Available online at www.sciencedirect.com

ScienceDirect

Discrete Mathematics 307 (2007) 971 – 974

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Packing of two digraphs into a transitive tournament

Monika Pilśniak

Faculty of Applied Mathematics AGH, Department of Discrete Mathematics, al.Mickiewicza 30, 30-059 Kraków, Poland

Received 22 September 2003; received in revised form 16 June 2004; accepted 22 November 2005
Available online 22 September 2006

Abstract

Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two oriented graphs of order n without directed cycles. Görlich, Pilśniak and Woźniak proved [A note on a packing problem in transitive tournaments, preprint Faculty of Applied Mathematics, AGH University of Science and Technology, No. 37/2002] that if the number of arcs in $\overrightarrow{G}$ is sufficiently small (not greater than $3(n-1)/4$) then two copies of $\overrightarrow{G}$ are packable into the transitive tournament TT_n. This bound is best possible.

In this paper we give a generalization of this result. We show that if the sum of sizes of $\overrightarrow{G}$ and $\overrightarrow{H}$ is not greater than $\frac{3}{2}(n-1)$ then the digraphs $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_n.

MSC: 05C70; 05C35

Keywords: Packing of digraphs; Transitive tournaments

1. Introduction

Let $\overrightarrow{G}$ be a digraph of order n with the vertex set $V(\overrightarrow{G})$ and the arc set $E(\overrightarrow{G})$. A digraph without directed cycles of the length of two is called an *oriented graph.* The replacement of every arc uv in an oriented graph $\overrightarrow{G}$ by an edge uv yields its *underlying graph*.

Let $\overrightarrow{G}$ be an oriented graph. For any vertex $v \in V(\overrightarrow{G})$ let us denote by $d^+(v)$ the *outdegree* of v, i.e. the number of vertices of $\overrightarrow{G}$ that are adjacent from v. By $d^-(v)$ we denote the *indegree* of v, i.e. the number of vertices adjacent to v. The *degree* of a vertex v, denoted by $d(v)$, is the sum $d(v)=d^-(v)+d^+(v)$. A vertex x such that $d(x)=d^+(x)$ is called *a source* and a vertex y such that $d(y)=d^-(y)$ is called *a sink.*

A digraph $\overrightarrow{G}$ is called *transitive* when it satisfies the condition of transitivity: if uv and vw are two arcs of $\overrightarrow{G}$ then uw is an arc, too.

A *tournament* is an oriented graph such that its underlying graph is complete. A transitive tournament of order n will be denoted by TT_n. As it is unique up to isomorphism, throughout the paper, we will view TT_n as shown in Fig. 1. We can denote the vertices in TT_n by consecutive integers in such way that if $i<j$ then ij is an arc of TT_n. The vertices 1, 2 and n will be called the *first*, the *second* and the *last vertex* of TT_n, respectively.

A *semipath* between two distinct vertices v_1 and v_k in an oriented graph $\overrightarrow{G}$ is a path between v_1 and v_k in the underlying graph G. The semipath is denoted by $v_1 \dots v_k$, where $v_1, \dots, v_k$ are vertices and $v_{i-1}v_i \in E(\overrightarrow{G})$ or $v_iv_{i-1} \in E(\overrightarrow{G})$ for $i \in \{2, \dots, k\}$.

E-mail address: pilsniak@agh.edu.pl.

0012-365X/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2005.11.038

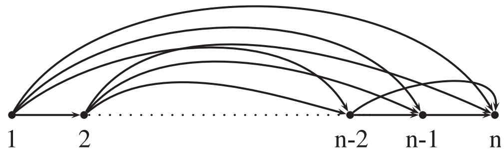

Fig. 1. Transitive tournament TT_n.

Let $\overrightarrow{G}$ be an oriented graph of order n. An *embedding of* $\overrightarrow{G}$ *into* TT_n is a couple (σ, σ') in which σ is a bijection $V(\overrightarrow{G}) \to \{1, \ldots, n\} = V(TT_n)$ and σ' is the injection $E(\overrightarrow{G}) \to E(TT_n)$ induced by σ (i.e. for any arc $ij \in E(\overrightarrow{G})$, $\sigma'(ij) = \sigma(i)\sigma(j)$). We will speak more simply of *the embedding* σ *of* $\overrightarrow{G}$. In the case $V(\overrightarrow{G}) = k < n$ we add $(n - k)$ isolated vertices to $\overrightarrow{G}$ and define an embedding of $\overrightarrow{G}$ into TT_n analogously. We say that $\overrightarrow{G}$ is embeddable into TT_n if $\overrightarrow{G'} := \overrightarrow{G} \cup \{\text{isolated points}\}$is embeddable.

A *packing* of two oriented graphs $\overrightarrow{G}$, $\overrightarrow{H}$ of order n into TT_n is a couple (σ, δ) in which σ and δ are embeddings of $\overrightarrow{G}$ and $\overrightarrow{H}$, respectively, such that the sets $\sigma'(E(\overrightarrow{G}))$ and $\delta'(E(\overrightarrow{H}))$ are disjoint. We say that $\overrightarrow{G}$ and $\overrightarrow{H}$ are *packable* into TT_n.

There are many results concerning packing of simple graphs. The basic result was proved, independently, in [2,3,7].

Theorem 1. *Let G, H be graphs of order n. If $|E(G)| \leqslant n - 2$ and $|E(H)| \leqslant n - 2$ then G and H are packable into K_n.*

Sauer and Spencer and independently Bollobás and Eldridge consider conditions on the sum of sizes of two graphs G and H that ensure the possibility of packing of G and H into the complete graph. They proved the following theorem [2,7].

Theorem 2. *Let G and H be graphs of order n. If $|E(G)| + |E(H)| \leqslant \frac{3}{2}(n - 1)$, then there is a packing G and H into K_n.*

They also give an example that the theorem is best possible.

The main references of this paper and of other packing problems are the last chapter of Bollobás's book [1], the 4th Chapter of Yap's book [9] as well the survey papers [8,10].

Görlich, Pilśniak, Woźniak [6] investigated the existence of a packing of two copies of a given oriented graph $\overrightarrow{G}$ into TT_n. More precisely, the following result was proved therein.

Theorem 3. *Let $\overrightarrow{G}$ be an oriented graph without any directed cycle and such that $|E(\overrightarrow{G})| \leqslant 3(n - 1)/4$. Then two copies of $\overrightarrow{G}$ are packable into TT_n.*

This bound is best possible.

In our paper we give a generalization of this result. At first we wanted to obtain that two oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$ of order n without any directed cycle and such that the size of each of $\overrightarrow{G}$ and $\overrightarrow{H}$ is not greater than $3(n - 1)/4$ are packable into TT_n.

In reality we are able to prove the following much stronger theorem.

Theorem 4. *Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two oriented graphs of order n without any directed cycle. If $|E(\overrightarrow{G})| + |E(\overrightarrow{H})| \leqslant \frac{3}{2}(n-1)$, then $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_n.*

2. Proof of Theorem 4

At the beginning we notice that an oriented graph $\overrightarrow{G}$ of order n is embeddable into TT_n iff $\overrightarrow{G}$ does not include any directed cycle.

Let $\overrightarrow{G}$ be a subgraph of TT_n of order n and size m_1 and $\overrightarrow{H}$ be a subgraph of TT_n of order n and size m_2 and such that $m_1 + m_2 \leqslant \frac{3}{2}(n - 1)$.

We use induction on the order n of the transitive tournament. We remark that for $n \leqslant 2$ at most one of the oriented graphs satisfying the assumption of Theorem 4 has one arc and, obviously, our theorem is true. For $n = 3$ the sizes of

$\overrightarrow{G}$ and $\overrightarrow{H}$ satisfying the assumption of Theorem 4 are at most 3 and 0 or 2 and 1. In both cases, it is easy to see, that $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_3.

Now, let $n \geqslant 4$ and assume that our result is true for all $n' < n$. The main idea of this part of the proof is to distinguish two cases:

Case A: Neither $\overrightarrow{G}$ nor $\overrightarrow{H}$ has any isolated vertex.

We can assume, without loss of generality, that $m_1 \leqslant m_2$, so $m_1 \leqslant \frac{3}{4}(n-1)$. It is easy to see that $m_1 \geqslant \lceil n/2 \rceil$ and $m_2 \geqslant \lceil n/2 \rceil$, because $\overrightarrow{G}$ and $\overrightarrow{H}$ do not have any isolated vertex. Moreover $m_2 \leqslant n - \frac{3}{2}$, because $m_1 \leqslant m_2$, so $\overrightarrow{H}$ has at least two non-trivial components. Hence at least two sources x_H and y_H are in $\overrightarrow{H}$, because every subgraph of TT_n has a source.

Now, let $\overrightarrow{G_1}$ be an isolated arc $x_G y_G$ and $\overrightarrow{G_2}$ be a semipath $x_G y_G z_G$. Because the size of $\overrightarrow{G}$ is sufficiently small, $\overrightarrow{G}$ has to contain a component isomorphic to $\overrightarrow{G_1}$ or $\overrightarrow{G_2}$. So we consider two subcases:

Subcase A1: If $\overrightarrow{G_1}$ is a component of $\overrightarrow{G}$ then we can pack $\overrightarrow{G'} = \overrightarrow{G} - \{x_G, y_G\}$ and $\overrightarrow{H'} = \overrightarrow{H} - \{x_H, y_H\}$ into TT_{n-2}, by induction. So let TT_{n-2} be a transitive tournament with the vertices numbered from 3 to n. Let σ' and δ' be embeddings of $\overrightarrow{G'}$ and $\overrightarrow{H'}$ in TT_{n-2}, respectively. Now, we define embeddings σ of $\overrightarrow{G}$ and δ of $\overrightarrow{H}$ into TT_n as follows: $\sigma(x_G) = \delta(x_H) = 1, \sigma(y_G) = \delta(y_H) = 2$ and $\sigma(v) = \sigma'(v)$, $\delta(v) = \delta'(v)$ for all of the remaining vertices. So $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_n.

Subcase A2: If $\overrightarrow{G_1}$ is no component of $\overrightarrow{G}$, then $\overrightarrow{G_2}$ is its component. Moreover $m_1 \geqslant \frac{2}{3}n$, $m_2 \leqslant n - (n+9)/6$ and so $\overrightarrow{H}$ has at least three non-trivial components. Let z_H be the third source in $\overrightarrow{H}$, then we can pack $\overrightarrow{G'} = \overrightarrow{G} - \{x_G, y_G, z_G\}$ and $\overrightarrow{H'} = \overrightarrow{H} - \{x_H, y_H, z_H\}$ into TT_{n-3} with the vertices numbered from 4 to n, by induction. We define an embedding δ of $\overrightarrow{H}$ into TT_n as follows: $\delta(x_H) = 1$, $\delta(y_H) = 2$, $\delta(z_H) = 3$. The embedding σ of the semipath $x_G y_G z_G$ into TT_3 is easy.

Case B: $\overrightarrow{G}$ has an isolated vertex y_G.

Let x_G be a source of $\overrightarrow{G}$ and x_H be a source of $\overrightarrow{H}$. Let x'_H be a vertex adjacent from x_H and x'_G be a vertex adjacent from x_G.

Subcase B1: If $d(x_H) \geqslant 2$, then we can pack $\overrightarrow{G'} = \overrightarrow{G} - \{y_G\}$ and $\overrightarrow{H'} = \overrightarrow{H} - \{x_H\}$ into TT_{n-1}, by induction. So let TT_{n-1} be a transitive tournament with the vertices numbered from 2 to n. Let σ' and δ' be embeddings of $\overrightarrow{G'}$ and $\overrightarrow{H'}$ in TT_{n-1}, respectively.

Now, we define the embeddings σ of $\overrightarrow{G}$ and δ of $\overrightarrow{H}$ into TT_n as follows: $\sigma(y_G) = \delta(x_H) = 1$ and $\sigma(v) = \sigma'(v)$, $\delta(v) = \delta'(v)$ for all of the remaining vertices. So $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_n.

Now, we can assume, that every source of $\overrightarrow{H}$ is of the degree one.

Subcase B2: If $d^-(x'_H) = 1$, then we consider three situations:

(a) If $d(x_G) \geqslant 2$ or $d(x'_H) \geqslant 2$, then we can pack $\overrightarrow{G'} = \overrightarrow{G} - \{x_G, y_G\}$ and $\overrightarrow{H'} = \overrightarrow{H} - \{x_H, x'_H\}$ into TT_{n-2}, by induction. We define the embeddings σ of $\overrightarrow{G}$ and δ of $\overrightarrow{H}$ as follows: $\sigma(x_G) = \delta(x_H) = 1$, $\sigma(y_G) = \delta(x'_H) = 2$ and $\sigma(v) = \sigma'(v)$, $\delta(v) = \delta'(v)$ for all of the remaining vertices. So $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_n.

Now, we can assume, that $\overrightarrow{H}$ has only isolated points or isolated arcs as its connected components.

(b) If z_G is a second source in $\overrightarrow{G}$, then we can pack $\overrightarrow{G}$ and $\overrightarrow{H}$ into TT_n similar to Subcase A1.

Now, we can assume, that only one source x_G is in $\overrightarrow{G}$.

(c) Let z_H be a second source in $\overrightarrow{H}$.

If $d(x'_G) \geqslant 3$, then we can pack $\overrightarrow{G'} = \overrightarrow{G} - \{x_G, y_G, x'_G\}$ and $\overrightarrow{H'} = \overrightarrow{H} - \{z_H, x_H, x'_H\}$ into TT_{n-3}, by induction. We define the embeddings σ of $\overrightarrow{G}$ and δ of $\overrightarrow{H}$ as follows: $\sigma(x_G) = \delta(z_H) = 1$, $\sigma(y_G) = \delta(x_H) = 2$, $\sigma(x'_G) = \delta(x'_H) = 3$ and $\sigma(v) = \sigma'(v)$, $\delta(v) = \delta'(v)$ for all of the remaining vertices. So $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_n.

If $d(x'_G) = 2$ and x''_G is adjacent from x'_G, then we can pack $\overrightarrow{G'} = \overrightarrow{G} - \{x_G, x'_G, x''_G\}$ and $\overrightarrow{H'} = \overrightarrow{H} - \{x_H, z_H, x'_H\}$ into TT_{n-3}, by induction. We define the embeddings σ of $\overrightarrow{G}$ and δ of $\overrightarrow{H}$ as follows: $\sigma(x_G) = \delta(x_H) = 1$, $\sigma(x'_G) = \delta(z_H) = 2$, $\sigma(x''_G) = \delta(x'_H) = 3$ and $\sigma(v) = \sigma'(v)$, $\delta(v) = \delta'(v)$ for all of the remaining vertices. So $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_n.

Subcase B3: Now $d^-(x'_H) = k > 1$ and let $x^1_H, \ldots, x^k_H$ be the sources of $\overrightarrow{H}$ adjacent to x'_H. We consider two situations:

(a) $\overrightarrow{G_1}$ is a non-trivial connected component of $\overrightarrow{G}$ of order $l \leqslant k$. Then we can pack $\overrightarrow{G'} = \overrightarrow{G} - \overrightarrow{G_1}$ and $\overrightarrow{H'} = \overrightarrow{H} - \{x^1_H, \ldots, x^l_H\}$ into TT_{n-l}, by induction (we remove at least $2l - 1$ arcs, so that is at least $\frac{3}{2}l$ for $l \geqslant 2$). We define the embedding δ of $\overrightarrow{H}$ as follows: $\delta(x_H^i) = i$, for $i = 1, \ldots, l$ and $\delta(v) = \delta'(v)$ for all of the remaining vertices. The embedding σ of $\overrightarrow{G_1}$ into TT_l is easy to see, because $\overrightarrow{G_1}$ is a subgraph of the transitive tournament. So $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_n.

(b) $\overrightarrow{G_1}$ is a connected component of $\overrightarrow{G}$ of order $l > k$. It is a subgraph of TT_l (by assumption), so we can put its vertices in order $x_G^1, \ldots, x_G^l$ such that if $x_G^i x_G^j$ is an arc of $\overrightarrow{G_1}$ then $i < j$.

(b1) We can pack $\overrightarrow{G'} = \overrightarrow{G} - \{x_G^1, \ldots, x_G^k, y_G\}$ and $\overrightarrow{H'} = \overrightarrow{H} - \{x_H^1, \ldots, x_H^k, x'_H\}$ into $TT_{n-(k+1)}$ for $k \geqslant 3$, by induction (we remove at least $2k$ arcs, so that is at least $\frac{3}{2}(k+1)$ for $k \geqslant 3$). We define the embeddings σ of $\overrightarrow{G}$ and δ of $\overrightarrow{H}$ as follows: $\sigma(x_G^i) = \delta(x_H^i) = i$, for $i = 1, \ldots, k$, $\sigma(y_G) = \delta(x'_H) = k+1$ and $\sigma(v) = \sigma'(v)$, $\delta(v) = \delta'(v)$ for all of the remaining vertices. So $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_n.

Now, we can assume $k = 2$ what implies $l \geqslant 3$. If $d^+(x_G^1) + d^+(x_G^2) \geqslant 3$ or $d(x'_H) \geqslant 3$ we can repeat the reasoning like above, so we assume that $d(x'_H) = 2$ and every source in $\overrightarrow{G}$ is of degree one.

(b2) If three sources x_G, w_G, z_G are in $\overrightarrow{G}$, then we can pack $\overrightarrow{G'} = \overrightarrow{G} - \{x_G, w_G, z_G\}$ and $\overrightarrow{H'} = \overrightarrow{H} - \{x_H^1, x_H^2, x'_H\}$ into TT_{n-3}, by induction. We define the embeddings σ of $\overrightarrow{G}$ and δ of $\overrightarrow{H}$ as follows: $\sigma(x_G) = \delta(x_H^1) = 1$, $\sigma(w_G) = \delta(x_H^2) = 2$, $\sigma(z_G) = \delta(x'_H) = 3$ and $\sigma(v) = \sigma'(v)$, $\delta(v) = \delta'(v)$ for all of the remaining vertices. So $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_n.

(b3) If only two sources x_G, z_G are in $\overrightarrow{G}$ and x'_G is adjacent from x_G, then let z_H be a source of $\overrightarrow{H}$ different from x_H^1 and x_H^2 (if such z_H does not exist then $\overrightarrow{H}$ is the semipath $x_H^1 x_{H'} x_H^2$ and a packing of $\overrightarrow{G}$ and $\overrightarrow{H}$ is easy). We can pack $\overrightarrow{G'} = \overrightarrow{G} - \{x_G, z_G, x'_G, y_G\}$ and $\overrightarrow{H'} = \overrightarrow{H} - \{z_H, x_H^1, x_H^2, x'_H\}$ into TT_{n-4}, by induction. We define the embeddings σ of $\overrightarrow{G}$ and δ of $\overrightarrow{H}$ as follows: $\sigma(x_G) = \delta(z_H) = 1$, $\sigma(z_G) = \delta(x_H^1) = 2$, $\sigma(x'_G) = \delta(x_H^2) = 3$, $\sigma(y_G) = \delta(x'_H) = 4$ and $\sigma(v) = \sigma'(v)$, $\delta(v) = \delta'(v)$ for all of the remaining vertices. So $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into TT_n.

(b4) If only one source x_G is in $\overrightarrow{G}$, x'_G is adjacent from x_G and z'_G is adjacent from z_G. Let z_H be a source of $\overrightarrow{H}$ different from x_H^1 and x_H^2 (if such z_H does not exist then $\overrightarrow{H}$ is the semipath $x_H^1 x_{H'} x_H^2$ and a packing of $\overrightarrow{G}$ and $\overrightarrow{H}$ is easy). It is easy to see that in this situation packing of $\overrightarrow{G}$ and $\overrightarrow{H}$ into TT_n is similar to Subcase b3.

Thus, by induction, the proof is complete. □

References

[1] B. Bollobás, Extremal Graph Theory, Academic Press, London, 1978.

[2] B. Bollobás, S.E. Eldridge, Packings of graphs and applications to computational complexity, J. Combin. Theory Ser. B 25 (1978) 105–124.

[3] D. Burns, S. Schuster, Every $(n, n-2)$ graph is contained in its complement, J. Graph Theory 1 (1977) 277–279.

[6] A. Görlich, M. Pilśniak, M. Woźniak, A note on a packing problem in transitive tournaments, preprint Faculty of Applied Mathematics, AGH University of Science and Technology, No. 37/2002.

[7] N. Sauer, J. Spencer, Edge disjoint placement of graphs, J. Combin. Theory Ser. B 25 (1978) 295–302.

[8] M. Woźniak, Packing of graphs, Dissertationes Math. 362 (1997) 1–78.

[9] H.P. Yap, Some Topics in Graph Theory, London Mathematical Society, Lecture Notes Series, vol. 108, Cambridge University Press, Cambridge, 1986.

[10] H.P. Yap, Packing of graphs—a survey, Discrete Math. 72 (1988) 395–404.