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A note on uniquely embeddable 2-factors

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A B S T R A C T

Let $C_{n_1} \cup C_{n_2} \cup \ldots \cup C_{n_k}$ be a 2-factor *i.e.* a vertex-disjoint union of cycles. In this note we completely characterize those 2-factors that are uniquely embeddable in their complement.

1. Introduction

We consider only finite, undirected graphs of order $n = |V(G)|$ and size $e(G) = |E(G)|$. All graphs will be assumed to have neither loops nor multiple edges.

We shall need some additional definitions in order to formulate the results. If a graph G has order n and size m, we say that G is an (n, m) graph.

Assume now that G_1 and G_2 are two graphs with disjoint vertex sets. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph is the union of n (≥ 2) disjoint copies of a graph H, then we write $G = nH$.

For our next operation, the conditions are quite different. Let now G_1 and G_2 be graphs with $V(G_1) = V(G_2)$ and $E(G_1) \cap E(G_2) = \emptyset$. The *edge sum* $G_1 \oplus G_2$ has $V(G_2 \oplus G_2)$ and $E(G_1 \oplus G_2)$.

An *embedding* of G (in its complement $\overline{G}$) is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$.

In other words, an embedding is an (edge-disjoint) *placement* (or *packing*) of two copies of G into a complete graph K_n.

The following theorem was proved, independently, in [1], [2] and [5].

Theorem 1. *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n - 2$ then G can be embedded in its complement $\overline{G}$.* ■

The example of the star $K_{1,n-1}$ shows that Theorem 1 cannot be improved by raising the size of G. The following theorem, proved in [3], gives the full characterization of graphs with order n and size $n - 1$ that are embeddable.

Theorem 2. *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n - 1$ then either G is embeddable or G is isomorphic to one of the following graphs: $K_{1,n-1}$, $K_{1,n-4} \cup K_3$ with $n \geq 8$, $K_1 \cup K_3$, $K_2 \cup K_3$, $K_1 \cup 2K_3$, $K_1 \cup C_4$.* ■

Let us consider now the problem of the uniqueness. First, we have to precise what we mean by *distinct* embeddings.

Let σ be an embedding of the graph $G = (V, E)$. We denote by $\sigma(G)$ the graph with the vertex set V and the edge set $\sigma^*(E)$ where the map σ^* is induced by σ. Since, by definition of an embedding, the sets E and $\sigma^*(E)$ are disjoint we may form the graph $G \oplus \sigma(G)$.

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Two embeddings σ_1, σ_2 of a graph G are said to be *distinct* if the graphs $G \oplus \sigma_1(G)$ and $G \oplus \sigma_2(G)$ are not isomorphic. A graph G is called *uniquely embeddable* if for all embeddings σ of G, all graphs $G \oplus \sigma(G)$ are isomorphic.

The problem of uniqueness has so far been the subject of two papers. The next theorem, proved in [6], characterizes all $(n, n-2)$ graphs that are uniquely embeddable.

Theorem 3. *Let G be a graph of order n and size $e(G) = n-2$. Then either G is not uniquely embeddable or G is isomorphic to one of the six following graphs: $K_2 \cup K_1$, $2K_2$, $K_3 \cup 2K_1$, $K_3 \cup K_2 \cup K_1$, $K_3 \cup 2K_2$, $2K_3 \cup 2K_1$.* ■

By *double star* $S(p,q)$ we mean a tree obtained from two stars S_{p+1} and S_{q+1} by joining their centers by an edge. By S'_n for $n = q+3$ we denote a double star $S(1,q)$. In [4] the following characterization of uniquely embeddable forests was proved.

Theorem 4. *Let F be a forest of order n having at least one edge. Then either F is not uniquely embeddable or F is isomorphic to one of the following graphs: $K_2 \cup K_1$, $2K_2$, $3K_2$, $S(p,q)$ or S'_n.* ■

Remark. The main references of the paper and of other packing problems are the following survey papers [9], [7] or [8].

The aim of this note is to consider the problem of uniqueness of embedding for 2-factors.

Theorem 5. *Let G be a union of vertex-disjoint cycles. Then G is uniquely embeddable if and only if G is one of C_5, C_6, $C_3 \cup C_4$, $C_3 \cup C_5$, $3C_3$ and $4C_3$.*

The proof of Theorem 5 is given in the next sections. Section 2 contains the case of cycles ($k = 1$), while Section 3 and Section 4 deal with union of two or three cycles and union of k cycles where $k \geq 4$ respectively.

Remark. We will often present packing of a graph G in figures. Therefore we need to introduce additional notation. We say that the first (initial) copy of a graph G is *black* and the second copy of G is *red* in the packing of G. In figures we draw with a continuous black line the first copy of G and we draw with a dashed red line the second copy of G.

We denote by $B(X,Y)$ graph which contains all edges between two disjoint sets X and Y. We will often use this notation for cycles C_4 in packings which contain these cycles. The following lemma will be useful through the proof.

Lemma 6. *If a graph $G = C_{n_1} \cup C_{n_2} \cup \ldots \cup C_{n_k}$ has a packing σ such that the graph $G \oplus \sigma(G)$ is not connected (a disconnected packing), then G has another packing $\hat{\sigma}$ such that the graph $G \oplus \hat{\sigma}(G)$ is connected (a connected packing). In particular, the graph G is not uniquely packable.*

Proof. Let's choose the packing σ with the least number of connected components. If $H = G \oplus \sigma(G)$ is connected, we're done. If not, let H_1, H_2 be two components of H.

Suppose y_1 is a vertex of H_1 such that removing the two red edges $y_1^- y_1$ and $y_1^+ y_1$ leaves H_1 connected where y_1^- and y_1^+ are neighbors of y_1 on the red cycle in H_1. In the same way, we select y_2, a vertex belonging to the component H_2.

If now instead of the edges $y_1^- y_1$ and $y_1^+ y_1$ we draw two red edges $y_2^- y_1$ and $y_2^+ y_1$, and instead of the edges $y_2^- y_2$ and $y_2^+ y_2$ we draw two red edges $y_1^- y_2$ and $y_1^+ y_2$, we get a new packing $\hat{\sigma}$ where two components H_1 and H_2 become one connected component, contradiction with the choice of packing σ.

To complete the proof, it suffices to show that for each connected component of the graph H one can choose a vertex as we did above.

Just take a vertex that is not cut vertex. Such a vertex exists in every connected component. For example, the last vertex on the longest component path. ■

2. Case $k = 1$

Let $G = C_n$ be a cycle of order n. It is easy to see that neither C_3 nor C_4 is embeddable.

The cycle C_5 is embeddable but for each embedding σ we have $C_5 \oplus \sigma(C_5) = K_5$. So, C_5 is uniquely embeddable.

The cycle C_6 is also embeddable. For each embedding σ the graph $C_6 \oplus \sigma(C_6)$ is a 4-regular subgraph of K_6. The complement of such a graph is a 1-factor in K_6. Thus, all these graphs are isomorphic. So, C_6 is uniquely embeddable.

Two distinct embeddings of C_7 are given in Fig. 1. In the first one, the complement of the graph $C_n \oplus \sigma(C_n)$ is isomorphic to C_7 while in the second one, to $C_3 \cup C_4$.

For $n \geq 8$ we shall show that there are at least two distinct embeddings of C_n:

A) One such that the graph $C_n \oplus \sigma(C_n)$ contains a clique K_4 and

B) another such that the graph $C_n \oplus \sigma(C_n)$ is K_4-free.

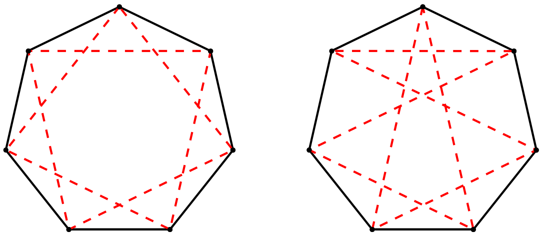

Fig. 1. Two distinct embeddings of C_7.

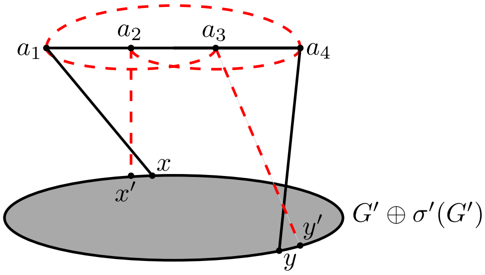

Fig. 2. If G' is embeddable, then $G \oplus \sigma(G)$ contains K_4.

Case A.

Denote by x, a_1, a_2, a_3, a_4, y six consecutive vertices of $G = C_n$ and by P the path joining x and y obtained from C_n by removing vertices $\{a_1, a_2, a_3, a_4\}$ *i.e.* $P = G' = G \setminus \{a_1, a_2, a_3, a_4\}$. Since $n \geq 8$ P, has at least four vertices. By Theorem 2 there is a permutation, say σ' being an embedding of P. Let $x' = \sigma'(x)$ and $y' = \sigma'(y)$. Fig. 2 shows how to extend σ' to get an embedding of C_n. Let us observe that the vertices $\{a_1, a_2, a_3, a_4\}$ induce a clique K_4.

We will use the above reasoning often, so it will be convenient to formulate it in the form of a lemma.

Lemma 7. *Denote by a_1, a_2, a_3, a_4 four vertices of G inducing a path in G. If the graph obtained from G by removing vertices $\{a_1, a_2, a_3, a_4\}$ is packable, then the graph G is also packable and there is a packing σ of G such that $G \oplus \sigma(G)$ contains a clique K_4.*

Case B. Denote by $v_1, v_2, v_3, \dots, v_n$ consecutive vertices of C_n. We shall consider two cases.

Subcase B1. n is odd.

Then, the edges $v_i v_{i+2} \pmod n$ define a cycle of length n. This cycle can be considered as an image of C_n by a permutation, say σ. We shall show that the graph $H = C_n \oplus \sigma(C_n)$ is K_4-free. Suppose that H contains a clique on four vertices. It has six edges and it is easy to see that three of them should belong to the first copy of C_n and the remaining three to the second copy of C_n, each of these triples forming a path of length three in the corresponding copy. But a path of length three in C_n should be induced by four consecutive vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3} \pmod n$. The fact that v_i, v_{i+3} is not an edge of the second (dashed) copy of C_n finishes the proof of this case.

Subcase B2. n is even.

It is easy to see that the edges of the form $v_i v_{i+r} \pmod n$ define a cycle of length n if r and n are coprime. In order to prove the existence of such an integer r we can use, for instance, the well-known Chebyshev's theorem saying that for each integer $k \geq 4$ there is a prime number between k and $2k - 2$. Denote by p such a number where $k = \frac{n}{2}$. Since a prime number p and n are surely coprime, r and n where $r = n - p$ are also coprime. Moreover, we have $3 \leq r \leq \frac{n}{2} - 1$. Similarly as above, it is easy to see that the graph formed by C_n and the edges of the form $v_i v_{i+r} \pmod n$ is K_4-free.

3. Case $k = 2$ or $k = 3$

3.1. Case $k=2$

Let $G = C_{n_1} \cup C_{n_2}$, where $n_1 \leq n_2$. If $n_1 \geq 5$ we have unconnected packing of G which consists of two components. Each of these components we obtain as a packing of two copies of a cycle in appropriate complete graph from the Case $k = 1$. Thus from Lemma 6 the graph G is not uniquely packable. Therefore we have to consider two subcases $G = C_3 \cup C_p$ where $p \geq 3$ and $G = C_4 \cup C_p$ where $p \geq 4$.

3.1.1. Subcase $G = C_3 \cup C_p$ where $p \geq 3$

It is easy to see that $C_3 \cup C_3$ is not embeddable. We show that $C_3 \cup C_4$ and $C_3 \cup C_5$ are uniquely embeddable. We start packing of $\mathbf{C_3 \cup C_4}$ from the black copy in which we denote by u_1, u_2, u_3 consecutive vertices of C_3 and by y_1, y_2, x_2, x_1 consecutive vertices

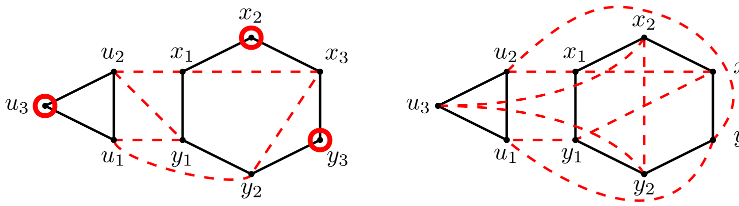

Fig. 3. Two distinct packings of $C_3 \cup C_6$. For the clarity of the drawing, the outer triangular face (connecting vertices marked with a circle) is not drawn.

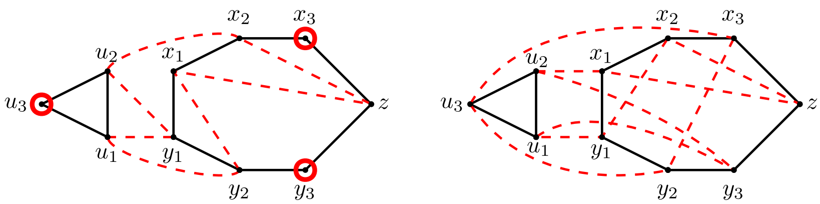

Fig. 4. Two distinct packings of $C_3 \cup C_7$. For the clarity of the drawing, the outer triangular face (connecting vertices marked with a circle) is not drawn.

of C_4. To draw a red triangle, we need to use one of the vertices of the black triangle and two opposite vertices from the black cycle C_4. Without loss of generality we can use vertices u_1, x_1, y_2, since all possibilities give isomorphic graphs. Now, we have only one possibility to draw the remaining red cycle C_4.

We start packing of $\mathbf{C_3 \cup C_5}$ from the black copy in which we denote by u_1, u_2, u_3 consecutive vertices of C_3 and by y_1, y_2, z, x_2, x_1 consecutive vertices of C_5. To draw a red triangle, we need to use one of the vertices of the black triangle and two opposite vertices from the black cycle C_5. Without loss of generality we can use vertices u_3, x_2, y_2. We start drawing remaining red C_5 from the vertex z. We can easily see that we cannot have edges zu_1 and zu_2 or zx_1 and zx_2 in this red C_5. Therefore we have to draw first red edge from z to the vertex on black C_5 and the second red edge to the vertex on black C_3. Thus we have only one possibility to draw the remaining red cycle C_5. Therefore the packing of $C_3 \cup C_5$ is unique.

Now we present plane and not planar packing of $\mathbf{C_3 \cup C_6}$ which could be extended to packings of $C_3 \cup C_p$, where p is from the set $\{8, 10, 12, \dots\}$. Two distinct packings of $C_3 \cup C_6$ are presented in Fig. 3. We can easily see that the first (left) packing of $C_3 \cup C_6$ is plane. We extend this packing by replacing the edge y_2y_3 by the path $y_2a_1a_2\dots a_ly_3$ and the edge x_3y_3 by the path $x_3b_1b_2\dots b_ly_3$, where $l = \frac{p-6}{2}$. Then we add a path $y_2b_1a_1b_2a_2\dots b_la_l$ and replace the edge u_1y_2 by the edge u_1a_l. Note that the packing which we obtain is plane. Now we prove that the second (right) packing of $C_3 \cup C_6$ is not planar. Vertices u_1, u_2 and u_3 induce a triangle. If we add to this triangle the vertex x_1 together with paths x_1u_2, $x_1y_1u_1$ and $x_1x_2u_3$ we obtain a subgraph homeomorphic to K_4. Thus if we add to this subgraph the vertex y_3 together with subsequent paths y_3u_1, y_3u_2, $y_3y_2u_3$ and $y_3x_3x_1$ we obtain a subgraph which is a subdivision of K_5. It follows from Kuratowski's theorem that the graph is not planar. We can extend this packing by replacing the edge y_1y_2 by the path $y_1a_1a_2\dots a_ly_2$ and the edge x_3y_3 by the path $x_3b_1b_2\dots b_ly_3$, where $l = \frac{p-6}{2}$. Then we add a path $y_1b_1a_1b_2a_2\dots b_la_l$ and replace the edge u_1y_1 by the edge u_1a_l. Note also that the packing which we obtain is not planar.

We show plane and not planar packing of $\mathbf{C_3 \cup C_7}$ which could be extended to packings of $C_3 \cup C_p$, where p is from the set $\{9, 11, 13, \dots\}$. Two distinct packings of $C_3 \cup C_7$ are presented in Fig. 4. We can easily see that the first (left) packing of $C_3 \cup C_7$ is plane. We extend this packing by replacing the edge x_1x_2 by the path $x_1a_1a_2\dots a_lx_2$ and the edge zx_3 by the path $zb_1b_2\dots b_lx_3$, where $l = \frac{p-7}{2}$. Then we replace the edge zx_2 by the edge za_1 and we add a path $a_1b_1a_2b_2\dots a_lb_lx_2$. Note that the packing which we obtain is plane. Now we prove that the second (right) packing of $C_3 \cup C_7$ is not planar. Vertices u_1, u_2 and u_3 induce a triangle. If we add to this triangle the vertex x_1 together with paths x_1u_2, $x_1y_1u_1$ and $x_1x_2x_3u_3$ we obtain a subgraph homeomorphic to K_4. Thus if we add to this subgraph the vertex y_3 together with subsequent paths y_3u_1, y_3u_2, $y_3y_2u_3$ and y_3zx_1 we obtain a subgraph which is a subdivision of K_5. It follows from Kuratowski's theorem that the graph is not planar. We can extend this packing by replacing the edge y_3z by the path $y_3a_1a_2\dots a_lz$ and the edge x_2x_3 by the path $x_2b_1b_2\dots b_lx_3$, where $l = \frac{p-7}{2}$. Then we replace the edge x_1z by the edge x_1a_1 and we add a path $a_1b_1a_2b_2\dots a_lb_lz$. Note also that the packing which we obtain is not planar.

3.1.2. Subcase $G = C_4 \cup C_p$ where $p \geq 4$

For a graph $G = \mathbf{C_4 \cup C_4}$ we show bipartite and not bipartite packing. In the graph G we denote by $x_1, \dots, x_4$ and $y_1, \dots, y_4$ vertices from two sets X and Y. In both packings we draw black cycles $B(\{x_1, x_2\}, \{y_1, y_2\})$ and $B(\{x_3, x_4\}, \{y_3, y_4\})$. In the first packing of G we draw red cycles $B(\{x_1, x_2\}, \{y_3, y_4\})$ and $B(\{x_3, x_4\}, \{y_1, y_2\})$. Note that X and Y are independent. In the second packing we draw red cycles $B(\{x_2, y_2\}, \{x_3, y_3\})$ and $B(\{x_1, y_1\}, \{x_4, y_4\})$. Note that each red cycle with two independent black edges induces a subgraph K_4. Therefore this packing of G is not bipartite.

We present plane and not planar packing of $\mathbf{C_4 \cup C_5}$. These two packings could be extended to packings of $C_4 \cup C_p$, where p is from the set $\{9, 11, 13 \dots\}$. Two distinct packings of $C_4 \cup C_5$ are presented below in Fig. 5. It is obvious that the first (left) packing of $C_4 \cup C_5$ is plane. We extend this packing by replacing the edge zy_2 by the path $za_1a_2\dots a_ly_2$ and the edge zx_2 by the path $zb_1b_2\dots b_lx_2$, where $l = \frac{p-5}{2}$ and $l > 1$. Then we replace the edge u_1z by the edge u_1a_1 and we add a path $a_1b_1a_2b_2\dots a_lb_lz$. Note

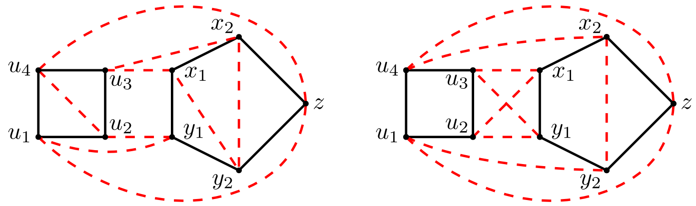

Fig. 5. Two distinct packings of $C_4 \cup C_5$.

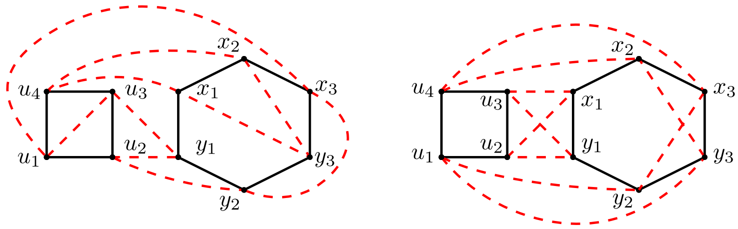

Fig. 6. Two distinct packings of $C_4 \cup C_6$.

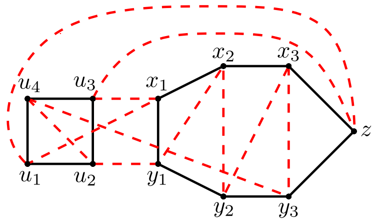

Fig. 7. The packing of $C_4 \cup C_7$ without K_4.

that the packing which we obtain is plane. Note also that the presented extension of plane packing does not work for $C_4 \cup C_7$. This case will be considered later.

Now we prove that the second (right) packing of $C_4 \cup C_5$ is not planar and contains a subgraph K_4. Vertices x_1, y_1, u_2 and u_3 induce K_4. If we add to this subgraph the vertex x_2 together with paths x_2x_1, $x_2y_2y_1$, $x_2u_4u_3$ and $x_2zu_1u_2$ we obtain a subgraph which is a subdivision of K_5. It follows from Kuratowski's theorem that the graph is not planar. We can extend this packing by replacing the edge y_1y_2 by the path $y_1a_1a_2\dots a_ly_2$ and the edge x_2z by the path $x_2b_1b_2\dots b_lz$, where $l=\frac{p-5}{2}$. Then we replace the edge x_2y_2 by the edge x_2a_1 and we add a path $a_1b_1a_2b_2\dots a_lb_ly_2$. Note also that the packing which we obtain is not planar.

We show plane and not planar packing of $\mathbf{C_4 \cup C_6}$ which could be extended to packings of $C_4 \cup C_p$, where p is from the set $\{8,10,12,\dots\}$. Two distinct packings of $C_4 \cup C_6$ are presented in Fig. 6. It is obvious that the first (left) packing of $C_4 \cup C_6$ is plane. We extend this packing by replacing the edge y_1x_1 by the path $y_1a_1a_2\dots a_lx_1$ and the edge y_2y_3 by the path $y_2b_1b_2\dots b_ly_3$, where $l=\frac{p-6}{2}$. Then we replace the edge y_2x_3 by the edge b_lx_3 and we add a path $y_2a_1b_1a_2b_2\dots a_lb_lx_3$. Note that the packing which we obtain is plane.

Now we prove that the second (right) packing of $C_4 \cup C_6$ is not planar. Vertices x_1, y_1, u_2 and u_3 induce K_4. If we add to this subgraph the vertex y_3 together with paths $y_3x_2x_1$, $y_3y_2y_1$, $y_3u_1u_2$ and $y_3x_3u_4u_3$ we obtain a subgraph which is a subdivision of K_5. It follows from Kuratowski's theorem that the graph is not planar. We can extend this packing by replacing the edge y_1y_2 by the path $y_1a_1a_2\dots a_ly_2$ and the edge x_1x_2 by the path $x_1b_1b_2\dots b_lx_2$, where $l=\frac{p-6}{2}$. Then we replace the edge u_1y_2 by the edge u_1a_1 and we add a path $a_1b_1a_2b_2\dots a_lb_ly_2$. Note also that the packing which we obtain is not planar.

Now, we show the remaining two distinct packings of $G=\mathbf{C_4 \cup C_7}$, the first with a subgraph K_4 and the second without K_4. The packing of G with a subgraph K_4 we obtain from Lemma 7. The packing of G without K_4 is presented in Fig. 7. Note that a subgraph K_4 in a packing of G could be obtained as a cycle C_4 from one copy of G and two independent edges from the other copy or as three consecutive edges of a cycle C_7 from the first copy of G and three consecutive edges of a cycle C_7 from the second copy. Therefore it suffices that we check vertices on cycles in black copy of G whether they induce K_4. We left this easy check to the reader.

3.2. Case $k=3$

Let $G=C_{n_1}\cup C_{n_2}\cup C_{n_3}$, where $n_1\le n_2\le n_3$. We can divide G into two subgraphs $G_1=C_{n_1}\cup C_{n_2}$, $G_2=C_{n_3}$ and from the previous cases ($k=2$ and $k=1$) we get a packing of G_1 and G_2 except for $G=C_3\cup C_4\cup C_4$, $G=C_4\cup C_4\cup C_4$ and $G=C_3\cup C_3\cup C_p$ where $p\ge 3$. Thus from Lemma 6 the graph G is not uniquely packable. Below we consider each of these exceptional graphs separately.

Two distinct packings of $\mathbf{C_3 \cup C_4 \cup C_4}$ are presented in Fig. 8. We can easily see that the first (left) packing of $C_3\cup C_4\cup C_4$

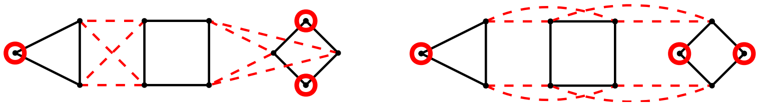

Fig. 8. Two distinct packings of $C_3 \cup C_4 \cup C_4$. For the clarity of the drawing, the cycles C_3 (connecting vertices marked with a circle) are not drawn.

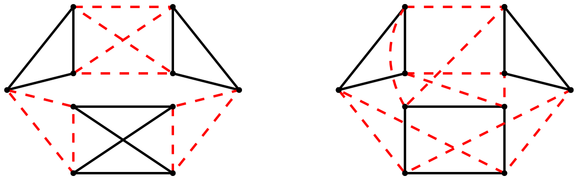

Fig. 9. Two distinct packings of $C_3 \cup C_3 \cup C_4$.

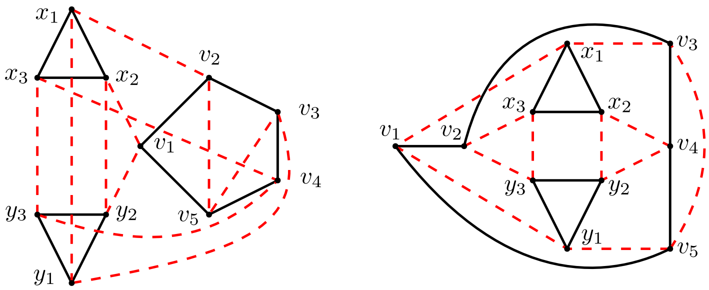

Fig. 10. Two distinct packings of $C_3 \cup C_3 \cup C_5$.

contains a subgraph K_4. The second (right) packing of $G = C_3 \cup C_4 \cup C_4$ is without a subgraph K_4. Note that the subgraph K_4 in a packing of G could be obtained as a cycle C_4 from one copy of G and two independent edges from the other copy. Therefore it suffices that we check cycles C_4 form both copies of G.

For a graph $G = \mathbf{C_4} \cup \mathbf{C_4} \cup \mathbf{C_4}$ we show bipartite and not bipartite packing. In the graph G we denote by $x_1, \ldots, x_6$ and $y_1, \ldots, y_6$ vertices from two sets X and Y. In both packings of G we draw black cycles $B(\{x_1, x_2\}, \{y_1, y_2\})$, $B(\{x_3, x_4\}, \{y_3, y_4\})$ and $B(\{x_5, x_6\}, \{y_5, y_6\})$. In the first packing of G we draw red cycles $B(\{x_1, x_2\}, \{y_3, y_4\})$, $B(\{x_3, x_4\}, \{y_5, y_6\})$ and $B(\{x_5, x_6\}, \{y_1, y_2\})$. Note that X and Y are independent. In the second packing we draw red cycles $B(\{x_2, y_2\}, \{x_3, y_3\})$, $B(\{x_4, y_4\}, \{x_5, y_5\})$ and $B(\{x_1, y_1\}, \{x_6, y_6\})$. Note that each red cycle with two independent black edges induces a subgraph K_4. Therefore this packing of G is not bipartite.

Now we consider $G = \mathbf{C_3} \cup \mathbf{C_3} \cup \mathbf{C_p}$ where $p \geq 3$. If $p = 3$ we start packing of G from the black copy in which we denote by x_1, x_2, x_3, y_1, y_2, y_3 and v_1, v_2, v_3 vertices of three cycles C_3. Then we draw red triangles $x_1 y_1 v_1$, $x_2 y_2 v_2$ and $x_3 y_3 v_3$. We can easily see that this packing is unique.

Two distinct packings of $G = C_3 \cup C_3 \cup C_4$ are presented in Fig. 9. We can easily see that the first (left) packing of G contains two subgraphs K_4. The second (right) packing of G is without a subgraph K_4. Note that a subgraph K_4 in a packing of G could be obtained as a cycle C_4 from one copy of G and two independent edges from the other copy. Therefore it suffices that we check cycles C_4 form both copies of G.

Two distinct packings of $G = C_3 \cup C_3 \cup C_5$ are presented in Fig. 10. In the first (left) packing of G there is a vertex v_5 such that its neighborhood induces a path of length three, while (as is relatively easy to check) the second (right) packing of G does not contain such a vertex.

Let $G = C_3 \cup C_3 \cup C_6$. It is easy to see that we can pack two black triangles with a red cycle C_6 and two red triangles with a black cycle C_6. Thus we have disconnected packing of G. Then from Lemma 6 we get connected packing of G.

We present two distinct packings of $C_3 \cup C_3 \cup C_p$ where $p \geq 7$, the first with a subgraph K_4 and the second without K_4. We start from the first packing of G. We denote by a_1, a_2, a_3, a_4 four consecutive vertices of a cycle C_p from G. Let $G' = G \setminus \{a_1, a_2, a_3, a_4\}$. We can easily see that $e(G') \leq |V(G')| - 1$. By Theorem 2 there is a packing of G'. Thus from Lemma 7 we get a packing of G with a subgraph K_4.

The second packing of $C_3 \cup C_3 \cup C_7$ and $C_3 \cup C_3 \cup C_8$ without K_4 is presented in Fig. 11. Note that a subgraph K_4 in a packing of $G = C_3 \cup C_3 \cup C_p$, where $p \in \{7, 8\}$, could be obtained as three consecutive edges of a cycle C_p from the black copy of G and three consecutive edges of a cycle C_p from the red copy. Note that then red edges are of length two and three with respect to the distance on black cycle. Therefore it suffices that we check vertices on a cycle C_p in one copy of G whether they induce K_4. We left this easy check to the reader.

We can extend the packing of $C_3 \cup C_3 \cup C_7$ to packings of $C_3 \cup C_3 \cup C_p$, where p is from the set $\{9, 11, 13, \ldots\}$. We replace the edge $v_3 v_2$ by the path $v_3 a_1 a_2 \ldots a_l v_2$ and the edge $v_5 v_6$ by the path $v_5 b_1 b_2 \ldots b_l v_6$, where $l = \frac{p-7}{2}$. Then we replace the edge

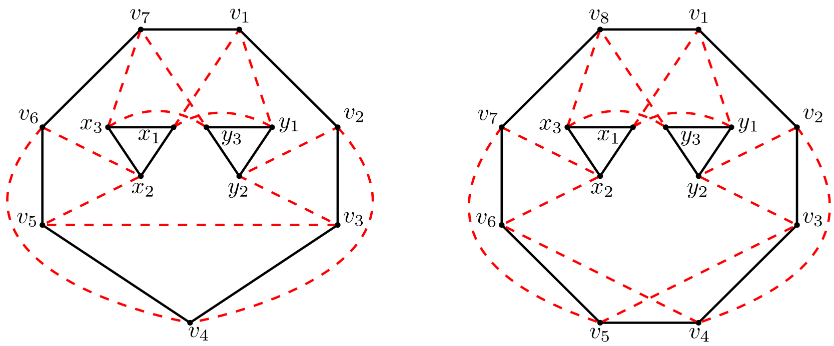

Fig. 11. The packing of $C_3 \cup C_3 \cup C_7$ and $C_3 \cup C_3 \cup C_8$ without K_4.

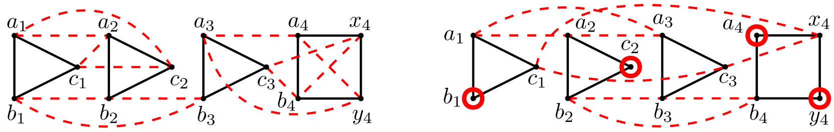

Fig. 12. Two distinct packings of $C_3 \cup C_3 \cup C_3 \cup C_4$. For the clarity of the drawing, the cycle C_4 (connecting vertices marked with a circle) is not drawn.

v_4v_2 by the edge v_4a_1 and we add a path $a_1b_1a_2b_2 \dots a_lb_lv_2$. Similarly we can extend the packing of $C_3 \cup C_3 \cup C_8$ to packings of $C_3 \cup C_3 \cup C_p$, where p is from the set $\{10, 12, 14, \dots\}$. We replace the edge v_3v_2 by the path $v_3a_1a_2 \dots a_lv_2$ and the edge v_6v_7 by the path $v_6b_1b_2 \dots b_lv_7$, where $l = \frac{p-8}{2}$. Then we replace the edge v_4v_2 by the edge v_4a_1 and we add a path $a_1b_1a_2b_2 \dots a_lb_lv_2$. Both presented extensions of $C_3 \cup C_3 \cup C_7$ and $C_3 \cup C_3 \cup C_8$ do not contain a clique K_4, because we added red edges between vertices with distance more than three with respect to black cycle.

4. Case $k \geq 4$

4.1. Case $k = 4$

Let $G = C_{n_1} \cup C_{n_2} \cup C_{n_3} \cup C_{n_4}$, where $n_1 \leq n_2 \leq n_3 \leq n_4$. If at least two n_i where $i \in \{1, 2, 3, 4\}$ are different from three then we can divide G into two parts $G = G_1 \cup G_2$ so that G_1 and G_2 have packing. Therefore from Lemma 6 the graph G is not uniquely packable. Similarly when $n_4 \geq 5$. Thus we have to consider two subcases $G = C_3 \cup C_3 \cup C_3 \cup C_3$ and $G = C_3 \cup C_3 \cup C_3 \cup C_4$.

We start packing of $G = \mathbf{C_3} \cup \mathbf{C_3} \cup \mathbf{C_3} \cup \mathbf{C_3}$ from the black copy in which we denote vertices creating triangle T_i by $\{a_i, b_i, c_i\}$ for $i \in \{1, 2, 3, 4\}$. Then we draw four red triangles with sets of vertices: $\{a_1, a_2, a_3\}$, $\{b_2, b_3, b_4\}$, $\{c_1, c_3, c_4\}$ and $\{b_1, c_2, a_4\}$. Now we show that this packing of G is unique (up to isomorphism). Note that all triangles in a packing of G are "real" i.e. have all edges form black or red copy of G. We claim that each three black triangles include exactly one red triangle. We take three arbitrary black triangles T_1, T_2 and T_3. First, suppose that these black triangles do not include any red triangle. Thus each red triangle has at least one vertex outside $T_1 \cup T_2 \cup T_3$, namely in T_4. We get a contradiction. Second, suppose that these black triangles include exactly two triangles. Then the remaining two red triangles can use at most two vertices from T_4. We also get a contradiction. Therefore from the fact that three black triangles do not include three red triangles we get a confirmation of our claim. Thus without loss of generality we can assume that the second red triangle includes vertices from black triangles T_2, T_3 and T_4. This implies that the packing of remaining two red triangles is determined. Therefore the packing of G is unique up to isomorphism.

Two distinct packings of $G = \mathbf{C_3} \cup \mathbf{C_3} \cup \mathbf{C_3} \cup \mathbf{C_4}$ are presented in Fig. 12. We can easily see that the first (left) packing of G is connected but the vertex b_3 is a cut vertex. Therefore it is not 2-connected. In the second (right) packing of G each vertex from black triangle has two red edges to different black cycles. Moreover each two vertices from black triangle have edges to three remaining cycles. Therefore removing one vertex does not disconnect the graph.

4.2. Case $k = 5$

Let $G = C_{n_1} \cup C_{n_2} \cup C_{n_3} \cup C_{n_4} \cup C_{n_5}$, where $n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5$. If $n_5 \geq 4$ we can divide G into two parts $G = G_1 \cup G_2$ so that $G_1 = C_{n_1} \cup C_{n_2} \cup C_{n_3}$ and $G_2 = C_{n_4} \cup C_{n_5}$ have packing. Therefore from Lemma 6 we know that the graph G is not uniquely packable. Thus we have to consider $G = 5C_3$.

We present two distinct packings of $G = 5C_3$. We start both packings of G from black copy in which we denote vertices creating triangle T_i by $\{a_i, b_i, c_i\}$ for $i \in \{1, \dots, 5\}$. Then in the first packing of G we draw five red triangles with sets of vertices: $\{a_1, a_2, a_3\}$, $\{b_1, a_4, a_5\}$, $\{c_1, c_2, b_5\}$, $\{b_2, b_3, b_4\}$ and $\{c_3, c_4, c_5\}$. In the second packing of G we draw five red triangles with sets of vertices: $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, $\{c_1, b_4, b_5\}$, $\{c_2, a_4, a_5\}$ and $\{c_3, c_4, c_5\}$. Note that all triangles in both packings of G are "real". We can easily see that in the first packing of G each nine vertices induce at most four triangles. In the second packing of G nine vertices from black triangles T_1, T_2 and T_3 induce also two red triangles. Thus there exists nine vertices which induce five triangles in the second packing of G.

4.3. Case $k \geq 6$

Let $G = C_{n_1} \cup C_{n_2} \cup \ldots \cup C_{n_k}$, where $n_1 \leq n_2 \leq \ldots \leq n_k$. We can divide G into two parts $G = G_1 \cup G_2$ so that $G_1 = C_{n_1} \cup C_{n_2} \cup C_{n_3}$ and $G_2 = C_{n_4} \cup C_{n_5} \cup \ldots \cup C_{n_k}$. From the previous cases and the fact that $k \geq 6$ we have packings of G_1 and G_2. Thus G has a disconnected packing. Therefore from Lemma 6 we get connected packing and we know that the graph G is not uniquely packable.

Data availability

No data was used for the research described in the article.

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