (17638), (18729), (19748), (28649), (59312), (69857) that of TT_9. This time, we partition the set $\{1, \dots, 8k\}$ into k subsets

$$U_i = \{2i - 1, 2i, 3k - i + 1, 3k + i, 5k - i + 1, 5k + i, 8k - 2i + 1, 8k - 2i + 2\},$$

$i = 1, \dots, k$. As in previous cases, to prove that every TT_n with $n = 8k$ or $n = 8k + 1$, is $C56$-decomposable, it suffices to show that a digraph $D''_{i,j}$ with the arc set

$$E(TT_n) \cap (U_i \times U_j \cup U_j \times U_i),$$

has a $C56$-decomposition. If we denote $U_i = \{a, b, c, d, e, f, g, h\}$ and $U_j = \{\breve{a}, \breve{b}, \breve{c}, \breve{d}, \breve{e}, \breve{f}, \breve{g}, \breve{h}\}$ for $i < j$, then the sequence

$$(ab\breve{a}\breve{b}\breve{c}cd\breve{d}\breve{e}ef\breve{f}\breve{g}\breve{h}gh)$$

is increasing. The decomposition of $D''_{i,j}$ may look like this: $(\breve{f}g\breve{a}a\breve{d})$, $(\breve{c}h\breve{g}a\breve{b})$, $(\breve{d}h\breve{h}a\breve{c})$, $(a\breve{e}c\breve{a}e)$, $(a\breve{f}d\breve{a}f)$, $(\breve{f}h\breve{a}b\breve{d})$, $(\breve{g}g\breve{h}b\breve{b})$, $(\breve{d}g\breve{e}b\breve{c})$, $(e\breve{f}c\breve{b}g)$, $(b\breve{g}d\breve{b}h)$, $(\breve{e}e\breve{d}c\breve{g})$, $(\breve{c}f\breve{d}d\breve{e})$, $(f\breve{g}e\breve{c}g)$, $(d\breve{h}f\breve{e}h)$, $(b\breve{f}f\breve{b}e)$, $(e\breve{h}c\breve{c}d)$. □

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