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# TOTAL AND PAIRED DOMINATION STABILITY IN PRISMS

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**Abstract**

A set $D$ of vertices in an isolate-free graph is a total dominating set if every vertex is adjacent to a vertex in $D$. If the set $D$ has the additional property that the subgraph induced by $D$ contains a perfect matching, then $D$ is a paired dominating set of $G$. The total domination number $\gamma_t(G)$ and the paired domination number $\gamma_{\rm pr}(G)$ of a graph $G$ are the minimum cardinalities of a total dominating set and a paired dominating set of $G$, respectively. The total domination stability (respectively, paired domination stability) of $G$, denoted $\mathrm{st}_{\gamma_t}(G)$ (respectively, $\mathrm{st}_{\gamma_{\rm pr}}(G)$), is the minimum size of a non-isolating set of vertices in $G$ whose removal changes the total domination number (respectively, paired domination number). In this paper, we study total and paired domination stability in prisms.

**Keywords:** total domination stability, paired domination stability, prism, hypercube.

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## 1. Introduction

A *dominating set* of a graph $G$ with vertex set $V(G)$ is a set $D$ of vertices of $G$ such that every vertex in $V(G) \setminus D$ is adjacent to a vertex in $D$. The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of $G$ having cardinality $\gamma(G)$ is called a $\gamma$-*set* of $G$.

The concept of domination stability in graphs was introduced in 1983 by Bauer, Harary, Nieminen and Suffel [5] and has been studied, for example, in [18]. Stability for other domination type parameters has been studied in the literature. For example, total domination stability, paired domination stability, 2-rainbow domination stability, exponential domination stability, and Roman domination stability are studied in [1, 4, 7, 11, 14].

An *isolate-free graph* is a graph with no isolated vertex. A *total dominating set*, abbreviated TD-set, of an isolate-free graph $G$ is a set $D$ of vertices of $G$ such that every vertex in $V(G)$ is adjacent to at least one vertex in $D$. The *total domination number* of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of $G$. A TD-set of $G$ having cardinality $\gamma_t(G)$ is called a $\gamma_t$-*set* of $G$. A vertex $v$ is *totally dominated* by a set $D$ in $G$ if the vertex $v$ has a neighbor in $D$. Total domination in graphs is well studied in the literature [13].

The total version of domination stability was first studied by Henning and Krzywkowski [11]. To define this formally, we call a set $S \subseteq V(G)$ of vertices in $G$ a *non-isolating set* if the graph $G - S$ is an isolate-free graph, where $G - S$ denotes the graph obtained from $G$ by removing $S$ and all edges incident with vertices in $S$. Let $\mathrm{NI}(G)$ denote the set of all non-isolating sets of vertices of $G$. The $\gamma_t^-$-stability of $G$, denoted $\mathrm{st}_{\gamma_t}^-(G)$, is the minimum size of a non-isolating set $S$ of vertices in $G$ whose removal decreases the total domination number. The $\gamma_t^+$-stability of $G$, denoted $\mathrm{st}_{\gamma_t}^+(G)$, is the minimum size of a non-isolating set of vertices in $G$ whose removal increases the total domination number, if such a set exists. If no such non-isolating set exists whose removal increases the total domination number, we define $\mathrm{st}_{\gamma_t}^+(G) = \infty$. As a trivial example, $\mathrm{st}_{\gamma_t}^-(P_7) = 2$ while $\mathrm{st}_{\gamma_t}^+(P_7) = \infty$. The *total domination stability* of $G$ (or the $\gamma_t$-*stability*), denoted $\mathrm{st}_{\gamma_t}(G)$, is the minimum size of a non-isolating set $S$ of vertices in $G$ whose removal changes the total domination number. Thus,

$$\mathrm{st}_{\gamma_t}(G) = \min_{S \in \mathrm{NI}(G)} \{|S| \colon \gamma_t(G - S) \neq \gamma_t(G)\} = \min \left\{ \mathrm{st}_{\gamma_t}^-(G), \mathrm{st}_{\gamma_t}^+(G) \right\}.$$

A *paired dominating set*, abbreviated PD-set, of an isolate-free graph $G = (V, E)$ is a set $D \subseteq V$ such that every vertex of $G$ is adjacent to some vertex in $D$ and the induced subgraph $G[D]$ contains a perfect matching $M$ (not necessarily induced). Two vertices joined by an edge of $M$ are said to be *paired*. The *paired domination number* of $G$, denoted by $\gamma_{\mathrm{pr}}(G)$, is the minimum cardinality of a PD-set of $G$, and a PD-set of $G$ having cardinality $\gamma_{\mathrm{pr}}(G)$ is called a $\gamma_{\mathrm{pr}}$-*set* of $G$.

Necessarily, the paired domination number of a graph is an even integer. Since every PD-set is a TD-set, we note that $\gamma_t(G) \le \gamma_{\rm pr}(G)$ for all graphs $G$ without isolated vertices. The concept of paired domination was first introduced and studied by Haynes and Slater in [10].

The paired version of domination stability was first studied by the authors in [7]. Unless otherwise stated, let $G$ be an isolate-free graph. The $\gamma_{\rm pr}^-$-stability of $G$, denoted ${\rm st}^-_{\gamma_{\rm pr}}(G)$, and the $\gamma_{\rm pr}^+$-stability of $G$, denoted ${\rm st}^+_{\gamma_{\rm pr}}(G)$, respectively, are defined analogously to the total versions. The *paired domination stability* of $G$ (or the *$\gamma_{pr}$-stability*), denoted ${\rm st}_{\gamma_{\rm pr}}(G)$, is the minimum size of a non-isolating set $S$ of vertices in $G$ whose removal changes the paired domination number. Thus,

$$ {\rm st}_{\gamma_{\rm pr}}(G) = \min_{S \in {\rm NI}(G)} \{|S| \colon \gamma_{\rm pr}(G-S) \neq \gamma_{\rm pr}(G)\} = \min\{{\rm st}^-_{\gamma_{\rm pr}}(G), {\rm st}^+_{\gamma_{\rm pr}}(G)\}. $$

Note that if no such non-isolating set exists whose removal increases the paired domination number, we define ${\rm st}^+_{\gamma_{\rm pr}}(G) = \infty$. As a trivial example, ${\rm st}^-_{\gamma_{\rm pr}}(P_5) = 1$ while ${\rm st}^+_{\gamma_{\rm pr}}(P_5) = \infty$. Following the original paper of Bauer *et al.* [5], we consider the *null graph* $K_0$ (also called the *order-zero graph*), which is the unique graph having no vertices and hence has order zero, as a graph. As observed in [7], considering the null graph, the total domination and paired domination stability of a non-trivial graph is always defined. If $G$ is a graph of order $n$ and $\gamma_t(G) = 2$, then ${\rm st}^-_{\gamma_t}(G) = n$ since removing all vertices from the graph $G$ produces the null graph with total domination number zero. Analogously, if $G$ is a graph of order $n$ and $\gamma_{\rm pr}(G) = 2$, then ${\rm st}^-_{\gamma_{\rm pr}}(G) = n$. Hence, it is only of interest for us to consider isolate-free graphs $G$ with $\gamma_t(G) \ge 3$ when determining ${\rm st}^-_{\gamma_t}(G)$, and with $\gamma_{\rm pr}(G) \ge 4$ when determining ${\rm st}^-_{\gamma_{\rm pr}}(G)$.

A *perfect dominating set* $D$ in a graph $G$ is a dominating set of $G$ in which every vertex of $G$ is dominated by exactly one vertex in $D$. Thus, if a graph $G$ has perfect dominating set, then the set $N_G[v]$ for all $v \in D$, partition the set $V(G)$. It is clear that a perfect dominating set for a graph $G$ is necessarily a $\gamma$-set of $G$.

The *Cartesian product* $G \,\Box\, H$ of graphs $G$ and $H$ is the graph whose vertex set is $V(G) \times V(H)$ and two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent in $G \,\Box\, H$ if either $g_1 = g_2$ and $h_1h_2$ is an edge in $H$, or $h_1 = h_2$ and $g_1g_2$ is an edge in $G$.

The *prism* of a graph $G$ is the graph $G \Box K_2$. Thus, it is defined by taking two disjoint copies $G_1$ and $G_2$ of $G$, called *layers*, and adding an edge between each pair of corresponding vertices. For each vertex $v$ in $G$ we denote its equivalent in $G_i$ by $v_i$ for $i \in [2]$, and refer to the vertices $v_1$ and $v_2$ as *partners*. If $G$ is a bipartite graph, then we call the prism $G \,\Box\, K_2$ the *bipartite prism* of $G$. If $G$ is a cycle, then we call the prism $G \,\Box\, K_2$ a *cycle prism* of $G$. It should be mentioned that the Cartesian products of graphs have wide applications to numerous problems of theoretical computer science, where the information nets are very often modelled by prisms.

The relationship between domination parameters in the graph and its prism have been studied extensively. In particular, we note that total domination and paired domination in prisms have been studied, for example, in [3, 6, 9, 16].

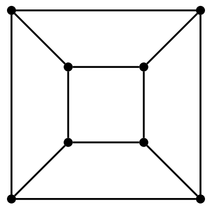

Figure 1. The hypercube $Q_3$.

We denote by $Q_n$ the $n$-dimensional hypercube, and so $Q_n$ can be represented as the $n^{\text{th}}$ power of $K_2$ with respect to the Cartesian product operation $\Box$, that is, $Q_1 = K_2$ and $Q_n = Q_{n-1} \,\Box\, K_2$ for $n \geq 2$. The vertices are represented by binary sequences of length $n$ and two vertices are adjacent if the corresponding sequences differ in exactly one coordinate. Note that the hypercube $Q_3$, illustrated in Figure 1, is the bipartite cycle prism of $C_4$.

For notation and graph theory terminology we generally follow [13].

## 2. Known Results and Motivation

We recall first trivial lower bounds on the total and paired domination numbers of a graph in terms of the maximum degree of the graph.

**Observation 1.** *If $G$ is an isolate-free graph of order $n$ and maximum degree $\Delta$, then $\gamma_t(G) \geq \lceil \frac{n}{\Delta} \rceil$ and $\gamma_{\rm pr}(G) \geq 2 \lceil \frac{n}{2\Delta} \rceil$.*

Let $G$ be a graph, and consider the prism $G\Box K_2$ formed by taking two disjoint copies $G_1$ and $G_2$ of $G$. Let $D$ be a $\gamma$-set of $G$, and let $D_i$ be the set of vertices in $G_i$ corresponding to $D$ for $i \in [2]$. The set $D_1 \cup D_2$ is a PD-set of $G$, with each vertex of $D_1$ paired with its neighbor in $D_2$. Thus, $\gamma_{\rm pr}(G) \leq |D_1|+|D_2| = 2|D| = 2\gamma(G)$. We state this observation formally.

**Observation 2.** *If $G$ is a graph, then $\gamma_t(G \,\Box\, K_2) \leq \gamma_{\rm pr}(G \,\Box\, K_2) \leq 2\gamma(G)$.*

Azarija, Henning, and Klavžar [3] proved that if $G$ is a bipartite graph, then we have equality throughout the inequality chain in Observation 2. A simple proof of this result was given in [6].

**Theorem 3** [3]**.** *If $G$ is a bipartite graph, then $\gamma_t(G\Box K_2) = \gamma_{\rm pr}(G\Box K_2) = 2\gamma(G)$.*

It is also shown in [3] that the bipartite condition in the statement of Theorem 3 is essential.

The $\gamma_t^-$-stability and the total domination stability of paths and cycles is computed in [11], while the $\gamma_{\rm pr}^-$-stability and the paired domination stability of paths and cycles is computed in [7]. In the introductory paper [7] on paired domination stability, it is shown that the paired domination stability of a graph can be very different from its domination or total domination stability.

Our aim in this paper is to study total and paired domination stability in prisms. We show that the difference between $\gamma_{\rm pr}$-stability and $\gamma_t$-stability is a small constant for some special but useful class of the Cartesian products. Additionally, $\gamma_{\rm pr}$ is very close to $\gamma_t$ for such graphs. In Section 3 we determine $\gamma_t^-$-stability of cycle prisms (Theorem 6) and establish upper bounds for $\gamma_t^+$-stability of these graphs (Theorems 7 and 8).

In Section 4 we determine $\gamma_{\rm pr}^-$-stability of cycle prisms (Theorem 19) and we establish next upper bounds on the $\gamma_{\rm pr}^+$-stability of these graphs (Theorem 21).

On the other hand, the following result establishes an upper bound on the $\gamma_t^-$-stability and $\gamma_{\rm pr}^-$-stability of a general graph in terms of its maximum degree.

**Theorem 4.** *If $G$ is a connected graph with maximum degree $\Delta$, then the following holds.*

(a) ([11]) *If $\gamma_t(G) \geq 3$, then $\mathrm{st}_{\gamma_t}^-(G) \leq 2\Delta - 1$, and this bound is sharp.*

(b) ([7]) *If $\gamma_{\rm pr}(G) \geq 4$, then $\mathrm{st}_{\gamma_{\rm pr}}^-(G) \leq 2\Delta$, and this bound is sharp.*

Both bounds in Theorem 4 are tight for an infinite family of trees, as shown in [7, 11]. In Section 4.1 we prove that the bound in Theorem 4(b) is tight for the family of hypercubes, $Q_n$, in the case when $n = 2^k$ for all $k \geq 1$ (Theorem 12).

## 3. Total Domination Stability in Cycle Prisms

In this section, we investigate total domination stability of a cycle prism. Recall that for $n \geq 3$, $\gamma(C_n) = \lceil \frac{n}{3} \rceil$. We begin by establishing the total domination number of a cycle prism.

**Proposition 5.** *For $n \geq 3$,*

$$\gamma_t(C_n \,\square\, K_2) = \begin{cases} 2\gamma(C_n) - 1 & \textit{if } n \equiv 1 \pmod 6, \\ 2\gamma(C_n) & \textit{otherwise.} \end{cases}$$

***Proof.*** If $n$ is even, then the result follows from Theorem 3. If $n \equiv 1 \pmod 6$, then the result is known cf. [3, Proposition 16]. Suppose that $n \pmod 6 \in \{3, 5\}$. If $n = 6k + 3$ for some $k \geq 0$, then $4k + 2 = \lceil (12k + 6)/3 \rceil = \lceil 2n/\Delta \rceil \leq$

$\gamma_t(C_n \,\square\, K_2) \le 2\gamma(C_n) = 2\lceil n/3 \rceil = 4k+2$. If $n = 6k+5$ for some $k \ge 0$, then $4k+4 = \lceil (12k+10)/3 \rceil = \lceil 2n/\Delta \rceil \le \gamma_t(C_n \,\square\, K_2) \le 2\gamma(C_n) = 2\lceil n/3 \rceil = 4k+4$. In both cases, we must have equality throughout the above inequality chains, implying that $\gamma_t(C_n \,\square\, K_2) = 2\gamma(C_n)$ when $n \,(\mathrm{mod}\ 6) \in \{3,5\}$. ■

We note that the vertices totally dominated by a set $D$ of vertices in a graph $G$ are precisely those vertices that have a neighbor in $D$. We are now in a position to present a proof of the following result determining the $\gamma_t^-$-stability of a cycle prism.

**Theorem 6.** *For $n \ge 4$,*

$$\mathrm{st}^-_{\gamma_t}(C_n \,\square\, K_2) = \begin{cases} 1 & \text{if } n \equiv 4 \,(\mathrm{mod}\ 6), \\ 2 & \text{if } n \,(\mathrm{mod}\ 6) \in \{1,2,5\}, \\ 4 & \text{if } n \,(\mathrm{mod}\ 6) \in \{0,3\}. \end{cases}$$

***Proof.*** For $n \ge 4$, let $G = C_n \,\square\, K_2$. Let $G_1$ and $G_2$ be the two layers of the prism $G$, where $G_1$ is the cycle $v_1v_2\cdots v_nv_1$ and $G_2$ is the cycle $u_1u_2\cdots u_nu_1$. Let $G$ be obtained from $G_1$ and $G_2$ by adding the edges $u_iv_i$ for $i \in [n]$. Suppose that $n \equiv 4 \,(\mathrm{mod}\ 6)$, and so $n = 6k+4$ for some $k \ge 0$. By Proposition 5, $\gamma_t(G) = 4k+4$. Letting $S = \{u_n\}$, the set $D = \{v_{n-1}\} \cup \bigcup_{i=0}^{2k}\{u_{3i+2}, v_{3i+2}\}$ is a TD-set of $G - S$, implying that $S$ is a non-isolating set of vertices in $G$ such that $\gamma_t(G-S) \le |D| = 4k+3 < \gamma_t(G)$. Hence, $\mathrm{st}^-_{\gamma_t}(G) \le |S| = 1$, implying that $\mathrm{st}^-_{\gamma_t}(G) = 1$.

Suppose that $n \equiv 1 \,(\mathrm{mod}\ 6)$, and so $n = 6k+1$ for some $k \ge 1$. By Proposition 5, $\gamma_t(G) = 4k+1$. If $|S| = 1$ and $S$ is a non-isolating set of vertices in $G$, then $|V(G-S)| = 12k+1$ and $\Delta(G-S) = 3$. Thus in this case, $\gamma_t(G-S) \ge \lceil (12k+1)/3 \rceil = 4k+1 = \gamma_t(G)$. Hence, if $S$ is a non-isolating set of vertices in $G$ such that $\gamma_t(G-S) \le 4k$, then $|S| \ge 2$, implying that $\mathrm{st}^-_{\gamma_t}(G) \ge 2$. Letting $S = \{u_n, v_n\}$, the set $D = \bigcup_{i=0}^{2k-1}\{u_{3i+2}, v_{3i+2}\}$ is a TD-set of $G - S$, implying that $\mathrm{st}^-_{\gamma_t}(G) \le 2$. Consequently, $\mathrm{st}^-_{\gamma_t}(G) = 2$.

Suppose that $n \equiv 2 \,(\mathrm{mod}\ 6)$, and so $n = 6k+2$ for some $k \ge 1$. By Proposition 5, $\gamma_t(G) = 4k+2$. Let $S$ be a minimum non-isolating set of $G$ such that $\gamma_t(G-S) \le 4k+1$. Let $D$ be a $\gamma_t$-set of $G-S$, and so $|D| = \gamma_t(G-S) \le 4k+1$. Adding vertices to $D$ if necessary, we may assume that $|D| = 4k+1$. Thus, $G[D]$ consists of at least one component of odd order at least 3. This implies that at least two vertices in $D$ have a common neighbor in $D$, and therefore that the number of vertices totally dominated by $D$ is at most $3|D|-1 = 3(4k+1)-1 = 12k+2$. Since the number of vertices totally dominated by $D$ is $|V(G-S)| = 12k+4-|S|$, we have $12k+4-|S| \le 12k+2$, implying that $\mathrm{st}^-_{\gamma_t}(G) = |S| \ge 2$. We now consider the set $S = \{v_1, u_8\}$. Let $D_1 = \{u_2, u_3, u_4, v_6, v_7\}$. If $k = 1$, let $D = D_1$, while if $k \ge 2$, let $D_k = D_1 \cup \bigcup_{i=3}^{2k}\{u_{3i+1}, v_{3i+1}\}$. The resulting set $D_k$ is a TD-set of

$G-S$, implying that $\gamma_t(G-S) \le |D_k| = 4k+1$. Hence, $\mathrm{st}^-_{\gamma_t}(G) \le 2$. Consequently, $\mathrm{st}^-_{\gamma_t}(G) = 2$.

Suppose that $n \equiv 5 \pmod 6$, and so $n = 6k+5$ for some $k \ge 0$. By Proposition 5, $\gamma_t(G) = 4k+4$. Let $S$ be a minimum non-isolating set of $G$ such that $\gamma_t(G-S) \le 4k+3$. Reasoning analogous to case when $n \equiv 2 \pmod 6$ allows us to state that $\mathrm{st}^-_{\gamma_t}(G) = |S| \ge 2$. We now consider the set $S = \{u_1, u_5\}$. Let $D_1 = \{v_2, v_3, v_4\}$. If $k = 0$, let $D = D_1$, while if $k \ge 1$, let $D_k = D_1 \cup \bigcup_{i=2}^{2k+1} \{u_{3i+1}, v_{3i+1}\}$. The resulting set $D_k$ is a TD-set of $G-S$, implying that $\gamma_t(G-S) \le |D_k| = 4k+3$. Hence, $\mathrm{st}^-_{\gamma_t}(G) \le 2$. Consequently, $\mathrm{st}^-_{\gamma_t}(G) = 2$.

Suppose that $n \equiv 0 \pmod 6$, and so $n = 6k$ for some $k \ge 1$. By Proposition 5, $\gamma_t(G) = 4k$. Let $S$ be a minimum non-isolating set of $G$ such that $\gamma_t(G-S) \le 4k-1$. As previously, an easy calculation suffices to show that $\mathrm{st}^-_{\gamma_t}(G) = |S| \ge 4$. We now consider the set $S = \{u_1, u_2, u_6, v_1\}$. Let $D_1 = \{v_3, v_4, v_5\}$. If $k = 1$, let $D = D_1$, while if $k \ge 2$, let $D_k = D_1 \cup \bigcup_{i=2}^{2k-1} \{u_{3i+2}, v_{3i+2}\}$. The resulting set $D_k$ is a TD-set of $G-S$, implying that $\gamma_t(G-S) \le |D_k| = 4k-1$. Hence, $\mathrm{st}^-_{\gamma_t}(G) \le 4$. Consequently, $\mathrm{st}^-_{\gamma_t}(G) = 4$.

Suppose that $n \equiv 3 \pmod 6$, and so $n = 6k+3$ for some $k \ge 1$. By Proposition 5, $\gamma_t(G) = 4k+2$. Once more, we omit the calculation showing that $\mathrm{st}^-_{\gamma_t}(G) = |S| \ge 4$. We now let $S = \{u_1, u_2, u_6, v_1\}$ and $D_k = \{v_3, v_4, v_5\} \cup \bigcup_{i=2}^{2k} \{u_{3i+2}, v_{3i+2}\}$. The resulting set $D_k$ is a TD-set of $G-S$, implying that $\gamma_t(G-S) \le |D_k| = 4k+1$. Hence, $\mathrm{st}^-_{\gamma_t}(G) \le 4$. Consequently, $\mathrm{st}^-_{\gamma_t}(G) = 4$. ■

We next establish upper bounds on the $\gamma_t^+$-stability of a cycle prism. For small values of $n \in \{3,4,5\}$, we note that $\mathrm{st}^+_{\gamma_t}(C_n \,\square\, K_2) = \infty$. Hence it is only of interest to consider values of $n \ge 6$. Firstly, we determine the exact value of the $\gamma_t^+$-stability of a cycle prism $C_n \,\square\, K_2$ when $n \equiv 0 \pmod 3$.

**Theorem 7.** *For $n \ge 6$ and $n \equiv 0 \pmod 3$,*

$$\mathrm{st}^+_{\gamma_t}(C_n \,\square\, K_2) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 6, \\ 4 & \text{if } n \equiv 3 \pmod 6. \end{cases}$$

***Proof.*** For $n \ge 6$ and $n \equiv 0 \pmod 3$, let $G = C_n \,\square\, K_2$. Let $G_1$ and $G_2$ be the two layers of the prism $G$, where $G_1$ is the cycle $v_1 v_2 \cdots v_n v_1$ and $G_2$ is the cycle $u_1 u_2 \cdots u_n u_1$. Let $G$ be obtained from $G_1$ and $G_2$ by adding the edges $u_i v_i$ for $i \in [n]$. We show firstly that $\mathrm{st}^+_{\gamma_t}(G) \ge 3$. By Proposition 5, $\gamma_t(G) = \frac{2}{3}n$. For $i \in [3]$, let $V_i = \{u_j, v_j \; : \; j \equiv i \pmod 3 \text{ and } j \in [n]\}$. Let $S$ be a minimum non-isolating set of $G$ such that $\gamma_t(G-S) > \frac{2}{3}n$. Since each of the sets $V_1$, $V_2$ and $V_3$ is a TD-set of $G$ of cardinality $\frac{2}{3}n = \gamma_t(G)$, we note that $|S \cap V_i| \ge 1$ for all $i \in [3]$, implying that $\mathrm{st}^+_{\gamma_t}(G) = |S| \ge 3$.

Suppose that $n \equiv 0 \pmod 6$, and so $n = 6k$ for some $k \ge 1$. We show that in this case, $\mathrm{st}^+_{\gamma_t}(G) \le 3$. By Proposition 5, $\gamma_t(G) = 4k$. Let $S = \{u_1, u_3, u_5\}$ and consider the graph $G-S$. If $k=1$, then $\gamma_t(G-S)=5>4=\gamma_t(G)$, implying that $\mathrm{st}^+_{\gamma_t}(G)\le|S|=3$. Hence we may assume that $k\ge 2$. Every TD-set of $G-S$ contains the two vertices $v_2$ and $v_4$. Let $D$ be a $\gamma_t$-set of $G-S$. We show that $|D|\ge 4k+1$. Let $V'=\bigcup_{i=6}^{6k}\{u_i,v_i\}$ and let $H=G[V']$. Note that $H$ is isomorphic to $P_{6(k-1)+1}\,\Box\,K_2$. Suppose that $v_3\in D$. If $v_5\in D$, then we can replace $v_5$ in $D$ with the vertex $u_6$ or $v_7$. If $v_{6k}\in D$, then we can replace $v_{6k}$ in $D$ with the vertex $u_{6k-1}$. Hence, we may choose $D$ so that $D\cap\{v_5,v_{6k}\}=\emptyset$, implying that $|D|\ge|\{v_2,v_3,v_4\}|+\gamma_t(H)=3+\gamma_t(P_{6(k-1)+1}\,\Box\,K_2)=3+2\gamma(P_{6(k-1)+1})=3+2(2(k-1)+1)=4k+1$.

Assume now that $v_3\notin D$. With this assumption, we can choose $D$ so that $\{v_1,v_2,v_4,v_5\}\subset D$. If $v_6\in D$, then we can replace $v_6$ in $D$ with the vertex $u_7$. If $v_{6k}\in D$, then we can replace $v_{6k}$ in $D$ with the vertex $u_{6k-1}$. Hence, we may choose $D$ so that $D\cap\{v_6,v_{6k}\}=\emptyset$. In order to totally dominate the vertices $u_6$ and $u_{6k}$, this implies that $u_7\in D$ and $u_{6k-1}\in D$, respectively. Let $V''=V'\setminus\{v_6,v_{6k}\}$, and let $D'=D\cap V'$. Thus, $\{u_7,u_{6k-1}\}\subset D'$ and the set $D'$ totally dominates the set $V''$. Since $|V''|=12(k-1)$ and each vertex of $D'$ totally dominates at most three vertices, we note that $|D'|\ge 4(k-1)$.

We show that $|D'|>4(k-1)$. Suppose, to the contrary, that $|D'|=4(k-1)$. This implies that each vertex of $D'$ uniquely totally dominates three vertices of $V'$. By our earlier observations, $u_7\in D'$. Since $u_6$ has only one neighbor in $V''$, and since $v_7$ has only two neighbors in $V''$, we note that $u_6\notin D'$ and $v_7\notin D'$, and therefore $u_8\in D'$. This in turn implies that $D'\cap\{v_8,u_9,v_9,u_{10}\}=\emptyset$. Therefore, $\{v_{10},v_{11}\}\subset D'$. If $k=2$, then as observed earlier, $u_{11}\in D'$, and so $|D'|\ge 5>4(k-1)$, a contradiction. Hence, $k\ge 3$. Since each vertex of $D'$ uniquely totally dominates three vertices of $V''$, and since $\{v_{10},v_{11}\}\subset D'$, we therefore have that $D'\cap\{u_{11},u_{12},v_{12},v_{13}\}=\emptyset$. Therefore, $\{u_{13},u_{14}\}\subset D'$. This in turn implies that $D'\cap\{v_{14},u_{15},v_{15},u_{16}\}=\emptyset$ and $\{v_{16},v_{17}\}\subset D'$. Continuing in this way, for each $i\in[k-1]$ we have $\{u_{6i+1},u_{6i+2},v_{6i+4},v_{6i+5}\}\subseteq D'$. However as observed earlier, $u_{6k-1}=u_{6(k-1)+5}\in D'$, implying that $|D'|\ge 4(k-1)+1$, a contradiction. Hence, $|D'|>4(k-1)$. Therefore, $|D|\ge 4+|D'|>4k=\gamma_t(G)$, implying that $\mathrm{st}^+_{\gamma_t}(G)\le|S|=3$.

Suppose that $n\equiv 3\,(\mathrm{mod}\ 6)$, and so $n=6k+3$ for some $k\ge 1$. We show that in this case, $\mathrm{st}^+_{\gamma_t}(G)\le 4$. By Proposition 5, $\gamma_t(G)=4k+2$. Let $S=\{u_1,u_3,u_5,u_7\}$ and consider the graph $G-S$. If $k=1$, then $\gamma_t(G-S)=7>6=\gamma_t(G)$, implying that $\mathrm{st}^+_{\gamma_t}(G)\le|S|=4$. Hence we may assume that $k\ge 2$. Every TD-set of $G-S$ contains the three vertices $v_2$, $v_4$ and $v_6$. Let $D$ be a $\gamma_t$-set of $G-S$. In order to totally dominate the vertex $v_4$, we can choose $D$ so that $v_3\in D$ or $v_5\in D$. By symmetry, we may assume that $v_5\in D$. With this assumption, we can choose $D$ so that $v_1\in D$ in order to totally dominate the vertex $v_2$. Thus, $\{v_1,v_2,v_4,v_5,v_6\}\subset D$. If $v_7\in D$, then we can replace $v_7$ in $D$ with the vertex $u_8$ or the vertex $v_9$. If $v_n=v_{6k+3}\in D$, then we can replace $v_{6k+3}$

in $D$ with the vertex $u_{6k+2}$. Hence, we may choose $D$ so that $D \cap \{v_7, v_{6k+3}\} = \emptyset$. Let $V' = \{u_{6k+3}\} \cup \bigcup_{i=8}^{6k+2} \{u_i, v_i\}$ and let $D' = D \cap V'$. Since each vertex of $D'$ totally dominates at most three vertices, in order to totally dominate the $12(k-1)+3$ vertices in $V'$ we note that $|D'| \geq 4(k-1)+1 = 4k-3$. We show that $|D'| > 4k-3$. Suppose, to the contrary, that $|D'| = 4k-3$. This implies that each vertex of $D'$ uniquely totally dominates three vertices of $V'$. Since each of $u_8$ and $v_8$ has only two neighbors in $V'$, this implies that $D' \cap \{u_8, v_8\} = \emptyset$ and therefore that $\{u_9, v_9\} \subset D'$. This in turn implies that $D' \cap \{u_{10}, v_{10}, u_{11}, v_{11}\} = \emptyset$, and therefore that $\{u_{12}, v_{12}\} \subset D'$. Continuing in this way, we have that $\{u_{3i}, v_{3i}\} \subset D'$ and $D' \cap \{u_{3i+1}, v_{3i+1}, u_{3i+2}, v_{3i+2}\} = \emptyset$ for all $i \in [2k] \setminus \{1,2\}$. This implies that the vertex $u_{6k+3}$ is not totally dominated by $D$, a contradiction. Hence, $|D'| > 4k-3$. Therefore, $|D| = 5 + |D'| > 5 + (4k-3) = 4k+2 = \gamma_t(G)$, implying that $\mathrm{st}^+_{\gamma_t}(G) \leq |S| = 4$.

It remains for us to show that $\mathrm{st}^+_{\gamma_t}(G) \geq 4$ in this case when $n = 6k+3$ for some $k \geq 1$. Let $S$ be a minimum non-isolating set of $G$ such that $\gamma_t(G-S) > \frac{2}{3}n$. As shown earlier, $|S| \geq 3$. We wish to show that $|S| \geq 4$. Suppose, to the contrary, that $|S| = 3$. If $k = 1$ (and so, $n = 9$), then this can be readily checked. Hence we may assume that $k \geq 2$. By our earlier observations, $|S \cap V_i| \geq 1$ for all $i \in [3]$ where recall $V_i$ is the set of all vertices of $G$ with subscript congruent to $i$ modulo 3. Thus, $|S \cap V_i| = 1$ for all $i \in [3]$.

Suppose that $S$ contains two adjacent vertices. Renaming vertices if necessary, we may assume that $\{u_1, u_2\} \subset S$. If $v_{6k+3} \notin S$, then $(V_1 \setminus \{u_1\}) \cup \{v_{6k+3}\}$ is a TD-set of $G-S$ of cardinality $|V_1| = \gamma_t(G)$. If $v_{6k+3} \in S$, then $(V_2 \setminus \{u_2\}) \cup \{v_3\}$ is a TD-set of $G-S$ of cardinality $|V_2| = \gamma_t(G)$. Hence, $\gamma_t(G-S) \leq \gamma_t(G)$, a contradiction. Thus, the set $S$ is an independent set in $G$. A detailed case analysis, which we omit, shows that $\gamma_t(G-S) \leq \gamma_t(G)$. We remark that our case analysis relies heavily on the fact that $n \equiv 3 \pmod 6$. To illustrate this, consider an arbitrary vertex $v$ of $G$. For notational convenience we may assume $v = u_3$. In this case, the set $D' = \{v_1, v_2, v_4, v_5\} \cup \left(\bigcup_{i=1}^{k} \{u_{6i+1}, u_{6i+2}\}\right) \cup \left(\bigcup_{i=1}^{k-1} \{v_{6i+4}, v_{6i+5}\}\right)$ is a $\gamma_t$-set of $G - u_3$. Thus for example, if $S = \{u_1, u_3, u_5\}$, then the set $D'$ is a TD-set of $G-S$, implying that $\gamma_t(G-S) \leq |D'| = \gamma_t(G)$, a contradiction. Indeed, the fact that $n \equiv 3 \pmod 6$ implies that there are many $\gamma_t$-sets of $G$, in addition to the sets $V_1$, $V_2$ and $V_3$, and one can guarantee that at least one such set is always a TD-set of $G-S$ in this case when $|S| = 3$, producing a contradiction. Hence, $|S| \geq 4$, and so $\mathrm{st}^+_{\gamma_t}(G) \geq 4$. By our earlier observation, $\mathrm{st}^+_{\gamma_t}(G) \leq 4$. Consequently, $\mathrm{st}^+_{\gamma_t}(G) = 4$ in this case when $n \equiv 3 \pmod 6$. ■

We present next a proof of the following result establishing upper bounds on the $\gamma_t^+$-stability of a cycle prism $C_n \,\square\, K_2$ when $n \geq 7$ and $n \not\equiv 0 \pmod 3$.

**Theorem 8.** *For $n \geq 7$ and $n \not\equiv 0 \pmod 3$,*

$$\mathrm{st}^+_{\gamma_t}(C_n \,\square\, K_2) \leq \begin{cases} 5 & \textit{if } n \equiv 1 \pmod 6, \\ 7 & \textit{if } n \pmod 6 \in \{2,5\}, \\ 8 & \textit{if } n \equiv 4 \pmod 6. \end{cases}$$

***Proof.*** For $n \geq 6$, let $G = C_n \,\square\, K_2$. Let $G_1$ and $G_2$ be the two layers of the prism $G$, where $G_1$ is the cycle $v_1v_2\cdots v_nv_1$ and $G_2$ is the cycle $u_1u_2\cdots u_nu_1$. Let $G$ be obtained from $G_1$ and $G_2$ by adding the edges $u_iv_i$ for $i \in [n]$.

Suppose that $n \equiv 1 \pmod 6$, and so $n = 6k+1$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k+1$. We let $S = \{u_1, u_3, v_1, v_3, v_5\}$ and $D$ be a $\gamma_t$-set of $G - S$. Necessarily, $\{u_2, u_4, v_2\} \subset D$ and we can choose the set $D$ so that $u_5 \in D$. Let $V' = \{v_6\} \cup \bigcup_{i=7}^{6k+1}\{u_i, v_i\}$.

In order to totally dominate the $12k-9$ vertices in the set $V'$, the set $D$ contains at least $4k-3$ vertices in addition to the four vertices in the set $\{u_2, u_4, u_5, v_2\}$. Suppose that $D$ contains exactly $4k-3$ additional vertices. In this case, each additional vertex uniquely totally dominates three new vertices. Hence neither $u_{6k+1}$ nor $v_{6k+1}$ belong to $D$ since both these vertices have degree 2 in $G-S$. Hence, $\{u_{6k}, v_{6k}\} \subset D$ in order to totally dominate the vertices $u_{6k+1}$ and $v_{6k+1}$. Since each of $u_{6k}$ and $v_{6k}$ uniquely totally dominates three vertices, this implies that $\{u_{6k-2}, u_{6k-1}, v_{6k-2}, v_{6k-1}\} \cap D = \emptyset$ and that $\{u_{6k-3}, v_{6k-3}\} \subset D$. Continuing this argument, we have that $\bigcup_{i=3}^{2k}\{u_{3i}, v_{3i}\} \subset D$.

At least two additional vertices are needed to totally dominate the three vertices $v_6$, $u_7$ and $v_7$, implying that in addition to the vertices in $\{u_2, u_4, u_5, v_2\}$, the set $D$ contains at least $2 + 4(k-1) = 4k-2$ additional vertices to totally dominate the vertices in $V'$, a contradiction. Therefore, the set $D$ contains at least $4k-2$ vertices in addition to the vertices in $\{u_2, u_4, u_5, v_2\}$, implying that $\gamma_t(G-S) = |D| \geq 4k+2 > \gamma_t(G)$. Thus, $\mathrm{st}^+_{\gamma_t}(G) \leq |S| = 5$.

Suppose that $n \equiv 2 \pmod 6$, and so $n = 6k+2$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k+2$. Let $S = \{u_1, v_1, u_3, v_3, u_5, v_5, v_7\}$ and consider the graph $G-S$. Let $D$ be a $\gamma_t$-set of $G-S$. Necessarily, $\{u_2, v_2, u_4, v_4, u_6\} \subset D$ and we can choose the set $D$ so that $u_7 \in D$. If $k = 1$, then the vertex $u_8$ is needed in $D$ to dominate the vertex $v_8$, implying that $\gamma_t(G-S) = |D| = 7 > 6 = \gamma_t(G)$. Suppose that $k \geq 2$ and let $V' = \{v_8\} \cup \bigcup_{i=9}^{6k+2}\{u_i, v_i\}$.

In order to totally dominate the $12k-11$ vertices in the set $V'$, the set $D$ contains at least $4k-3$ vertices in addition to the six vertices in the set $\{u_2, v_2, u_4, v_4, u_6, u_7\}$. Thus, $|D| \geq 6 + (4k-3) = 4k+3 > \gamma_t(G)$. Hence, $\mathrm{st}^+_{\gamma_t}(G) \leq |S| = 7$.

Suppose that $n \equiv 5 \pmod 6$, and so $n = 6k+5$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k+4$. As in the previous case when $n \equiv 2 \pmod 6$, we let $S = \{u_1, v_1, u_3, v_3, u_5, v_5, v_7\}$ and consider the graph $G-S$. Let $D$ be a $\gamma_t$-set of

$G - S$. Necessarily, $\{u_2, v_2, u_4, v_4, u_6\} \subset D$ and we can choose the set $D$ so that $u_7 \in D$. Let $V' = \{v_8\} \cup \bigcup_{i=9}^{6k+5} \{u_i, v_i\}$.

In order to totally dominate the $12k - 5$ vertices in the set $V'$, the set $D$ contains at least $4k - 1$ vertices in addition to the six vertices in the set $\{u_2, v_2, u_4, v_4, u_6, u_7\}$. Thus, $|D| \geq 6 + (4k - 1) = 4k + 5 > \gamma_t(G)$. Hence, $\mathrm{st}^+_{\gamma_t}(G) \leq |S| = 7$.

Suppose that $n \equiv 4 \,(\mathrm{mod}\ 6)$, and so $n = 6k + 4$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k + 4$. Let $S = \{u_1, v_1, u_3, v_3, u_5, v_5, v_7, v_9\}$ and let $D$ be a $\gamma_t$-set of $G - S$. We show that $|D| \geq 4k + 5$. Necessarily, $\{u_2, v_2, u_4, v_4, u_6, u_8\} \subset D$ and we can choose the set $D$ so that $u_7 \in D$. If $k = 1$, then $\{u_9, u_{10}\} \subseteq D$ or $\{u_{10}, v_{10}\} \subseteq D$, and so $|D| \geq 9 = 4k + 5$, as claimed. Hence we may assume that $k \geq 2$. Suppose that $u_9 \notin D$. In this case, $|D| = 7 + \gamma_t(P_{6k-5} \,\square\, K_2) = 7 + 2\gamma(P_{6k-5}) = 7 + 2(2k - 1) = 4k + 5 > \gamma_t(G)$. Assume now that $u_9 \in D$ and let $V' = \{v_{10}\} \cup \bigcup_{i=11}^{6k+4} \{u_i, v_i\}$.

In order to totally dominate the $12k - 11$ vertices in the set $V'$, the set $D$ contains at least $4k - 3$ vertices in addition to the eight vertices in the set $\{u_2, v_2, u_4, v_4, u_6, u_7, u_8, u_9\}$. Thus, $|D| \geq 8 + (4k - 3) = 4k + 5$. Hence in all cases, $|D| \geq 4k + 5 > \gamma_t(G)$, implying that $\mathrm{st}^+_{\gamma_t}(G) \leq |S| = 8$. ■

As an immediate consequence of Theorems 6, 7 and 8 we have the following result on the total domination stability of a cycle prism.

**Corollary 9.** *For $n \geq 4$,*

$$\mathrm{st}_{\gamma_t}(C_n \,\square\, K_2) = \begin{cases} 1 & \text{if } n \equiv 4 \,(\mathrm{mod}\ 6), \\ 2 & \text{if } n \,(\mathrm{mod}\ 6) \in \{1, 2, 5\}, \\ 3 & \text{if } n \equiv 0 \,(\mathrm{mod}\ 6), \\ 4 & \text{if } n \equiv 3 \,(\mathrm{mod}\ 6). \end{cases}$$

***Proof.*** By Theorems 6, 7 and 8, the only remaining case that needs to be covered now is the fact that the $\gamma_t^+$-stability of $C_n \,\square\, K_2$ is greater than 1 in cases when $n \,(\mathrm{mod}\ 6) \in \{1, 2, 5\}$. Let $v$ be an arbitrary vertex of $C_n \,\square\, K_2$. We remark that removing one vertex from $C_n \,\square\, K_2$ does not produce isolated vertices. It is easily seen that there is a $\gamma_t$-set $D$ of $C_n \,\square\, K_2$ such that $v \notin D$. Hence, $D$ is a TD-set for $(C_n \,\square\, K_2) - \{v\}$, so we conclude that $\mathrm{st}^+_{\gamma_t}(C_n \,\square\, K_2) > 1$. ■

It remains an open problem to determine the exact value of the $\gamma_t^+$-stability of a cycle prism $C_n \,\square\, K_2$ for $n \geq 7$ and $n \not\equiv 0 \,(\mathrm{mod}\ 3)$.

## 4. Paired Domination Stability in Prisms

In this section, we investigate paired domination stability of a prism. We consider three types of prisms, namely hypercubes, bipartite prisms, and cycle prisms.

### 4.1. Hypercubes

In this subsection, we consider the class of prisms called *hypercubes* $Q_n$ for $n \geq 1$. Recall that $Q_1 = K_2$ and $Q_n = Q_{n-1} \,\square\, K_2$ for $n \geq 2$. To determine $\gamma(Q_n)$ turns out to be an intrinsically difficult problem. To date, exact values are only known for $n \leq 9$, and for two infinite families of hypercubes. These results are summarized in Table 1 and Theorem 10. The result $\gamma(Q_9) = 62$ in Table 1 due to Östergård and Blass [17] actually presented a breakthrough back in 2001.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma(Q_n)$ | 1 | 2 | 2 | 4 | 7 | 12 | 16 | 32 | 62 |

Table 1. Domination numbers of hypercubes up to dimension 9.

**Theorem 10** [3]. *If $k \geq 1$, then $\gamma(Q_{2^k-1}) = 2^{2^k-k-1}$ and $\gamma(Q_{2^k}) = 2^{2^k-k}$.*

A *code* in a graph $G = (V, E)$ is a subset $C \subset V$ such that any two vertices of $C$ are at distance at least 3 in $G$. A *perfect code* is a code $C$ with the property that $C$ is a dominating set in $G$, cf. [15]. The first assertion of Theorem 10 is based on the fact that hypercubes $Q_{2^k-1}$ contain perfect codes, cf. [8], and the domination number of a graph with a perfect code is equal to the size of such a code. We note that $Q_n$ contains a perfect code if and only if $n = 2^k - 1$ for some $k \geq 1$. The second assertion of Theorem 10 is due to van Wee [19].

As a consequence of Table 1 and Theorem 3, the exact values of $\gamma_{\rm pr}(Q_n)$ for $n \leq 10$ are given in Table 2.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma_{\rm pr}(Q_n)$ | 2 | 4 | 4 | 8 | 14 | 24 | 32 | 64 | 124 |

Table 2. Paired domination numbers of hypercubes up to dimension 10.

Arumugam and Kala [2] established the following upper bound on the domination number of a hypercube.

**Theorem 11** [2]. *For all $n \geq 7$, we have $\gamma(Q_n) \leq 2^{n-3}$.*

Total domination in Cartesian products has been studied in [12]. We focus on the paired domination and paired domination stability in hypercubes.

We are now in a position to prove the following theorem, that establishes the existence of a $2^k$-regular connected graph $G$, where $k$ can be chosen arbitrarily large, satisfying ${\rm st}^-_{\gamma_{\rm pr}}(G) = 2\Delta(G)$. Thus, the family of hypercubes $Q_n$, where $n = 2^k$ for some $k \geq 1$, shows that the upper bound in Theorem 4(b) is tight.

**Theorem 12.** *If $Q_n$ is a hypercube such that $n = 2^k$, for $k \geq 1$, then*

$$
{\rm st}^-_{\gamma_{\rm pr}}(Q_n) = 2\Delta(Q_n) = 2^{k+1}.
$$

***Proof.*** Let $n = 2^k$ for some integer $k \geq 1$, and consider the hypercube $Q_n$. We note that $|V(Q_n)| = 2^n$ and $\Delta(Q_n) = n = 2^k$. By Theorem 3 and Theorem 10, we have that $\gamma_{\rm pr}(Q_n) = 2\gamma(Q_{n-1}) = 2 \cdot 2^{n-k-1} = 2^{n-k}$. Suppose that there exists a non-isolating subset $S$ of vertices in the hypercube $Q_n$ such that $\gamma_{\rm pr}(Q_n - S) < \gamma_{\rm pr}(Q_n)$ and $|S| \leq 2\Delta(Q_n) - 1 = 2^{k+1} - 1$. Thus, $|V(Q_n - S)| = |V(Q_n)| - |S| \geq 2^n - 2^{k+1} + 1$. We note that $\Delta(Q_n - S) \leq \Delta(Q_n) = 2^k$. By Observation 1, we therefore have

$$\gamma_{\rm pr}(Q_n - S) \geq \frac{|V(Q_n - S)|}{\Delta(Q_n - S)} \geq \left\lceil \frac{2^n - 2^{k+1} + 1}{2^k} \right\rceil = 2^{n-k} - 1.$$

Since the paired domination number of a graph is an even integer, this implies that $\gamma_{\rm pr}(Q_n - S) \geq 2^{n-k} = \gamma_{\rm pr}(Q_n)$, a contradiction. Hence, every non-isolating subset $S$ of vertices in the hypercube $Q_n$ such that $\gamma_{\rm pr}(Q_n - S) < \gamma_{\rm pr}(Q_n)$ has cardinality at least $2\Delta(Q_n)$, that is, $\mathrm{st}^-_{\gamma_{\rm pr}}(Q_n) \geq 2\Delta(Q_n)$. By Theorem 4, $\mathrm{st}^-_{\gamma_{\rm pr}}(Q_n) \leq 2\Delta(Q_n)$. Consequently, $\mathrm{st}^-_{\gamma_{\rm pr}}(Q_n) = 2\Delta(Q_n) = 2^{k+1}$. ■

We establish next the upper bound on the $\gamma^+_{\rm pr}$-stability of a class of connected bipartite prisms.

**Proposition 13.** *If $G = (X, Y; E)$ is a connected bipartite graph, with $\gamma(G) < \min\{|X|, |Y|\}$, then $\mathrm{st}^+_{\gamma_{\rm pr}}(G \,\square\, K_2) \leq \min\{|X|, |Y|\}$.*

***Proof.*** Let $G = (X, Y; E)$ be a connected bipartite graph with $\gamma(G) < \min\{|X|, |Y|\}$. Without the loss of generality, let $\min\{|X|, |Y|\} = |X| = k$ and let $G_1$ and $G_2$ be two copies of $G$ that form a graph $G \,\square\, K_2$ by adding a perfect matching between corresponding vertices in $G_1$ and $G_2$. Let $X_i$ and $Y_i$ be the two partite sets of $G_i$ for $i \in [2]$. Renaming the sets if necessary, we may assume that there is a perfect matching between the vertices of $X_1$ and $X_2$ (respectively, between $Y_1$ and $Y_2$) in $G \,\square\, K_2$. Note that $\gamma_{\rm pr}(G \,\square\, K_2) < 2k$ since we can choose a dominating set in $G_i$ and its corresponding set in $G_{3-i}$. Let $H$ be the (connected) graph obtained from $G \,\square\, K_2$ by removing the vertices in the set $X_1$, and so $H = G \,\square\, K_2 - X_1$. We note that each vertex in $Y_1$ has degree 1 in $H$. Further, the (unique) neighbor in $H$ of each vertex in $Y_1$ belongs to the set $Y_2$. Thus, each vertex of $Y_2$ is a support vertex in $H$, and therefore belongs to every PD-set of $H$. Since the set $Y_2$ is an independent set, this implies that every PD-set of $H$ has cardinality at least $2|Y_2| \geq 2|X_2| = 2k > \gamma_{\rm pr}(G \,\square\, K_2)$. Hence, the set $X_1$ is a non-isolating set of vertices of $G \,\square\, K_2$ such that $\gamma_{\rm pr}(G \,\square\, K_2 - X_1) > \gamma_{\rm pr}(G \,\square\, K_2)$, implying that $\mathrm{st}^+_{\gamma_{\rm pr}}(G \,\square\, K_2) \leq k = |X_1|$. ■

From Proposition 13 we can immediately conclude the following result.

**Corollary 14.** *For $n \geq 4$, we have $\mathrm{st}^+_{\gamma_{\rm pr}}(Q_n) \leq 2^{n-2}$.*

***Proof.*** Let $G = Q_n$ for some $n \geq 4$. We show that $\gamma_{\rm pr}(G) < 2^{n-1}$ for all $n \geq 4$. By Table 2, this is true for small $n \in \{4, 5, \dots, 10\}$. For $n \geq 11$, by Theorems 3 and 11, we have that $\gamma_{\rm pr}(G) = 2\gamma(Q_{n-1}) \leq 2 \cdot 2^{n-4} < 2^{n-1}$. Thus, $\gamma_{\rm pr}(G) < 2^{n-1}$ for all $n \geq 4$. Further, with reasoning as in the proof of the Proposition 13 we conclude that the result follows. ■

It remains an open problem, however, to determine the exact value of the $\gamma_{\rm pr}^+$-stability of a hypercube $Q_n$ for $n \geq 4$.

### 4.2. Bipartite prisms

In this section, we study bipartite prisms. For this purpose, we first determine the $\gamma^-$-stability of a regular graph.

**Lemma 15.** *For $r \geq 2$, if $G$ is an $r$-regular graph that contains a perfect dominating set, then* $\mathrm{st}_\gamma^-(G) = r + 1$.

***Proof.*** Let $G$ be an $r$-regular graph of order $n$ that contains a perfect dominating set. Since any perfect dominating set is necessarily a minimum dominating set, we have $n = k(r+1)$ for some integer $k \geq 1$, and $\gamma(G) = k$. Let $S$ be a set of vertices of $G$ such that $\gamma(G - S) \leq \gamma(G) - 1 = k - 1$. We note that $|V(G - S)| = n - |S|$ and $\Delta(G - S) \leq \Delta(G) = r$. Thus,

$$k - 1 \geq \gamma(G - S) \geq \frac{|V(G - S)|}{\Delta(G - S) + 1} \geq \frac{n - |S|}{r + 1},$$

and so, $|S| \geq n - (k - 1)(r + 1) = k(r + 1) - (k - 1)(r + 1) = r + 1$, implying that $\mathrm{st}_\gamma^-(G) \geq r + 1$. However, if $D$ is a perfect dominating set of $G$ and $v \in D$, then removing $v$ and all its neighbors from $G$ produces a graph with domination number $|D| - 1 = \gamma(G) - 1$, implying that $\mathrm{st}_\gamma^-(G) \leq r + 1$. Consequently, $\mathrm{st}_\gamma^-(G) = r + 1$. ■

We are now ready to prove a sharp upper bound on the $\gamma_{\rm pr}^-$-stability of a bipartite prism.

**Theorem 16.** *If $G$ is a bipartite graph, then*

$$\mathrm{st}_{\gamma_{\rm pr}}^-(G \,\square\, K_2) \leq 2\mathrm{st}_\gamma^-(G),$$

*and this bound is sharp.*

***Proof.*** Let $G$ be a bipartite graph, and consider the prism $G \,\square\, K_2$ formed by taking two disjoint copies $G_1$ and $G_2$ of $G$. Let $S$ be a non-isolating set of vertices in $G$ such that $\gamma(G - S) < \gamma(G)$ and $|S| = \mathrm{st}_\gamma^-(G)$. Let $D$ be a $\gamma$-set of $G - S$. Let $D_i$ and $S_i$ be the set of vertices in $G_i$ corresponding to the sets $D$ and $S$, respectively, in $G$ for $i \in [2]$.

We note that the set $S_1 \cup S_2$ is a non-isolating set of vertices in $G \,\square\, K_2$. Further, we note that the graph $(G\square K_2)-(S_1\cup S_2)$ is isomorphic to $(G-S)\square K_2$. Additionally, in the graph $(G \,\square\, K_2) - (S_1 \cup S_2)$ every vertex of the set $D_1$ is adjacent to its partner in $D_2$. Hence, $D_1 \cup D_2$ is a PD-set of $(G - S) \,\square\, K_2$. Therefore, $S_1 \cup S_2$ is a non-isolating set in $G \,\square\, K_2$ such that $\gamma_{\rm pr}((G \,\square\, K_2) - (S_1 \cup S_2)) \le |D_1 \cup D_2| = 2|D| = 2\gamma(G - S) < 2\gamma(G) = \gamma_{\rm pr}(G \,\square\, K_2)$. Consequently, ${\rm st}^-_{\gamma_{\rm pr}}(G \,\square\, K_2) \le |S_1 \cup S_2| = 2|S| = 2{\rm st}^-_{\gamma}(G)$.

It remains to show that this bound is tight. This may be seen by taking, for example, $G$ to be a hypercube $Q_n$, where $n = 2^k - 1$ for some $k \ge 1$. In this case the graph $G \,\square\, K_2$ is isomorphic to $Q_{n+1}$, where $n + 1 = 2^k$. Thus by Theorem 12, we have ${\rm st}^-_{\gamma_{\rm pr}}(G \,\square\, K_2) = {\rm st}^-_{\gamma_{\rm pr}}(Q_{n+1}) = 2\Delta(Q_{n+1}) = 2(n + 1)$. As observed earlier, the hypercube $G = Q_{2^k-1}$ contain a perfect code, that is, the graph $G$ contains a perfect dominating set. This implies by Lemma 15 that ${\rm st}^-_{\gamma}(G) = {\rm st}^-_{\gamma}(Q_n) = \Delta(Q_n) + 1 = n + 1$. Thus, ${\rm st}^-_{\gamma_{\rm pr}}(G \,\square\, K_2) = 2{\rm st}^-_{\gamma}(G)$. ■

We present next an additional example of a class of graphs $G$ achieving equality in the upper bound of Theorem 16. For $k \ge 1$, let $G_{2k}$ be the graph constructed as follows. Consider two copies of the path $P_{4k}$ with respective vertex sequences $a_1b_1a_2b_2 \cdots a_{2k}b_{2k}$ and $c_1d_1c_2d_2 \cdots c_{2k}d_{2k}$. For each $i \in [2k]$, join $a_i$ to $d_i$ and $b_i$ to $c_i$. To complete the construction of the graph $G_{2k}$, add the two edges $a_1b_{2k}$ and $c_1d_{2k}$. Let $\mathcal{G} = \{G_{2k} : k \ge 1\}$. The graph $G_4 \in \mathcal{G}$ is illustrated in Figure 2, where the framed vertices form a $\gamma$-set of $G_4$. We note that $G_{2k}$ is a cubic graph of order $8k$ for all $k \ge 1$, and that $G_{2k}$ contains a perfect dominating set. In particular, $\gamma(G_{2k}) = 2k$.

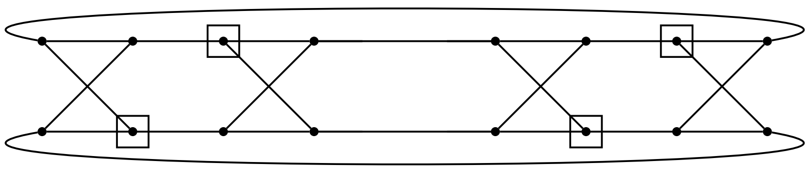

Figure 2. A graph $G_4 \in \mathcal{G}$.

**Proposition 17.** *If $G \in \mathcal{G}$, then ${\rm st}^-_{\gamma_{\rm pr}}(G \,\square\, K_2) = 2{\rm st}^-_{\gamma}(G)$.*

***Proof.*** Let $G$ be an arbitrary graph in the family $\mathcal{G}$, and so $G = G_{2k}$ for some $k \ge 1$. Thus, $\gamma(G) = 2k$. The graph $G$ is a 3-regular graph that contains a perfect dominating set, and so by Lemma 15, we have ${\rm st}^-_{\gamma}(G) = 4$.

We show next that ${\rm st}^-_{\gamma_{\rm pr}}(G \,\square\, K_2) = 8$. Let $G_1$ and $G_2$ be the two layers of the prism $G \,\square\, K_2$. Let $D$ be a $\gamma$-set of $G$, and let $D_i$ be the set of vertices in $G_i$ corresponding to the set $D$ in $G$ for $i \in [2]$. The set $D_1 \cup D_2$ is a PD-set of $G \,\square\, K_2$, and so $\gamma_{\rm pr}(G \,\square\, K_2) \le |D_1| + |D_2| = 2|D| = 2\gamma(G) = 4k$. Since $G\square K_2$ is a 4-regular graph of order $16k$, by Observation 1 we have $\gamma_{\rm pr}(G\square K_2) \ge$

$|V(G\Box K_2)|/\Delta(G\Box K_2) = 16k/4 = 4k$. Consequently, $\gamma_{\rm pr}(G\Box K_2) = 4k$. Let $S$ be a non-isolating set of vertices in $G\Box K_2$ such that $\gamma_{\rm pr}((G\Box K_2)-S) < \gamma_{\rm pr}(G\Box K_2)$, and so $\gamma_{\rm pr}((G\,\Box\,K_2)-S) \le 4k-2$. We note that $|V((G\,\Box\,K_2)-S)| = 16k-|S|$ and $\Delta((G\,\Box\,K_2)-S) \le \Delta(G\,\Box\,K_2) = 4$. By Observation 1,

$$4k-2 \ge \gamma_{\rm pr}((G\,\Box\,K_2)-S) \ge \frac{|V((G\,\Box\,K_2)-S)|}{\Delta((G\,\Box\,K_2)-S)} \ge \frac{16k-|S|}{4},$$

and so, $|S| \ge 8$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(G\Box K_2) \ge 8$. However, removing from the $\gamma_{\rm pr}$-set $D_1 \cup D_2$ of $G\,\Box\,K_2$ a vertex $v_1 \in D_1$ and its partner $v_2 \in D_2$, and all their neighbors in $G\,\Box\,K_2$, produces a graph with PD-set $(D_1 \cup D_2) \setminus \{v_1, v_2\}$. Thus, there exists a set of eight vertices whose removal from $G\Box K_2$ decreases the paired domination number, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(G\Box K_2) \le 8$. Consequently, $\mathrm{st}^-_{\gamma_{\rm pr}}(G\Box K_2) = 8$. ■

### 4.3. Cycle prisms

As observed earlier, for $n \ge 3$, $\gamma(C_n) = \lceil n/3 \rceil$. We show first that the result of Theorem 3 can be extended to (odd) cycles.

**Proposition 18.** *If $G$ is a cycle, then $\gamma_{\rm pr}(G\,\Box\,K_2) = 2\gamma(G)$.*

***Proof.*** For $n \ge 3$, let $G$ be a cycle $C_n$ and consider the cycle prism of $G$, namely $G\,\Box\,K_2$. We note that $|V(G\,\Box\,K_2)| = 2n$ and that $G\,\Box\,K_2$ is a cubic graph. By Observations 1 and 2, we have

$$2\left\lceil \frac{n}{3} \right\rceil = 2\left\lceil \frac{2n}{2\cdot 3} \right\rceil = 2\left\lceil \frac{|V(G\,\Box\,K_2)|}{2\cdot\Delta(G\,\Box\,K_2)} \right\rceil \le \gamma_{\rm pr}(G\,\Box\,K_2) \le 2\gamma(G) = 2\left\lceil \frac{n}{3} \right\rceil.$$

Consequently, we must have equality throughout the above inequality chain. In particular, $\gamma_{\rm pr}(G\,\Box\,K_2) = 2\gamma(G)$. ■

We are now in a position to determine the $\gamma^-_{\rm pr}$-stability of a cycle prism.

**Theorem 19.** *For $n \ge 4$,*

$$\mathrm{st}^-_{\gamma_{\rm pr}}(C_n\,\Box\,K_2) = \begin{cases} 2 & \textit{if } n \pmod 6 \in \{1,4\}, \\ 4 & \textit{if } n \pmod 6 \in \{2,5\}, \\ 6 & \textit{if } n \pmod 6 \in \{0,3\}. \end{cases}$$

***Proof.*** For $n \ge 4$, let $G = C_n\,\Box\,K_2$. Let $G_1$ and $G_2$ be the two layers of the prism $G$ used to build the graph $G$, where $G_1$ is the cycle $v_1v_2\cdots v_nv_1$ and $G_2$ is the cycle $u_1u_2\cdots u_nu_1$. Let $G$ be obtained from $G_1$ and $G_2$ by adding the edges $u_iv_i$ for $i \in [n]$.

Suppose that $n \equiv 1 \pmod 6$, and so $n = 6k+1$ for some $k \ge 1$. By Proposition 18, $\gamma_{\rm pr}(G) = 2\gamma(C_{6k+1}) = 2\left\lceil \frac{6k+1}{3} \right\rceil = 4k+2$. If $|S| = 1$, then $|V(G-S)|$

$= 12k + 1$ and $\Delta(G - S) = 3$. Thus in this case, $\gamma_{\rm pr}(G - S) \geq 2\left\lceil \frac{12k+1}{2\cdot 3} \right\rceil = 4k + 2 = \gamma_{\rm pr}(G)$, a contradiction. Hence, $|S| \geq 2$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \geq 2$. Letting $S = \{u_1, v_1\}$, the graph $G - S$ is isomorphic to $P_{6k} \,\Box\, K_2$. In this case, by Theorem 3 we have $\gamma_{\rm pr}(G - S) = 2\gamma(P_{6k}) = 2\left\lceil \frac{6k}{3} \right\rceil = 4k < 4k + 2 = \gamma_{\rm pr}(G)$. Thus, $S$ is a non-isolating set of vertices in $G$ such that $\gamma_{\rm pr}(G - S) < \gamma_{\rm pr}(G)$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \leq 2$. Consequently, $\mathrm{st}^-_{\gamma_{\rm pr}}(G) = 2$.

Suppose that $n \equiv 4 \pmod 6$. With computations analogous to the case when $n \equiv 1 \pmod 6$ we obtain the desired result.

Suppose that $n \equiv 2 \pmod 6$, and so $n = 6k + 2$ for some $k \geq 1$. By Proposition 18, $\gamma_{\rm pr}(G) = 2\gamma(C_{6k+2}) = 2\left\lceil \frac{6k+2}{3} \right\rceil = 4k + 2$. If $|S| \leq 3$, then $|V(G - S)| \geq 12k + 1$ and $\Delta(G - S) = 3$. Thus in this case, $\gamma_{\rm pr}(G - S) \geq 2\left\lceil \frac{12k+1}{2\cdot 3} \right\rceil = 4k + 2 = \gamma_{\rm pr}(G)$, a contradiction. Hence, $|S| \geq 4$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \geq 4$. Letting $S = \{u_1, v_1, u_2, v_2\}$, the graph $G - S$ is isomorphic to $P_{6k} \,\Box\, K_2$. Thus, by Theorem 3 we have $\gamma_{\rm pr}(G - S) = 2\gamma(P_{6k}) = 2\left\lceil \frac{6k}{3} \right\rceil = 4k < 4k + 2 = \gamma_{\rm pr}(G)$. Hence, $S$ is a non-isolating set of vertices in $G$ such that $\gamma_{\rm pr}(G - S) < \gamma_{\rm pr}(G)$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \leq 4$. Consequently, $\mathrm{st}^-_{\gamma_{\rm pr}}(G) = 4$.

Suppose that $n \equiv 5 \pmod 6$. The result is obtained by analogous reasoning as in the case when $n \equiv 2 \pmod 6$.

Suppose that $n \equiv 0 \pmod 6$, and so $n = 6k$ for some $k \geq 1$. By Proposition 18, $\gamma_{\rm pr}(G) = 2\gamma(C_{6k}) = 2\left\lceil \frac{6k}{3} \right\rceil = 4k$. If $|S| \leq 5$, then $|V(G - S)| \geq 12k - 5$ and $\Delta(G - S) = 3$. Thus in this case, $\gamma_{\rm pr}(G - S) \geq 2\left\lceil \frac{12k-5}{2\cdot 3} \right\rceil = 4k = \gamma_{\rm pr}(G)$, a contradiction. Hence, $|S| \geq 6$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \geq 6$. Letting $S = \{u_1, v_1, u_2, v_2, u_3, v_3\}$, the graph $G - S$ is isomorphic to $P_{6k-3} \,\Box\, K_2$. Thus, by Theorem 3 we have $\gamma_{\rm pr}(G - S) = 2\gamma(P_{6k-3}) = 2\left\lceil \frac{6k-3}{3} \right\rceil = 4k - 2 < 4k = \gamma_{\rm pr}(G)$. Hence, $S$ is a non-isolating set of vertices in $G$ such that $\gamma_{\rm pr}(G - S) < \gamma_{\rm pr}(G)$, implying that $\mathrm{st}^-_{\gamma_{\rm pr}}(G) \leq 6$. Consequently, $\mathrm{st}^-_{\gamma_{\rm pr}}(G) = 6$.

Suppose that $n \equiv 3 \pmod 6$. The calculations are analogous to the case when $n \equiv 0 \pmod 6$. ■

We next consider the $\gamma^+_{\rm pr}$-stability of a cycle prism $C_n \,\Box\, K_2$. For small values of $n \in \{3, 4, 5, 7\}$, we note that $\mathrm{st}^+_{\gamma_{\rm pr}}(C_n \,\Box\, K_2) = \infty$. Hence it is only of interest to consider values of $n$ where $n \geq 6$ and $n \neq 7$. For this purpose, we prove an additional lemma.

Let $G$ be a path $P_k$ for some $k \geq 1$, and consider the prism $G \,\Box\, K_2$ formed by taking two disjoint copies $G_1$ and $G_2$ of $G$. Thus, $G_1$ and $G_2$ are the two layers of the prism, $G \,\Box\, K_2$. If $k = 1$, then $G_1$ is isomorphic to $K_1$, and we let $L_k$ be obtained from $G \,\Box\, K_2$ by attaching two leaf neighbors to the vertex of $G_1$. Thus, $L_1$ is isomorphic to $K_{1,3}$. For $k \geq 2$, let $L_k$ be a graph of order $2k + 2$ obtained from $G \,\Box\, K_2$ by attaching a leaf neighbor to each end-vertex of the path $G_1$. Let $\mathcal{L}$ be the family of all such graphs $L_k$, and so $\mathcal{L} = \{L_k : k \geq 1\}$. We proceed further with the following lemma that computes the paired domination number of a graph that belongs to the family $\mathcal{L}$.

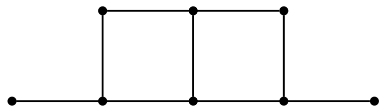

Figure 3. The graph $L_3$ from the family $\mathcal{L}$.

**Lemma 20.** *If $G$ is a graph in the family $\mathcal{L}$ with order $n$, then*

$$\gamma_{\rm pr}(G) = \begin{cases} 2\left\lceil \frac{n}{6} \right\rceil & \text{if } n \not\equiv 0 \pmod{12}, \\ \frac{n}{3}+2 & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

***Proof.*** Let $G \in \mathcal{L}$ have order $n$, and so $G = L_k$ for some $k \geq 1$. Let $G_1$ and $G_2$ be the two layers of the prism $P_k \square K_2$ used to build the graph $G$, where $G_1$ is the path $v_1v_2 \cdots v_k$ and $G_2$ is the path $u_1u_2 \cdots u_k$. Let $G$ be obtained from $G_1$ and $G_2$ by adding the two new vertices $v_0$ and $v_{k+1}$, and adding the edges $v_0v_1$, $v_kv_{k+1}$, and the edges $u_iv_i$ for $i \in [k]$. We note that $\Delta(G) = 3$ and $|V(G)| = n = 2k+2$. Thus, by Observation 1, we have $\gamma_{\rm pr}(G) \geq 2\left\lceil \frac{n}{6} \right\rceil = 2\left\lceil \frac{k+1}{3} \right\rceil$.

**Claim 1.** *If $n \not\equiv 0 \pmod{12}$, then $\gamma_{\rm pr}(G) = 2\left\lceil \frac{k+1}{3} \right\rceil$.*

***Proof.*** Suppose that $n \not\equiv 0 \pmod{12}$, implying that $k \not\equiv 5 \pmod 6$. We show that $\gamma_{\rm pr}(G) \leq 2\left\lceil \frac{k+1}{3} \right\rceil$. We proceed by induction on $k \geq 1$. If $k = 1$ or $k = 2$, then let $D = \{v_1, v_2\}$. If $k = 3$, then let $D = \{u_1, v_1, u_3, v_3\}$. If $k = 4$, then let $D = \{u_1, v_1, u_4, v_4\}$. If $k = 6$, then let $D = \{u_1, v_1, u_4, v_4, u_6, v_6\}$. In all four cases, the resulting set $D$ is a PD-set of $G$ and satisfies $|D| = 2\left\lceil \frac{k+1}{3} \right\rceil$. This establishes the base cases. Let $k \geq 7$, and assume that if $G' = L_{k'}$ for some $k'$ where $1 \leq k' < k$ and $k' \not\equiv 5 \pmod 6$, then $\gamma_{\rm pr}(G') = 2\left\lceil \frac{k'+1}{3} \right\rceil$. We now consider the graph $G = L_k$. Let $S = \{v_0, v_1, v_2, v_3, v_{k-2}, v_{k-1}, v_k, v_{k+1}\} \cup \{u_1, u_2, u_{k-1}, u_k\}$.

We note that the set $D = \{v_1, v_2, v_{k-1}, v_k\}$ is a PD-set of the subgraph, $G[S]$, of $G$ induced by the set $S$. Let $G' = G - S$, and note that $G'$ is isomorphic to $L_{k'}$, where $k' = k - 6$. Thus, $1 \leq k' < k$ and $k' \not\equiv 5 \pmod 6$. Applying the inductive hypothesis to $G'$, we have $\gamma_{\rm pr}(G') = 2\left\lceil \frac{k'+1}{3} \right\rceil = 2\left\lceil \frac{k+1}{3} \right\rceil - 4$. Thus, $\gamma_{\rm pr}(G) \leq \gamma_{\rm pr}(G') + \gamma_{\rm pr}(G[S]) \leq (2\left\lceil \frac{k+1}{3} \right\rceil - 4) + 4 = 2\left\lceil \frac{k+1}{3} \right\rceil$, as desired. Hence, by induction, $\gamma_{\rm pr}(G) \leq 2\left\lceil \frac{k+1}{3} \right\rceil$. As observed earlier, we have $\gamma_{\rm pr}(G) \geq 2\left\lceil \frac{k+1}{3} \right\rceil$. Consequently, $\gamma_{\rm pr}(G) = 2\left\lceil \frac{k+1}{3} \right\rceil$. This completes the proof of Claim 1. □

By Claim 1, we may assume that $n \equiv 0 \pmod{12}$, for otherwise the desired result follows. With this assumption, $k \geq 5$ and $k \equiv 5 \pmod 6$.

Suppose that $k = 5$, and so $G = L_5$. Let $D$ be a $\gamma_{\rm pr}$-set of $G$. Since $D$ contains all support vertices of $G$, we have $\{v_1, v_5\} \subseteq D$. In order to dominate the vertex $u_3$, the set $D$ contains a vertex $v \in N_G[u_3]$. Thus, letting $A = \{v, v_1, v_5\}$, we note that the independent set $A$ is a subset of $D$, implying that $\gamma_{\rm pr}(G) \geq 2|A| = 6$.

Conversely, the set $\{v_1, v_2, u_3, v_3, v_4, v_5\}$ is a PD-set of $G$ (with $v_1$ and $v_2$ paired, $u_3$ and $v_3$ paired, and $v_4$ and $v_5$ paired), and so $\gamma_{\rm pr}(G) \le 6$. Consequently, $\gamma_{\rm pr}(G) = 6 = \frac{n}{3} + 2$ and this establishes the base case as we will further proceed with induction on $k$.

We show firstly that when $k \ge 5$ and $k \equiv 5 \pmod 6$, we have $\gamma_{\rm pr}(G) \le \frac{n}{3} + 2$. Let $k \ge 11$, and assume that if $5 \le k' < k$ where $k' \equiv 5 \pmod 6$, then $\gamma_{\rm pr}(L_{k'}) \le \frac{2k'+2}{3} + 2$. Let the set $S$ be defined as in the proof of Claim 1. As before, we note that $\gamma_{\rm pr}(G[S]) \le 4$. Let $G' = G - S$, and note that $G'$ is isomorphic to $L_{k'}$, where $k' = k - 6$. Thus, $5 \le k' < k$ and $k' \equiv 5 \pmod 6$. Applying the inductive hypothesis to $G'$, we have $\gamma_{\rm pr}(G') \le \frac{2k'+2}{3} + 2 = \frac{2k+2}{3} - 2 = \frac{n}{3} - 2$. Thus, $\gamma_{\rm pr}(G) \le \gamma_{\rm pr}(G') + \gamma_{\rm pr}(G[S]) \le (\frac{n}{3} - 2) + 4 = \frac{n}{3} + 2$, as desired.

We show next that when $k \ge 5$ and $k \equiv 5 \pmod 6$, we have $\gamma_{\rm pr}(G) \ge \frac{n}{3} + 2$. Let $k \ge 11$, and assume that if $5 \le k' < k$ where $k' \equiv 5 \pmod 6$, then $\gamma_{\rm pr}(L_{k'}) \ge \frac{2k'+2}{3} + 2$. Let $D$ be a $\gamma_{\rm pr}$-set of $G$. Since $D$ contains all support vertices of $G$, we have $\{v_1, v_k\} \subseteq D$.

We show that we can choose $D$ so that $v_2 \in D$ and the vertices $v_1$ and $v_2$ are paired in $D$. Suppose that the vertex $v_1$ is paired with $u_1$ in $D$. (The case when $v_1$ is paired with $v_0$ is analogous.) If $v_2 \notin D$, then we can simply replace $u_1$ in $D$ with the vertex $v_2$, and in the resulting set pair $v_1$ and $v_2$, as desired. Hence, we may assume that $v_2 \in D$. If $v_2$ is paired with $u_2$, then we can simply replace the pairs $v_1$ and $u_1$, and $v_2$ and $u_2$, with the new pairing $v_1$ and $v_2$, and $u_1$ and $u_2$, to yield the desired result. Hence, we may assume that $v_2$ is paired with $v_3$. If now $u_3 \in D$, then we contradict the minimality of the set $D$. Hence, $u_3 \notin D$. In this case, the set $D' = (D \setminus \{u_1\}) \cup \{u_3\}$ is a new $\gamma_{\rm pr}$-set of $G$, with $v_1$ and $v_2$ paired, and $u_3$ and $v_3$ paired in $D'$, once again yielding a $\gamma_{\rm pr}$-set of $G$ with the desired property that $v_1$ and $v_2$ are paired in the set. Hence, we may choose $D$ so that $v_2 \in D$ and the vertices $v_1$ and $v_2$ are paired in $D$.

With this choice of $D$, we note that $v_0 \notin D$. If $u_1 \in D$, then $u_2 \in D$ and $u_1$ and $u_2$ are partners in $D$. In this case, the vertex $u_1$ is only needed to partner the vertex $u_2$, and we can replace $u_1$ in $D$ with the vertex $u_3$. Hence, we may assume that $u_1 \notin D$. If $v_3 \in D$, then we can replace $v_3$ in $D$ with the vertex $u_4$. Hence, we may further assume that $v_3 \notin D$. If $u_2 \in D$, then $u_3 \in D$ with $u_2$ and $u_3$ paired, and we can replace $u_2$ in $D$ with the vertex $u_4$. Hence, we can choose the set $D$ so that $D \cap \{v_0, v_1, v_2, v_3, u_1, u_2\} = \{v_1, v_2\}$. Analogously, we can choose the set $D$ so that $D \cap \{v_{k-2}, v_{k-1}, v_k, v_{k+1}, u_{k-1}, u_k\} = \{v_{k-1}, v_k\}$.

Let the set $S$ be defined as in the proof of Claim 1, and consider the graph $G' = G - S$. We note that $G'$ is isomorphic to $L_{k'}$, where $k' = k - 6$. Thus, $5 \le k' < k$ and $k' \equiv 5 \pmod 6$. By our choice of the set $D$, we note that the set $D' = D \setminus \{v_1, v_2, v_{k-1}, v_k\}$ is a PD-set of $G'$, and so $\gamma_{\rm pr}(G') \le |D'| = |D| - 4$. Applying the inductive hypothesis to $G'$, we have $|D| - 4 \ge \gamma_{\rm pr}(G') \ge \frac{2k'+2}{3} + 2 = \frac{2k+2}{3} - 2 = \frac{n}{3} - 2$. Thus, $\gamma_{\rm pr}(G) = |D| \ge \frac{n}{3} + 2$, as desired. As proven earlier,

$\gamma_{\rm pr}(G) \le \frac{n}{3} + 2$. Consequently, $\gamma_{\rm pr}(G) = \frac{n}{3} + 2$. This completes the proof of Lemma 20. ■

We are now in a position to establish upper bounds on the $\gamma_{\rm pr}^+$-stability of a cycle prism $C_n \,\Box\, K_2$ when $n \ge 6$ and $n \ne 7$.

**Theorem 21.** *For $n \ge 6$ and $n \ne 7$,*

$$\mathrm{st}^+_{\gamma_{\rm pr}}(C_n \,\Box\, K_2) \le \begin{cases} 3 & \text{if } n \equiv 0 \pmod 6, \\ 4 & \text{if } n \pmod 6 \in \{2,3\}, \\ 5 & \text{if } n \pmod 6 \in \{4,5\}, \\ 6 & \text{if } n \equiv 1 \pmod 6. \end{cases}$$

***Proof.*** For $n \ge 6$ and $n \ne 7$, let $G = C_n \,\Box\, K_2$. Let $G_1$ and $G_2$ be the two layers of the prism $G$ used to build the graph $G$, where $G_1$ is the cycle $v_1v_2 \cdots v_nv_1$ and $G_2$ is the cycle $u_1u_2 \cdots u_nu_1$. Let $G$ be obtained from $G_1$ and $G_2$ by adding the edges $u_iv_i$ for $i \in [n]$.

Suppose that $n \equiv 0 \pmod 6$. In this case, $n = 6k$ for some $k \ge 1$, and $\gamma_{\rm pr}(G) = 4k$. We consider the non-isolating set $S = \{u_1, u_3, u_5\}$. Let $D$ be a $\gamma_{pr}$-set of $G - S$. Since $D$ contains all support vertices of $G - S$, we have $\{v_2, v_4\} \subset D$. We can clearly choose the partner of $v_2$ in $D$ as the vertex $v_1$, and the partner of $v_4$ in $D$ as the vertex $v_5$. Thus, $D \cap \{u_2, v_3, u_4\} = \emptyset$. If $k = 1$, then $v_6 \in D$ in order to dominate the vertex $u_6$, implying that $\gamma_{\rm pr}(G - S) = 6 > 4 = 4k = \gamma_{\rm pr}(G)$, and so $\mathrm{st}^+_{\gamma_{\rm pr}}(G) \le |S| = 3$. Hence, we may assume now that $k \ge 2$.

If $v_n \in D$, then we can replace $v_n$ in $D$ with the vertex $u_{n-1}$. If $v_6 \in D$, then we can replace $v_6$ in $D$ with the vertex $u_7$. Hence, we may further choose $D$ to contain neither $v_6$ nor $v_n$. Let $D_1 = \{v_1, v_2, v_4, v_5\}$ and let $G' = G - (S \cup N_G[D_1])$. We note that $G'$ is isomorphic to $L_{6k-7} \in \mathcal{L}$, and so $G'$ has order $n' = 12(k-1)$. By Lemma 20, $\gamma_{\rm pr}(G') = 2\left\lceil \frac{n'}{6} \right\rceil + 2 = 2\left\lceil \frac{12(k-1)}{6} \right\rceil + 2 = 4k - 2$. By our choice of the set $D$, the set $D' = D \setminus D_1$ is a PD-set of $G'$, and so $\gamma_{\rm pr}(G') \le |D'| = |D| - 4 = \gamma_{\rm pr}(G - S) - 4$. Thus, $\gamma_{\rm pr}(G - S) \ge \gamma_{\rm pr}(G') + 4 = 4k + 2 > 4k = \gamma_{\rm pr}(G)$. This implies that $\mathrm{st}^+_{\gamma_{\rm pr}}(G) \le |S| = 3$.

Suppose that $n \equiv 2 \pmod 6$. In this case, $n = 6k + 2$ for some $k \ge 1$, and $\gamma_{\rm pr}(G) = 4k + 2$. We consider the non-isolating set $S = \{u_1, u_3, u_5, u_7\}$. Let $D$ be a $\gamma_{\rm pr}$-set of $G - S$. Since $D$ contains all support vertices of $G - S$, we have $\{v_2, v_4, v_6\} \subset D$. We can choose $D$ so that the partner of $v_2$ in $D$ as the vertex $v_1$, the partner of $v_6$ in $D$ as the vertex $v_7$, and the partner of $v_4$ in $D$ as either the vertex $v_3$ or $v_5$, say $v_3$. If $k = 1$, then $v_8 \in D$ in order to dominate the vertex $u_8$, implying that $\gamma_{\rm pr}(G - S) = 8 > 6 = 4k + 2 = \gamma_{\rm pr}(G)$, and so $\mathrm{st}^+_{\gamma_{\rm pr}}(G) \le |S| = 4$. Hence, we may assume now that $k \ge 2$.

If $v_n \in D$, then we can replace $v_n$ in $D$ with the vertex $u_{n-1}$. If $v_8 \in D$, then we can replace $v_8$ in $D$ with the vertex $u_9$. Hence, we may further choose

$D$ to contain neither $v_8$ nor $v_n$. Let $D_1 = \{v_1, v_2, v_3, v_4, v_6, v_7\}$ and let $G' = G - (S \cup N_G[D_1])$. We note that $G'$ is isomorphic to $L_{6k-7} \in \mathcal{L}$, and so $G'$ has order $n' = 12(k-1)$. By Lemma 20, $\gamma_{\rm pr}(G') = 4k - 2$. By our choice of the set $D$, the set $D' = D \setminus D_1$ is a PD-set of $G'$, and so $\gamma_{\rm pr}(G') \le |D'| = |D| - 6 = \gamma_{\rm pr}(G - S) - 6$. Thus, $\gamma_{\rm pr}(G-S) \ge \gamma_{\rm pr}(G') + 6 = 4k + 4 > 4k + 2 = \gamma_{\rm pr}(G)$. This implies that $\mathrm{st}^+_{\gamma_{\rm pr}}(G) \le |S| = 4$.

Suppose that $n \equiv 3 \pmod 6$. The reasoning is analogous as in the case when $n \equiv 2 \pmod 6$.

Suppose that $n \equiv 4 \pmod 6$. In this case, $n = 6k + 4$ for some $k \ge 1$, and $\gamma_{\rm pr}(G) = 4k + 4$. We consider the non-isolating set $S = \{u_1, u_3, u_5, u_7, u_9\}$. Let $D$ be a $\gamma_{pr}$-set of $G - S$. Since $D$ contains all support vertices of $G - S$, we have $\{v_2, v_4, v_6, v_8\} \subset D$. Let $D_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9\}$. We can choose $D$ so that $D_1 \subset D$, where $v_1$ and $v_2$ are paired, $v_3$ and $v_4$ are paired, $v_5$ and $v_6$ are paired, and $v_8$ and $v_9$ are paired. If $k = 1$, then $v_{10} \in D$ in order to dominate the vertex $u_{10}$, implying that $\gamma_{\rm pr}(G - S) = 10 > 8 = 4k + 4 = \gamma_{\rm pr}(G)$, and so $\mathrm{st}^+_{\gamma_{\rm pr}}(G) \le |S| = 5$. Hence, we may assume now that $k \ge 2$. We can now further choose $D$ to contain neither $v_{10}$ nor $v_n$. Let $G' = G - (S \cup N_G[D_1])$. We note that $G'$ is isomorphic to $L_{6k-7} \in \mathcal{L}$, and so $G'$ has order $n' = 12(k-1)$. By Lemma 20, $\gamma_{\rm pr}(G') = 4k - 2$. By our choice of the set $D$, the set $D' = D \setminus D_1$ is a PD-set of $G'$, and so $\gamma_{\rm pr}(G') \le |D'| = |D| - 8 = \gamma_{\rm pr}(G - S) - 8$. Thus, $\gamma_{\rm pr}(G - S) \ge \gamma_{\rm pr}(G') + 8 = 4k + 6 > 4k + 4 = \gamma_{\rm pr}(G)$. This implies that $\mathrm{st}^+_{\gamma_{\rm pr}}(G) \le |S| = 5$.

Suppose that $n \equiv 5 \pmod 6$. The case is analogous to the case when $n \equiv 4 \pmod 6$.

Suppose that $n \equiv 1 \pmod 6$. In this case, $n = 6k + 1$ for some $k \ge 2$, and $\gamma_{\rm pr}(G) = 4k + 2$. We consider the non-isolating set $S = \{u_1, u_3, u_5, u_7, u_9, u_{11}\}$. Let $D$ be a $\gamma_{pr}$-set of $G - S$. Let $D_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_{10}, v_{11}\}$. We can choose $D$ so that $D_1 \subseteq D$, where $v_1$ and $v_2$ are paired, $v_3$ and $v_4$ are paired, $v_5$ and $v_6$ are paired, $v_7$ and $v_8$ are paired, and $v_{10}$ and $v_{11}$ are paired. If $k = 2$, then two additional vertices are needed in $D$, implying that $\gamma_{\rm pr}(G - S) \ge 12 > 10 = 4k + 2 = \gamma_{\rm pr}(G)$, and so $\mathrm{st}^+_{\gamma_{\rm pr}}(G) \le |S| = 6$. Hence, we may assume now that $k \ge 3$. We can now further choose $D$ to contain neither $v_{12}$ nor $v_n$. We let $G' = G - (S \cup N_G[D_1])$, and note that $G'$ is isomorphic to $L_{6(k-2)} \in \mathcal{L}$, and so $G'$ has order $n' = 12k - 22$. By Lemma 20, $\gamma_{\rm pr}(G') = 2\left\lceil \frac{n'}{6} \right\rceil = 2\left\lceil \frac{12k-22}{6} \right\rceil = 4k - 6$. By our choice of the set $D$, the set $D' = D \setminus D_1$ is a PD-set of $G'$, and so $\gamma_{\rm pr}(G') \le |D'| = |D| - 10 = \gamma_{\rm pr}(G - S) - 10$. Thus, $\gamma_{\rm pr}(G - S) \ge \gamma_{\rm pr}(G') + 10 = 4k + 4 > 4k + 2 = \gamma_{\rm pr}(G)$. This implies that $\mathrm{st}^+_{\gamma_{\rm pr}}(G) \le 6$. ■

As an immediate consequence of Theorems 19 and 21, we have the following result on the paired domination stability of a cycle prism.

**Corollary 22.** *For $n \geq 4$, $\mathrm{st}_{\gamma_{\mathrm{pr}}}(C_n \Box K_2) \leq 4$, with strict inequality if $n \pmod 6 \in \{0, 1, 4\}$.*

It remains an open problem to determine the exact value of the $\gamma^+_{\mathrm{pr}}$-stability of a cycle prism $C_n \Box K_2$ for $n \geq 6$ and $n \neq 7$.

## References


[1] M. Amraee, N. Jafari Rad and M. Maghasedi, *Roman domination stability in graphs*, Math. Rep. (Bucur.) **21(71)** (2019) 193–204.

[2] S. Arumugam and R. Kala, *Domination parameters of hypercubes*, J. Indian Math. Soc. (N.S.) **65** (1998) 31–38.

[3] J. Azarija, M.A. Henning and S. Klavžar, (*Total*) *domination in prisms*, Electron. J. Combin. **24(1)** (2017) #P1.19.
https://doi.org/10.37236/6288

[4] A. Aytaç and B. Atay Atakul, *Exponential domination critical and stability in some graphs*, Internat. J. Found. Comput. Sci. **30** (2019) 781–791.
https://doi.org/10.1142/S0129054119500217

[5] D. Bauer, F. Harary, J. Nieminen and C.L. Suffel, *Domination alternation sets in graphs*, Discrete Math. **47** (1983) 153–161.
https://doi.org/10.1016/0012-365X(83)90085-7

[6] W. Goddard and M.A. Henning, *A note on domination and total domination in prisms*, J. Comb. Optim. **35** (2018) 14–20.
https://doi.org/10.1007/s10878-017-0150-0

[7] A. Gorzkowska, M.A. Henning, M. Pilśniak and E. Tumidajewicz, *Paired domination stability in graphs,* (2020), manuscript.

[8] F. Harary and M. Livingston, *Independent domination in hypercubes*, Appl. Math. Lett. **6** (1993) 27–28.
https://doi.org/10.1016/0893-9659(93)90027-K

[9] T.W. Haynes, M.A. Henning and L.C. van der Merwe, *Domination and total domination in complementary prisms*, J. Comb. Optim. **18** (2009) 23–37.
https://doi.org/10.1007/s10878-007-9135-8

[10] T.W. Haynes and P.J. Slater, *Paired domination in graphs*, Networks **32** (1998) 199–206.
https://doi.org/10.1002/(SICI)1097-0037(199810)32:3¡199::AID-NET4¿3.0.CO;2-F

[11] M.A. Henning and M. Krzywkowski, *Total domination stability in graphs*, Discrete Appl. Math. **236** (2018) 246–255.
https://doi.org/10.1016/j.dam.2017.07.022

[12] M.A. Henning and D.F. Rall, *On the total domination number of Cartesian products of graphs*, Graphs Combin. **21** (2005) 63–69.
https://doi.org/10.1007/s00373-004-0586-8

[13] M.A. Henning and A. Yeo, Total Domination in Graphs (Springer Monographs in Mathematics, 2013).
https://doi.org/10.1007/978-1-4614-6525-6

[14] Z. Li, Z. Shao and S.J. Xu, 2-*rainbow domination stability of graphs*, J. Comb. Optim. **38** (2019) 836–845.
https://doi.org/10.1007/s10878-019-00414-0

[15] M. Mollard, *On perfect codes in Cartesian product of graphs*, European J. Combin. **32** (2011) 398–403.
https://doi.org/10.1016/j.ejc.2010.11.007

[16] C.M. Mynhardt and M. Schurch, *Paired domination in prisms of graphs*, Discuss. Math. Graph Theory **31** (2011) 5–23.
https://doi.org/10.7151/dmgt.1526

[17] P.R.J. Östergård and U. Blass, *On the size of optimal binary codes of length* 9 *and covering radius* 1, IEEE Trans. Inform. Theory **47** (2001) 2556–2557.
https://doi.org/10.1109/18.945268

[18] N. Jafari Rad, E. Sharifi and M. Krzywkowski, *Domination stability in graphs*, Discrete Math. **339** (2016) 1909–1914.
https://doi.org/10.1016/j.disc.2015.12.026

[19] G.J.M. van Wee, *Improved sphere bounds on the covering radius of codes*, IEEE Trans. Inform. Theory **34** (1988) 237–245.
https://doi.org/10.1109/18.2632

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