A note on packing of two copies of a hypergraph

Monika Pilśniak
Faculty of Applied Mathematics
AGH University of Science and Technology
ul. Mickiewicza 30, 30-059 Kraków, Poland
e-mail: pilśniak @agh.edu.pl

Mariusz Woźniak*
Institute of Mathematics
Polish Academy of Sciences
ul. Św. Tomasza 30, Kraków, Poland
e-mail: mwozniak @agh.edu.pl

Abstract

A 2-packing of a hypergraph $\mathcal{H}$ is a permutation $\sigma$ on $V(\mathcal{H})$ such that if an edge $e$ belongs to $E(\mathcal{H})$, then $\sigma(e)$ does not belong to $E(\mathcal{H})$.

We prove that a hypergraph which does not contain neither empty edge $\emptyset$ nor complete edge $V(\mathcal{H})$ and has at most $\frac{2}{3}n$ edges is 2-packable.

A 1-uniform hypergraph of order $n$ with more than $\frac{1}{3}n$ edges shows that this result cannot be improved by increasing the size of $\mathcal{H}$.

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*On leave from Faculty of Applied Mathematics, AGH University of Science and Technology
1 Introduction

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph where $V$ is the vertex set and $\mathcal{E} \subseteq 2^V$ is the edge set. We shall assume that $V$ and $\mathcal{E}$ are non-empty but allow in general empty edges for technical reasons. So, a complete simple hypergraph of order $n$ has $2^n$ edges. We consider only finite hypergraphs.

An edge $e \in \mathcal{E}$ is called a singleton if $|e| = 1$. A vertex is isolated if no edge contains it. The number $d(v)$ of edges containing a vertex $v$ is called the degree of $v \in V$. A hypergraph is $t$-uniform if $|e| = t$ for all $e \in \mathcal{E}$.

Let $\mathcal{H}$ be a hypergraph of order $n$. A packing of two copies of $\mathcal{H}$ $(2$-packing of $\mathcal{H})$ is a permutation $\sigma$ on $V(\mathcal{H})$ such that if an edge $e = \{x_1, \ldots, x_k\}$ belongs to $\mathcal{E}(\mathcal{H})$, then the edge $\sigma(e) = \{\sigma(x_1), \ldots, \sigma(x_k)\}$ does not belong to $\mathcal{E}(\mathcal{H})$. Such a permutation (a packing permutation) is called also an embedding of $\mathcal{H}$ into its complement.

Let us consider a hypergraph $\mathcal{H}$ and a permutation $\sigma$ on $V$. We have $\sigma(V) = V$ and $\sigma(\emptyset) = \emptyset$. So, if $V \in \mathcal{E}$ or $\emptyset \in \mathcal{E}$ then $\mathcal{H}$ cannot be packable. The hypergraph $\mathcal{H}$ such that neither $\emptyset \in \mathcal{E}(\mathcal{H})$ nor $V \in \mathcal{E}(\mathcal{H})$ is called admissible. We consider only admissible hypergraphs.

Let $\mathcal{H}$ be an admissible hypergraph. Let us consider a hypergraph $\tilde{\mathcal{H}} = (V, \tilde{\mathcal{E}})$ with the same vertex set $V$ and the edge set, $\tilde{\mathcal{E}}$ obtained from $\mathcal{E}$ in the following way: if $e \in \mathcal{E}$ has at most $\frac{n}{2}$ vertices then $e$ belongs to $\tilde{\mathcal{E}}$ and if $e$ has more than $\frac{n}{2}$ vertices, then $e$ is replaced by $V \setminus e$ with convention that each double edge eventually created in this way is replaced by a single one.

**Remark.** Let $\mathcal{H}$ be an admissible hypergraph of order $n$ and size at most $\frac{1}{2}n$. Let us observe that a hypergraph $\tilde{\mathcal{H}}$ is 2-packable iff the hypergraph $\mathcal{H}$ is 2-packable.

Therefore, we will consider 2-packing of $\tilde{\mathcal{H}}$ in the proof of Theorem 2, although we shall write $\mathcal{H}$.

A 2-uniform hypergraph is called a graph. The packing problems for graphs have been studied for about thirty years (see for instance chapters in the books by B. Bollobás or H. P. Yap ([2], [7]) or survey papers by H. P. Yap or M. Wozniak ([8], [5], [6] and [4])). One of the first results in this area was the following theorem (see [3]).

**Theorem 1** A graph $G$ of order $n$ and size at most $n - 2$ is 2-packable.
2 Main result

The aim of this note is to prove the following theorem.

**Theorem 2** An admissible hypergraph of order \( n \) and size at most \( \frac{1}{2}n \) is 2-packable.

First, let us observe that this bound is best possible. Namely, if \( \mathcal{H} \) is a hypergraph of order \( n \) and has more than \( \frac{1}{2}n \) edges and each edge is a singleton, then evidently \( \mathcal{H} \) is not packable.

**Proof of Theorem 2.** It is easy to see that the theorem is true for \( n = 2 \) and \( n = 3 \). So, let \( n \geq 4 \).

By Remark in the previous section, we may consider only hypergraphs which have only edges of cardinality at most \( \frac{n}{2} \). Let \( \mathcal{H} \) be an admissible hypergraph. Denote by \( m_k \) the number of edges of cardinality \( k \) and let \( m \) be the size of \( \mathcal{H} \). Thus

\[
\frac{n}{2} \geq m = m_1 + m_2 + \ldots + m_{\lfloor \frac{n}{2} \rfloor}.
\]

The proof will be divided into two parts.

**Case 1.** \( m_1 = 0 \)

First, by using a ‘probabilistic’ argument we shall show that the packing permutation exists if \( \mathcal{H} \) has no singleton.

Let \( e \) and \( f \) be two edges of \( \mathcal{H} \) of the same cardinality and let \( \sigma \) be a random permutation on \( V \). We say that an edge \( e \) covers an edge \( f \) (with respect to \( \sigma \)), if \( \sigma(e) = f \). We write: \( (e \sim f) \).

Let \( e \) and \( f \) be two edges of cardinality \( k \). The probability of the event \( A \) that \( e \) covers \( f \) (denoted by \( A(e \sim f) \)) is equal to

\[
Pr(A(e \sim f)) = \frac{k!(n-k)!}{n!} = \left( \frac{n}{k} \right)^{-1}.
\]

Let us observe, that the number of events that an edge of cardinality \( k \) cover some edge in \( \mathcal{H} \) of cardinality \( k \) is equal to \( m_k^2 \).

So, we have

\[
Pr \left( \bigcup_{e,f \in \mathcal{H}} A(e \sim f) \right) \leq \sum_{e,f \in \mathcal{H}} Pr(A(e \sim f)) =
\]
\[ m_2 \frac{n}{2}^{-1} + m_3 \frac{n}{3}^{-1} + \ldots + m_{\lfloor \frac{n}{2} \rfloor} \left( \frac{n}{\lfloor \frac{n}{2} \rfloor} \right)^{-1}. \]

Since \( k \leq \frac{n}{2} \), the sequence \( \left( \frac{n}{2} \right)^{-1}, \left( \frac{n}{3} \right)^{-1}, \ldots \) is decreasing and we have

\[ m_2 \frac{n}{2}^{-1} + m_3 \frac{n}{3}^{-1} + \ldots + m_{\lfloor \frac{n}{2} \rfloor} \left( \frac{n}{\lfloor \frac{n}{2} \rfloor} \right)^{-1} \leq \left( \frac{n}{2} \right)^{-1} \left( m_2^2 + m_3^2 + \ldots + m_{\lfloor \frac{n}{2} \rfloor}^2 \right) \leq \left( \frac{n}{2} \right)^{-1} \left( \frac{n}{2} \right)^2 = \frac{n}{2(n-1)}. \]

It is easy to see that \( \frac{n}{2(n-1)} < 1 \) for \( n > 2 \). In consequence, there exists a packing of an admissible hypergraph \( \mathcal{H} \) of order \( n \) and size at most \( \frac{1}{2}n \) into its complement, if \( \mathcal{H} \) does not have any singletons.

**Case 2.** \( m_1 \geq 1 \)

In this case we use the induction with respect to \( n \). Let \( \mathcal{H} \) be an admissible hypergraph of order \( n \) and suppose that the theorem holds for \( n' < n \).

Let \( \{x\} \) be a singleton and let \( y \) be a vertex of \( \mathcal{H} \) such that \( \{y\} \notin \mathcal{E} \) and \( \{x, y\} \notin \mathcal{E} \). Such a \( y \) exists. For, otherwise each vertex other than \( x \) would be either a singleton or the end of an edge joining it with \( x \), and we would get a contradiction with the size of \( \mathcal{H} \).

Now, we construct a hypergraph \( \mathcal{H}' = (V', \mathcal{E}') \) such that \( V' = V - \{x, y\} \) and the set of edges is obtained from \( \mathcal{E} \) as follows: we delete the edge \( \{x\} \) and we replace all edges containing \( x \) or \( y \) (or \( x \) and \( y \)) by new edges without these vertices. So \( \mathcal{H}' \) has one edge and two vertices less than \( \mathcal{H} \). If \( m_1 \neq 0 \) in \( \mathcal{H}' \) then a packing permutation \( \sigma' \) exists by the induction hypothesis. If \( m_1 = 0 \) in \( \mathcal{H}' \) then a packing permutation \( \sigma' \) exists by Case 1.

By the choice of \( x \) and \( y \) and the property of \( \sigma' \), it is easy to see that the permutation \( \sigma = \sigma' \circ (xy) \) where \( (xy) \) denotes a transposition, is a packing permutation of \( \mathcal{H} \)

\[ \square \]

### 3 Some open problems

One of the objectives of this paper is to draw attention of the reader to some open problems.

4
I. It would be interesting to consider an analogous problem for uniform hypergraphs. For $t = 1$ the above result is still best possible. For $t = 2$ the answer is given by Theorem 1. In the case $t = 3$ we suppose (together with E. Győri) that the right bound of the size of a packable hypergraph is $k = \frac{1}{6}(n - 2)(n + 3)$. It is easy to see that this bound cannot be improved.

II. Does Theorem 2 remain true if instead of packing of two copies of the same hypergraph of order $n$ we pack two distinct hypergraphs of order $n$, both of size at most $\frac{1}{2}n$?

III. A. Benhocine and A. P. Wojda in [1] proved that a graph of order $n$ and size at most $n - 1$ is 2-packable if and only if $G$ is embeddable into a self complementary graph of the same order (for $n \equiv 0, 1 \pmod{4}$). It would be interesting to get an analogous result for hypergraphs.

References


