Packing of two digraphs into a transitive tournament

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Received 22 September 2003; received in revised form 16 June 2004; accepted 22 November 2005
Available online 22 September 2006

Abstract

Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two oriented graphs of order $n$ without directed cycles. Görlich, Pilśniak and Woźniak proved [A note on a packing problem in transitive tournaments, preprint Faculty of Applied Mathematics, AGH University of Science and Technology, No. 37/2002] that if the number of arcs in $\overrightarrow{G}$ is sufficiently small (not greater than $3(n-1)/4$) then two copies of $\overrightarrow{G}$ are packable into the transitive tournament $TT_n$. This bound is best possible.

In this paper we give a generalization of this result. We show that if the sum of sizes of $\overrightarrow{G}$ and $\overrightarrow{H}$ is not greater than $3/2(n-1)$ then the digraphs $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into $TT_n$.

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MSC: 05C70; 05C35

Keywords: Packing of digraphs; Transitive tournaments

1. Introduction

Let $\overrightarrow{G}$ be a digraph of order $n$ with the vertex set $V(\overrightarrow{G})$ and the arc set $E(\overrightarrow{G})$. A digraph without directed cycles of the length of two is called an oriented graph. The replacement of every arc $uv$ in an oriented graph $\overrightarrow{G}$ by an edge $uv$ yields its underlying graph.

Let $\overrightarrow{G}$ be an oriented graph. For any vertex $v \in V(\overrightarrow{G})$ let us denote by $d^+(v)$ the outdegree of $v$, i.e. the number of vertices of $\overrightarrow{G}$ that are adjacent from $v$. By $d^-(v)$ we denote the indegree of $v$, i.e. the number of vertices adjacent to $v$. The degree of a vertex $v$, denoted by $d(v)$, is the sum $d(v) = d^-(v) + d^+(v)$. A vertex $x$ such that $d(x) = d^+(x)$ is called a source and a vertex $y$ such that $d(y) = d^-(y)$ is called a sink.

A digraph $\overrightarrow{G}$ is called transitive when it satisfies the condition of transitivity: if $uv$ and $vw$ are two arcs of $\overrightarrow{G}$ then $uw$ is an arc, too.

A tournament is an oriented graph such that its underlying graph is complete. A transitive tournament of order $n$ will be denoted by $TT_n$. As it is unique up to isomorphism, throughout the paper, we will view $TT_n$ as shown in Fig. 1. We can denote the vertices in $TT_n$ by consecutive integers in such way that if $i < j$ then $ij$ is an arc of $TT_n$. The vertices 1, 2 and $n$ will be called the first, the second and the last vertex of $TT_n$, respectively.

A semipath between two distinct vertices $v_1$ and $v_k$ in an oriented graph $\overrightarrow{G}$ is a path between $v_1$ and $v_k$ in the underlying graph $G$. The semipath is denoted by $v_1 \ldots v_k$, where $v_1, \ldots, v_k$ are vertices and $v_{i-1}v_i \in E(\overrightarrow{G})$ or $v_iv_{i-1} \in E(\overrightarrow{G})$ for $i \in \{2, \ldots, k\}$.

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doi:10.1016/j.disc.2005.11.038
Theorem 1. Let $G$ and $H$ be graphs of order $n$. If $|E(G)| \leq n-2$ and $|E(H)| \leq n-2$ then $G$ and $H$ are packable into $K_n$. Sauer and Spencer and independently Bollobás and Eldridge consider conditions on the sum of sizes of two graphs $G$ and $H$ that ensure the possibility of packing of $G$ and $H$ into the complete graph. They proved the following theorem [2,7].

Theorem 2. Let $G$ and $H$ be graphs of order $n$. If $|E(G)| + |E(H)| \leq \frac{3}{2}(n-1)$, then there is a packing $G$ and $H$ into $K_n$.

They also give an example that the theorem is best possible.

The main references of this paper and of other packing problems are the last chapter of Bollobás’s book [1], the 4th Chapter of Yap’s book [9] as well the survey papers [8,10]. Görlich, Pilśniak, Woźniak [6] investigated the existence of a packing of two copies of a given oriented graph $\overrightarrow{G}$ into $TT_n$. More precisely, the following result was proved therein.

Theorem 3. Let $\overrightarrow{G}$ be an oriented graph without any directed cycle and such that $|E(\overrightarrow{G})| \leq 3(n-1)/4$. Then two copies of $\overrightarrow{G}$ are packable into $TT_n$.

This bound is best possible.

In our paper we give a generalization of this result. At first we wanted to obtain that two oriented graphs $\overrightarrow{G}$ and $\overrightarrow{H}$ of order $n$ without any directed cycle and such that the size of each of $\overrightarrow{G}$ and $\overrightarrow{H}$ is not greater than $3(n-1)/4$ are packable into $TT_n$.

In reality we are able to prove the following much stronger theorem.

Theorem 4. Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two oriented graphs of order $n$ without any directed cycle. If $|E(\overrightarrow{G})| + |E(\overrightarrow{H})| \leq \frac{3}{2}(n-1)$, then $\overrightarrow{G}$ and $\overrightarrow{H}$ are packable into $TT_n$.

2. Proof of Theorem 4

At the beginning we notice that an oriented graph $\overrightarrow{G}$ of order $n$ is embeddable into $TT_n$ iff $\overrightarrow{G}$ does not include any directed cycle.

Let $\overrightarrow{G}$ be a subgraph of $TT_n$ of order $n$ and size $m_1$ and $\overrightarrow{H}$ be a subgraph of $TT_n$ of order $n$ and size $m_2$ and such that $m_1 + m_2 \leq \frac{3}{2}(n-1)$.

We use induction on the order $n$ of the transitive tournament. We remark that for $n \leq 2$ at most one of the oriented graphs satisfying the assumption of Theorem 4 has one arc and, obviously, our theorem is true. For $n = 3$ the sizes of...
\( \overrightarrow{G} \) and \( \overrightarrow{H} \) satisfying the assumption of Theorem 4 are at most 3 and 0 or 2 and 1. In both cases, it is easy to see, that \( \overrightarrow{G} \) and \( \overrightarrow{H} \) are packable into \( TT_3 \).

Now, let \( n \geq 4 \) and assume that our result is true for all \( n' < n \). The main idea of this part of the proof is to distinguish two cases:

**Case A:** Neither \( \overrightarrow{G} \) nor \( \overrightarrow{H} \) has any isolated vertex.

We can assume, without loss of generality, that \( m_1 \leq m_2 \), so \( m_1 \leq \frac{3}{4}(n - 1) \). It is easy to see that \( m_1 \geq \lceil n/2 \rceil \) and \( m_2 \geq \lceil n/2 \rceil \), because \( \overrightarrow{G} \) and \( \overrightarrow{H} \) do not have any isolated vertex. Moreover \( m_2 \leq n - \frac{3}{2} \), because \( m_1 \leq m_2 \), so \( \overrightarrow{H} \) has at least two non-trivial components. Hence at least two sources \( x_H \) and \( y_H \) are in \( \overrightarrow{H} \), because every subgraph of \( TT_n \) has a source.

Now, let \( \overrightarrow{G}_1 \) be an isolated arc \( x_Gy_G \) and \( \overrightarrow{G}_2 \) be a semipath \( x_Gy_Gz_G \). Because the size of \( \overrightarrow{G} \) is sufficiently small, \( \overrightarrow{G} \) has to contain a component isomorphic to \( \overrightarrow{G}_1 \) or \( \overrightarrow{G}_2 \). So we consider two subcases:

**Subcase A1:** If \( \overrightarrow{G}_1 \) is a component of \( \overrightarrow{G} \) then we can pack \( \overrightarrow{G} = \overrightarrow{G} - \{x_G, y_G\} \) and \( \overrightarrow{H} = \overrightarrow{H} - \{x_H, y_H\} \) into \( TT_{n-2} \), by induction. So let \( TT_{n-2} \) be a transitive tournament with the vertices numbered from 3 to \( n \). Let \( \sigma' \) and \( \delta' \) be embeddings of \( \overrightarrow{G} \) and \( \overrightarrow{H} \) in \( TT_{n-2} \), respectively. Now, we define embeddings \( \sigma \) of \( \overrightarrow{G} \) and \( \delta \) of \( \overrightarrow{H} \) into \( TT_n \) as follows: \( \sigma(x_G) = \delta(x_H) = 1, \sigma(y_G) = \delta(y_H) = 2 \) and \( \sigma(v) = \sigma'(v), \delta(v) = \delta'(v) \) for all of the remaining vertices. So \( \overrightarrow{G} \) and \( \overrightarrow{H} \) are packable into \( TT_n \).

**Subcase A2:** If \( \overrightarrow{G}_1 \) is no component of \( \overrightarrow{G} \), then \( \overrightarrow{G}_2 \) is its component. Moreover \( m_1 \geq \frac{3}{2} n, m_2 \leq n - (n + 9)/6 \) and \( \overrightarrow{H} \) has at least three non-trivial components. Let \( z_H \) be the third source in \( \overrightarrow{H} \), then we can pack \( \overrightarrow{G} = \overrightarrow{G} - \{x_G, y_G, z_G\} \) and \( \overrightarrow{H} = \overrightarrow{H} - \{x_H, y_H, z_H\} \) into \( TT_{n-3} \) with the vertices numbered from 4 to \( n \), by induction. We define an embedding \( \delta \) of \( \overrightarrow{H} \) into \( TT_n \) as follows: \( \delta(x_H) = 1, \delta(y_H) = 2, \delta(z_H) = 3 \). The embedding \( \sigma \) of the semipath \( x_Gy_Gz_G \) into \( TT_3 \) is easy.

**Case B:** \( \overrightarrow{G} \) has an isolated vertex \( y_G \).

Let \( x_G \) be a source of \( \overrightarrow{G} \) and \( x_H \) be a source of \( \overrightarrow{H} \). Let \( x'_G \) be a vertex adjacent from \( x_H \) and \( x'_H \) be a vertex adjacent from \( x_G \).

**Subcase B1:** If \( d(x_H) \geq 2 \), then we can pack \( \overrightarrow{G} = \overrightarrow{G} - \{y_G\} \) and \( \overrightarrow{H} = \overrightarrow{H} - \{x_H\} \) into \( TT_{n-1} \), by induction. So let \( TT_{n-1} \) be a transitive tournament with the vertices numbered from 2 to \( n \). Let \( \sigma \) and \( \delta \) be embeddings of \( \overrightarrow{G} \) and \( \overrightarrow{H} \) in \( TT_{n-1} \), respectively.

Now, we define the embeddings \( \sigma \) of \( \overrightarrow{G} \) and \( \delta \) of \( \overrightarrow{H} \) into \( TT_n \) as follows: \( \sigma(y_G) = \delta(x_H) = 1 \) and \( \sigma(v) = \sigma'(v), \delta(v) = \delta'(v) \) for all of the remaining vertices. So \( \overrightarrow{G} \) and \( \overrightarrow{H} \) are packable into \( TT_n \).

Now, we can assume, that every source of \( \overrightarrow{H} \) is of the degree one.

**Subcase B2:** If \( d(x'_H) = 1 \), then we consider three situations:

(a) If \( d(x_G) \geq 2 \) or \( d(x'_G) \geq 2 \), then we can pack \( \overrightarrow{G} = \overrightarrow{G} - \{x_G, y_G\} \) and \( \overrightarrow{H} = \overrightarrow{H} - \{x_H, x'_H\} \) into \( TT_{n-2} \), by induction. We define the embeddings \( \sigma \) of \( \overrightarrow{G} \) and \( \delta \) of \( \overrightarrow{H} \) as follows: \( \sigma(x_G) = \delta(x_H) = 1, \sigma(y_G) = \delta(x_H) = 2 \) and \( \sigma(v) = \sigma'(v), \delta(v) = \delta'(v) \) for all of the remaining vertices. So \( \overrightarrow{G} \) and \( \overrightarrow{H} \) are packable into \( TT_n \).

Now, we can assume, that \( \overrightarrow{H} \) has only isolated points or isolated arcs as its connected components.

(b) If \( y_G \) is a second source in \( \overrightarrow{G} \), then we can pack \( \overrightarrow{G} \) and \( \overrightarrow{H} \) into \( TT_n \) similar to Subcase A1.

Now, we can assume, that only one source \( x_G \) is in \( \overrightarrow{G} \).

(c) Let \( z_H \) be a second source in \( \overrightarrow{H} \).

If \( d(x'_G) \geq 3 \), then we can pack \( \overrightarrow{G} = \overrightarrow{G} - \{x_G, y_G, x'_G\} \) and \( \overrightarrow{H} = \overrightarrow{H} - \{z_H, x_H, x'_H\} \) into \( TT_{n-3} \), by induction. We define the embeddings \( \sigma \) of \( \overrightarrow{G} \) and \( \delta \) of \( \overrightarrow{H} \) as follows: \( \sigma(x_G) = \delta(z_H) = 1, \sigma(y_G) = \delta(x_H) = 2, \sigma(x'_G) = \delta(x'_H) = 3 \) and \( \sigma(v) = \sigma'(v), \delta(v) = \delta'(v) \) for all of the remaining vertices. So \( \overrightarrow{G} \) and \( \overrightarrow{H} \) are packable into \( TT_n \).

If \( d(x'_G) = 2 \) and \( x'_G \) is adjacent from \( x_G \), then we can pack \( \overrightarrow{G} = \overrightarrow{G} - \{x_G, y_G, x'_G\} \) and \( \overrightarrow{H} = \overrightarrow{H} - \{x_H, z_H, x'_H\} \) into \( TT_{n-3} \), by induction. We define the embeddings \( \sigma \) of \( \overrightarrow{G} \) and \( \delta \) of \( \overrightarrow{H} \) as follows: \( \sigma(x_G) = \delta(x_H) = 1, \sigma(x'_G) = \delta(z_H) = 2, \sigma(x'_G) = \delta(x'_H) = 3 \) and \( \sigma(v) = \sigma'(v), \delta(v) = \delta'(v) \) for all of the remaining vertices. So \( \overrightarrow{G} \) and \( \overrightarrow{H} \) are packable into \( TT_n \).

**Subcase B3:** Now \( d(x'_H) = k \) and let \( x_H^1, \ldots, x_H^k \) be the sources of \( \overrightarrow{H} \) adjacent to \( x'_H \). We consider two situations:

(a) \( \overrightarrow{G}_1 \) is a non-trivial connected component of \( \overrightarrow{G} \) of order \( l \leq k \). Then we can pack \( \overrightarrow{G} = \overrightarrow{G} - \overrightarrow{G}_1 \) and \( \overrightarrow{H} = \overrightarrow{H} - \{x_H^1, \ldots, x_H^l\} \) into \( TT_{n-1} \), by induction (we remove at least \( 2l - 1 \) arcs, so that is at least \( k \)). We define the
embedding \( \delta \) of \( \overrightarrow{H} \) as follows: \( \delta(x_H^i) = i \), for \( i = 1, \ldots, l \) and \( \delta(v) = \delta'(v) \) for all of the remaining vertices. The embedding \( \sigma \) of \( G_1 \) into \( TT_l \) is easy to see, because \( G_1 \) is a subgraph of the transitive tournament. So \( G \) and \( H \) are packable into \( TT_n \).

(b) \( G_1 \) is a connected component of \( G \) of order \( l > k \). It is a subgraph of \( TT_l \) (by assumption), so we can put its vertices in order \( x_G^1, \ldots, x_G^l \) such that if \( x_G^i \) is an arc of \( G_1 \) then \( i < j \).

(b1) We can pack \( G = G_1 - \{x_G^1, \ldots, x_G^k, y_G\} \) and \( H = H - \{x_H^1, \ldots, x_H^k, x'_H\} \) into \( TT_{n-(k+1)} \) for \( k \geq 3 \), by induction (we remove at least \( 2k \) arcs, so that is at least \( \frac{3}{2}(k+1) \) for \( k \geq 3 \)). We define the embeddings \( \sigma \) of \( G \) and \( \delta \) of \( H \) as follows: \( \sigma(x_G^i) = \delta(x_H^i) = i \), for \( i = 1, \ldots, k \), \( \sigma(y_G) = \delta(x_H^k) = k + 1 \) and \( \sigma(v) = \sigma'(v) \), \( \delta(v) = \delta'(v) \) for all of the remaining vertices. So \( G \) and \( H \) are packable into \( TT_n \).

Now, we can assume \( k = 2 \) what implies \( l \geq 3 \). If \( d^+(x_G^1) + d^+(x_G^2) \geq 3 \) or \( d(x'_H) \geq 3 \) we can repeat the reasoning like above, so we assume that \( d(x'_H) = 2 \) and every source in \( G \) is of degree one.

(b2) If three sources \( x_G, w_G, z_G \) are in \( G \), then we can pack \( G = G - \{x_G, w_G, z_G\} \) and \( H = H - \{x_H^1, x'_H, x_H^2\} \) into \( TT_{n-3} \), by induction. We define the embeddings \( \sigma \) of \( G \) and \( \delta \) of \( H \) as follows: \( \sigma(x_G) = \delta(x_H^1) = 1 \), \( \sigma(w_G) = \delta(x_H^2) = 2 \), \( \sigma(z_G) = \delta(x_H^3) = 3 \) and \( \sigma(v) = \sigma'(v) \), \( \delta(v) = \delta'(v) \) for all of the remaining vertices. So \( G \) and \( H \) are packable into \( TT_n \).

(b3) If only two sources \( x_G, z_G \) are in \( G \) and \( x'_H \) is adjacent from \( x_G \), then let \( z_H \) be a source of \( H \) different from \( x_H^1 \) and \( x_H^2 \) (if such \( z_H \) does not exist then \( H \) is the semipath \( x_H^1x_H^2x'_H \) and a packing of \( G \) and \( H \) is easy). We can pack \( G = G - \{x_G, z_G, x'_H, y_G\} \) and \( H = H - \{x_H^1, x_H^2, x'_H\} \) into \( TT_{n-4} \), by induction. We define the embeddings \( \sigma \) of \( G \) and \( \delta \) of \( H \) as follows: \( \sigma(x_G) = \delta(z_H) = 1 \), \( \sigma(z_G) = \delta(x_H^1) = 2 \), \( \sigma(x'_H) = \delta(x_H^2) = 3 \), \( \sigma(y_G) = \delta(x_H^3) = 4 \) and \( \sigma(v) = \sigma'(v) \), \( \delta(v) = \delta'(v) \) for all of the remaining vertices. So \( G \) and \( H \) are packable into \( TT_n \).

(b4) If only one source \( x_G \) is in \( G \), \( x'_H \) is adjacent from \( x_G \) and \( z'_G \) is adjacent from \( z_G \). Let \( z_H \) be a source of \( H \) different from \( x_H^1 \) and \( x_H^2 \) (if such \( z_H \) does not exist then \( H \) is the semipath \( x_H^1x_H^2x'_H \) and a packing of \( G \) and \( H \) is easy). It is easy to see that in this situation packing of \( G \) and \( H \) into \( TT_n \) is similar to Subcase b3.

Thus, by induction, the proof is complete.

References