Majority Edge-Colorings of Graphs

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Abstract

We propose the notion of a majority k-edge-coloring of a graph G, which is an edge-coloring of G with k colors such that, for every vertex u of G, at most half the edges of G incident with u have the same color. We show the best possible results that every graph of minimum degree at least 2 has a majority 4-edge-coloring, and that every graph of minimum degree at least 4 has a majority 3-edge-coloring. Furthermore, we discuss a natural variation of majority edge-colorings and some related open problems.

Mathematics Subject Classifications: 05C15

1 Introduction

Motivated by similar notions considered for vertex-colorings, we propose and study *majority edge-colorings* of graphs: For a (finite, simple, and undirected) graph G, an edge-coloring $c : E(G) \to [k]$ is a *majority k-edge-coloring* if, for every vertex u of G and every color α in $[k]$, at most half the edges incident with u have the color α.

Before we present our results, we discuss some related research. Lovász [9] showed that every graph G has a 2-vertex-coloring such that, for every vertex u of G, at most half the neighbors of u have the same color as u. For infinite graphs, this leads to the *Unfriendly Partition Conjecture* [2]. Kreutzer, Oum, Seymour, van der Zypen, and Wood [8] showed that every digraph D has a 4-vertex-coloring such that, for every vertex u of D, at most half the out-neighbors of u have the same color as u, and they conjecture that 3 colors suffice. Anholcer, Bosek, and Grytczuk [4] studied a choosability version for digraphs. It follows from a result of Wood [13] that every digraph D has a 4-arc-coloring such that, for every vertex u of D, at most half the arcs leaving u have the same color. Further related research concerns *defective* or *frugal* edge-colorings [1, 3, 7], where maximum degree conditions are imposed on the subgraphs formed by edges having the same color.

Our first result is that 2 colors almost suffice for a majority edge-coloring.

Theorem 1. *Let G be a connected graph.*

(i) If G has an even number of edges or G contains vertices of odd degree, then G has a 2-edge-coloring such that, for every vertex u of G, at most $\left\lceil \frac{d_G(u)}{2} \right\rceil$ of the edges incident with u have the same color.

(ii) If G has an odd number of edges, all vertices of G have even degree, and u_G is any vertex of G, then G has a 2-edge-coloring such that, for every vertex u of G distinct from u_G, exactly $\frac{d_G(u)}{2}$ of the edges incident with u have the same color, and exactly $\frac{d_G(u_G)}{2} + 1$ of the edges incident with u_G have the same color.

Using Vizing's bound [12] on the chromatic index leads to our second result.

Theorem 2. *Every graph of minimum degree at least 2 has a majority 4-edge-coloring.*

Clearly, a graph containing a vertex of degree 1 does not have a majority edge-coloring, which motivates the minimum degree condition in Theorem 2. Furthermore, since graphs of minimum degree at least 2, maximum degree 3, and chromatic index 4 have no majority 3-edge-coloring, the number of colors in Theorem 2 is best possible under this minimum degree condition. In fact, if a graph G of minimum degree at least 2 has an induced subgraph H such that H is a graph of maximum degree 3 and chromatic index 4 such that all vertices of H have degree 2 or 3 in G, then G has no majority 3-edge-coloring. We conjecture that all graphs for which 4 colors are needed contain an induced subgraph of maximum degree 3 and chromatic index 4.

Our third result supports this conjecture.

Theorem 3. *Every graph of minimum degree at least 4 has a majority 3-edge-coloring.*

Since a graph containing a vertex of odd degree at least 3 does not have a majority 2-edge-coloring, the number of colors in Theorem 3 is best possible under the minimum degree condition in that result. In Section 2 we prove our results, and in a conclusion we discuss a variation of majority edge-colorings.

2 Proofs

Theorem 1 is a consequence of *Euler's Theorem* [6].

Proof of Theorem 1.

- (i) Let the multigraph G' arise from G by adding the edges of a perfect matching M on the possibly empty set of vertices of odd degree. Clearly, the multigraph G' is connected and every vertex has even degree in G'. Let $e_0 e_1 \cdots e_{m-1}$ be an *Euler tour* of G', where, provided that M is not empty, we may assume that $e_{m-1} \in M$. Setting $c(e_i) = (i \bmod 2) + 1$ for every index i such that e_i belongs to G, yields the desired 2-edge-coloring of G.

- (ii) Let $e_0 e_1 \cdots e_{m-1}$ be an Euler tour of G such that e_0 is incident with u_G. Now, setting $c(e_i) = (i \bmod 2) + 1$ for every index i, yields the desired 2-edge-coloring of G. □

Theorem 2 is a consequence of *Vizing's Theorem* [12].

Proof of Theorem 2. Let G be a graph of minimum degree at least 2. If u is a vertex of degree d, and $d = d_1 + \cdots + d_k$ is a partition of d into positive integers d_i, then the graph H *arises from* G *by splitting* u *into vertices of degrees* $d_1, \ldots, d_k$ if there is a partition $N_G(u) = N_1 \cup \cdots \cup N_k$ of $N_G(u)$ with $|N_i| = d_i$ for $i \in [k]$, $V(H) = (V(G) \setminus \{u\}) \cup \{u_1, \ldots, u_k\}$ for $u_1, \ldots, u_k \notin V(G)$, and $E(H) = E(G - u) \cup \bigcup_{i \in [k]} \{u_i v : v \in N_i\}$. See Figure 1 for an illustration.

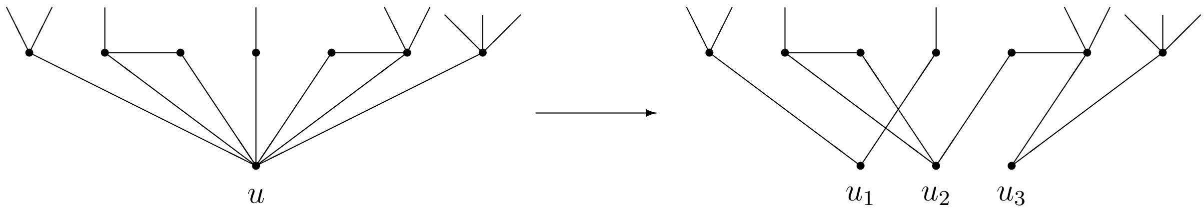

Figure 1: Splitting a vertex u of degree 7 into vertices of degrees 2, 2, and 3.

Now, let G^* arise from G by splitting every vertex of degree $d > 3$ into vertices of degrees

- $3, \ldots, 3$, if $d \equiv 0 \bmod 3$,
- $2, 2, 3, \ldots, 3$, if $d \equiv 1 \bmod 3$, and
- $2, 3, \ldots, 3$, if $d \equiv 2 \bmod 3$.

Note that there is a natural bijection between the edges of G and those of G^*. By Vizing's Theorem [12], the graph G^* has a proper 4-edge-coloring, which yields a majority 4-edge-coloring of G. In fact, we obtain an edge-coloring of G such that, for every vertex of degree d at least 4, at most $(d+2)/3$ of the incident edges have the same color. □

We proceed to the proof of Theorem 3.

Proof of Theorem 3. Let G be a graph of minimum degree δ at least 4. Let $V(G) = D \cup A \cup C$ be the *Gallai-Edmonds decomposition* of G, that is, D is the set of all vertices of G that are missed by some maximum matching, A is the set of all vertices of G outside of D that have a neighbor in D, and C contains the remaining vertices, cf. [10].

Let D' be the set of isolated vertices in $G[D]$.

Claim 4. *It is possible to select, for every vertex u in D', exactly one edge incident with u in such a way that every vertex v in A is incident with at most $\left\lfloor \frac{d_G(v)}{2} \right\rfloor$ of the selected edges.*

Proof of Claim 4. Let H_0 be the bipartite subgraph of G with partite sets D' and A whose edges are exactly all edges of G between D' and A. Let H arise from H_0 by replacing each vertex v in A by $\left\lfloor \frac{d_G(v)}{2} \right\rfloor$ copies having the same neighbors in D' as v. Clearly, the desired statement follows if H has a matching saturating all vertices in D'. Suppose, for a contradiction, that such a matching does not exist. By *Hall's Theorem* [5], there is a subset S of D' with $|S| > |N_H(S)|$. By the definition of D' and the construction of H, we have $|N_H(S)| = \sum\limits_{v \in N_G(S)} \left\lfloor \frac{d_G(v)}{2} \right\rfloor$. Let m denote the number of edges of G between S and $N_G(S)$. Since every vertex in D' has all its neighbors in A, we have $m \geqslant \delta|S|$. Furthermore, $m \leqslant \sum\limits_{v \in N_G(S)} d_G(v)$. Combining these estimates, we obtain

$$\sum_{v \in N_G(S)} \delta \left\lfloor \frac{d_G(v)}{2} \right\rfloor = \delta|N_H(S)| < \delta|S| \leqslant m \leqslant \sum_{v \in N_G(S)} d_G(v). \tag{1}$$

For integers δ and d with $3 \leqslant \delta \leqslant d$, it is easy to verify that $\delta \left\lfloor \frac{d}{2} \right\rfloor \geqslant d$, which yields a contradiction to (1). This completes the proof of Claim 4. □

The properties of the *Gallai-Edmonds decomposition* imply that $G[C]$ has a perfect matching M_C, that there is a matching M_A using edges between A and D that connects each vertex from A to a distinct component of $G[D]$, and that every component of $G[D]$ is *factor-critical*; recall that a graph H is factor-critical if $H - u$ has a perfect matching for every vertex u of H.

We now construct a subset E_1 of the edge set $E(G)$ of G as follows, starting with the empty set:

- We add to E_1 all $|D'|$ selected edges as in Claim 4.
- We add M_C to E_1.
- For every vertex v from A that is not incident with a selected edge, we add to E_1 the unique edge from M_A incident with v. Let M'_A be the subset of M_A added to E_1.

- For every component K of $G[D]$ of order at least 3 such that some vertex x of K is incident with an edge from M'_A, we add to E_1 a perfect matching of $K - x$.
- For every component K of $G[D]$ of order at least 3 such that no vertex of K is incident with an edge from M'_A, we add to E_1 a perfect matching of $K - x$ for some vertex x of K as well as one further edge of K incident with x.

Up to some small modifications explained below, this completes the description of E_1.

By construction, the spanning subgraph G_1 of G with edge set E_1 satisfies

$$1 \leqslant d_{G_1}(u) \leqslant \left\lfloor \frac{d_G(u)}{2} \right\rfloor \text{ for every vertex } u \text{ of } G. \tag{2}$$

Let G_2 be the spanning subgraph of G with edge set $E(G) \setminus E_1$.

For every component K of G_2 such that all vertices of K have even degree in G_2, K has an odd number of edges, and all vertices from $V(K)$ have degree 1 in G_1, we select any edge e_K from K and move it from G_2 to G_1. Note that $K - e_K$ contains exactly two vertices of odd degree, and, hence, is still connected. Furthermore, since G has minimum degree at least 4, it follows that (2) still holds after each such modification. Having performed these modifications for each such component of G_2, every component K of (the modified) G_2 now

- either contains at least one vertex of odd degree in K,
- or all vertices of K have even degrees in K, and the number of edges of K is even,
- or all vertices of K have even degrees in K, the number of edges of K is odd, and K contains a vertex u_K such that the degree of u_K in G_1 is at least 2.

The components of G_2 as in the final point are called *type 2* components, and the remaining components of G_2 are called *type 1* components.

We are now in a position to describe a majority 3-edge-coloring $c : E(G) \to [3]$.

- For all edges e of G_1, let $c(e) = 3$.
- For every component K of G_2 that is of type 1, let $c : E(K) \to [2]$ be as in Theorem 1(i) (applied to K as G).
- For every component K of G_2 that is of type 2, let $c : E(K) \to [2]$ be as in Theorem 1(ii) (applied to K and u_K as G and u_G).

It is now easy to verify that c is a majority 3-edge-coloring of G, which completes the proof. □

3 Conclusion

The most natural question motivated by our results is which graphs of minimum degree at least 2 do not have a majority 3-edge-coloring.

As a variation of majority edge-colorings, we propose the study of *α-majority edge-colorings* for $\alpha \in (0,1)$, where at most an α-fraction of the edges incident with each vertex are allowed to have the same color. If k is a positive integer at least 2, then every positive integer at least $k(k-1)$ can be written as a non-negative integral linear combination of k and $k+1$. Using this fact, a straightforward adaptation of the proof of Theorem 2 yields the following statement: *If a graph G has minimum degree at least $k(k-1)$, then G has a $\frac{1}{k}$-majority $(k+2)$-edge-coloring.* A probabilistic argument implies that, for a sufficiently large minimum degree, one color less suffices.

Theorem 5. *For every integer k at least 2, there is a positive integer δ_k such that every graph of minimum degree at least δ_k has a $\frac{1}{k}$-majority $(k+1)$-edge-coloring.*

Proof. Let G be a graph of minimum degree δ at least δ_k, where we specify δ_k later. Let $c : E(G) \to [k+1]$ be a random $(k+1)$-edge-coloring, where we choose the color of each edge uniformly and independently at random. For every vertex u of G, we consider the *bad event* A_u that more than $\frac{1}{k}d_G(u)$ of the edges incident with u have the same color.

For $d = d_G(u)$, the *union bound* and the *Chernoff inequality*, cf. [11], imply

$$\begin{aligned}
\mathbb{P}[A_u] &\leqslant (k+1)\mathbb{P}\left[\mathrm{Bin}\left(d, \frac{1}{k+1}\right) > \frac{d}{k}\right] && \text{(union bound)}\\
&= (k+1)\mathbb{P}\left[\mathrm{Bin}\left(d, \frac{1}{k+1}\right) > \left(1+\frac{1}{k}\right)\frac{d}{k+1}\right]\\
&\leqslant (k+1)e^{-\frac{d}{3k^2(k+1)}}. && \text{(Chernoff inequality)}
\end{aligned}$$

For every vertex u of G, the event A_u is determined only by the colors of the edges incident with u, which are chosen uniformly and independently at random. Therefore, the event A_u is mutually independent of all events A_v with $v \in V(G) \setminus (\{u\} \cup N_G(u))$. In order to complete the proof, we use the *weighted Lovász Local Lemma*, cf. [11], which states that with positive probability none of the bad events A_u occurs provided that there is a positive integer t_u for every vertex u of G and there is some real p with $0 \leqslant p \leqslant \frac{1}{4}$ such that

- $\mathbb{P}[A_u] \leqslant p^{t_u}$ for every vertex u of G and
- $\sum\limits_{v \in N_G(u)} (2p)^{t_v} \leqslant \frac{t_u}{2}$ for every vertex u of G.

Let $p = (k+1)e^{-\frac{\delta}{3k^2(k+1)}}$ and, for every vertex u of G, let $t_u = \left\lfloor \frac{d_G(u)}{\delta} \right\rfloor$. Note that $d_G(u) \geqslant \delta$ implies that t_u is a positive integer, and that $2t_u = 2\left\lfloor \frac{d_G(u)}{\delta} \right\rfloor \geqslant \frac{d_G(u)}{\delta}$.

Choosing δ_k sufficiently large, we may ensure that $p \leqslant \frac{1}{4}$, and, hence, $\mathbb{P}[A_u] \leqslant p^{\frac{d_G(u)}{\delta}} \leqslant p^{t_u}$. Furthermore, we obtain

$$\sum_{v \in N_G(u)} (2p)^{t_v} \leqslant 2pd_G(u) \leqslant 4p\delta t_u = \underbrace{\left(4(k+1)e^{-\frac{\delta}{3k^2(k+1)}}\delta\right)}_{\to 0 \ for \ \delta \to \infty} t_u,$$

which is at most $t_u/2$ for δ_k sufficiently large.

Altogether, choosing δ_k sufficiently large, the weighted Lovász Local Lemma implies that with positive probability none of the bad events A_u occurs, which implies the existence of a $\frac{1}{k}$-majority $(k+1)$-edge-coloring and completes the proof. □

The estimates in the above proof allow to show that δ_k can be chosen to be $O(k^3 \log k)$. Our Theorem 3 implies that 4 is the smallest possible value for δ_2.

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