SIMULATION OF STEADY STATE THERMAL PROCESSES

Thermal phenomena occurring in the steady state are described by the Fourier equation in the following form:

$$div(k(t)grad(t)) + Q = 0$$

$$\frac{\partial}{\partial x}\left(k_x(t)\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y(t)\frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z(t)\frac{\partial t}{\partial z}\right) + Q = 0\,, \tag{5.1}$$

where:
$k_x(t)$, $k_y(t)$, $k_z(t)$ – anisotropic coefficients of heat conduction dependent on temperature *t;*
Q – heat generation rate.

The solution of equation (5.1) is reduced to the task of finding the minimum of such functional for which Equation (5.1) is the Euler equation. According to the variance account [5, 14, 17], the functional would be:

$$J = \int_V \frac{1}{2}\left(k_x(t)\left(\frac{\partial t}{\partial x}\right)^2 + k_y(t)\left(\frac{\partial t}{\partial y}\right)^2 + k_z(t)\left(\frac{\partial t}{\partial z}\right)^2 - 2Qt\right)dV\,. \tag{5.2}$$

$$J = \int_V \frac{k(t)}{2}\left(\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2 - 2Qt\right)dV\,. \tag{5.3}$$

The function *t(x, y, z)* must satisfy the boundary conditions on the surface of the area. The heat flux q is determined on the surface according to the law of convection:

$$k(t)\left(\frac{\partial t}{\partial x}a_x + \frac{\partial t}{\partial y}a_y + \frac{\partial t}{\partial z}a_z\right) = \alpha_{konw}\left(t - t_\infty\right), \tag{5.4}$$

where:
a_x, a_y, a_z – directional cosines of normal vector to the surface;
t_∞ – ambient temperature;
α_{konw} – convective heat transfer coefficient;

Expression $\alpha(t_\infty - t)$ refers to the exchange of heat with the environment. The coefficient α is define according to the existing conditions. It is possible to exchange heat with gas, air or cooling medium on free surfaces. Introducing boundary conditions into function (5.3) directly is not possible. In practice, these conditions are imposed by adding to the function (5.3) the integral in the form:

$$\int_S \frac{\alpha}{2}(t-t_\infty)^2\,dS + \int_S qtdS \quad , \tag{5.8}$$

where:

S – surface on which the boundary conditions are set.

Combining the boundary conditions (5.8) with the Fourier equation (5.3) gives:

$$J = \int_V \left(\frac{k(t)}{2} \left(\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2 \right) - Qt \right) dV + \\ + \int_S \frac{\alpha}{2}(t-t_\infty)^2\,dS + \int_S qtdS \quad . \tag{5.9}$$

Discretization of the presented problem is based on dividing the area in separated elements and representing the temperature inside the element as a function of nodal values according to the following relation:

$$t = \sum_{i=1}^{n} N_i t_i = \{N\}^T \{t\} \cdot \tag{5.10}$$

By introducing dependence (5.10) to the function (5.9) we obtain:

$$J = \int_V \left(\frac{k}{2} \left(\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}^T \{t\} \right)^2 + \left(\left\{\frac{\partial\{N\}}{\partial y}\right\}^T \{t\} \right)^2 + \left(\left\{\frac{\partial\{N\}}{\partial z}\right\}^T \{t\} \right)^2 \right) - Q\{N\}^T\{t\} \right) dV + \\ + \int_S \frac{\alpha}{2}\left(\{N\}^T\{t\} - t_\infty\right)^2 dS + \int_S q\{N\}^T\{t\}dS. \tag{5.11}$$

The minimization of functional (5.11) is based on partial derivatives of this function with respect to the node values of temperature *{t}*, which results in the following set of equations:

$$\frac{\partial J}{\partial\{t\}} = \int_V \left(k\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T + \left\{\frac{\partial\{N\}}{\partial z}\right\}\left\{\frac{\partial\{N\}}{\partial z}\right\}^T \right)\{t\} - Q\{N\} \right) dV + \\ + \int_S \alpha\left(\{N\}^T\{t\} - t_\infty\right)\{N\}dS + \int_S q\{N\}dS = 0 \tag{5.12}$$

The system of equations (5.12) written in matrix form has the following form:

$$[H]\{t\}+\{P\}=0\,. \tag{5.13}$$

where:

$$[H]=\int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T+\left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T+\left\{\frac{\partial\{N\}}{\partial z}\right\}\left\{\frac{\partial\{N\}}{\partial z}\right\}^T\right)dV+$$
$$+\int_S \alpha\{N\}\{N\}^T dS, \tag{5.14}$$

$$\{P\}=-\int_S \alpha\{N\}t_\infty dS-\int_V Q\{N\}dV+\int_S q\{N\}dS\,. \tag{5.15}$$

In other way, the minimization of the function (5.11) can be done by direct selection of the node temperature values by one of the existing minimization methods.

1.1 DETERMINATION OF TEMPERATURE FIELD IN ROD – STEADY STATE

Consider the process of steady heat conduction in a rod. Suppose that heat transfer will be effected only by the two ends of the rod (Fig. 5.1). For the fixed end of the rod the heat flux q is added. At the free end of the rod heat exchange takes place by convection. Convection heat transfer coefficient is equal α, while the ambient temperature is equal to t_∞.

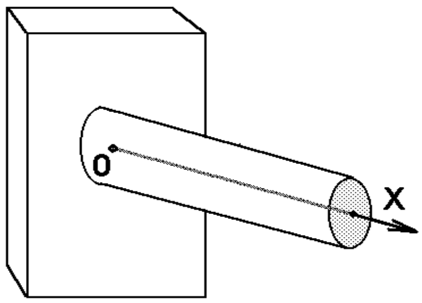

Fig. 5.1. Scheme of the problem of determining the temperature field in the rod

Rod length is equal to L. Lets consider the differential Fourier equation for the case of one-dimensional problem:

$$k\frac{d^2t}{dx^2}=0\,, \tag{5.16}$$

with the boundary conditions:

$$k\frac{dt}{dx}+q=0, \text{ when } x=0; \tag{5.17}$$

$$k\frac{dt}{dx}+\alpha(t-t_\infty)=0, \text{ when } x=L. \tag{5.18}$$

Heat flux q is a positive if heat is taken away from the rod. So functional (5.9) for the case under consideration can be written in the following way:

$$J=\int_V \frac{1}{2}\left(k\left(\frac{\partial t}{\partial x}\right)^2\right)dV+\int_S \frac{\alpha}{2}(t-t_\infty)^2\,dS+\int_S tq\;dS \quad . \tag{5.19}$$

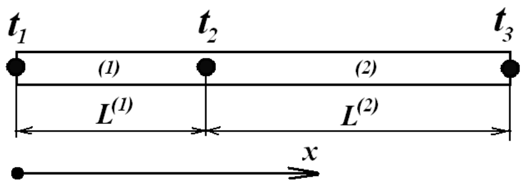

Fig. 5.2. *Division of the computational domain into two FEM elements.*

Consider the process of minimizing the functional (5.19). Rod split into two finite elements with the nodes 1, 2, 3 (Fig. 5.2), L $^{(1)}$ and L $^{(2)}$ are the lengths of the elements. The nodal temperature values t1, t2, t3 are the unknowns. The temperature inside the elements is defined as follows:

$$t^{(1)}=N_1^{(1)}t_1+N_2^{(1)}t_2\,, \tag{5.20}$$

$$t^{(2)}=N_2^{(2)}t_2+N_3^{(2)}t_3\,, \tag{5.21}$$

$$N_1^{(1)}=\frac{x_2-x}{L^{(1)}},\quad N_2^{(1)}=\frac{x-x_1}{L^{(1)}}\,, \tag{5.22}$$

$$N_2^{(2)}=\frac{x_3-x}{L^{(2)}}\,,\quad N_3^{(2)}=\frac{x-x_2}{L^{(2)}}\,. \tag{5.23}$$

Considered integrals in functional (5.19):

$$\int_{S_1} qtdS=qt_1S_1\,, \tag{5.24}$$

$$\int_{S_3}\frac{\alpha}{2}(t-t_\infty)^2dS=\frac{\alpha S_3}{2}\left(t_3^2-2t_3t_\infty+t_\infty^2\right), \tag{5.25}$$

where:

S1 and *S3* – rod cross section in the nodes 1 and 3.

To determine the functional volume integrals (5.19) calculated temperature derivatives with respect to x:

$$\frac{dt^{(1)}}{dx}=\frac{(-t_1+t_2)}{L^{(1)}}, \tag{5.26}$$

$$\frac{dt^{(2)}}{dx}=\frac{(-t_2+t_3)}{L^{(2)}} \tag{5.27}$$

Given that $dV=S^{(e)}dx$ using the algebraic transformation obtained:

$$\begin{gathered}\int_V \frac{k}{2}\left(\frac{dt}{dX}\right)^2 dV = \\ =\frac{k^{(1)}S^{(1)}}{2L^{(1)}}(t_2-t_1)^2+\frac{k^{(2)}S^{(2)}}{2L^{(2)}}(t_3-t_2)^2.\end{gathered} \tag{5.28}$$

Coefficient of thermal conductivity k can be different for each finite element. Summarize the formulas (5.24, 5.25 and 5.28) and obtain the functional (5.19) as a function of the nodal values of temperature:

$$\begin{gathered}J=\frac{C^{(1)}}{2}(t_1^2-2t_1t_2+t_2^2)+\frac{C^{(2)}}{2}(t_2^2-2t_2t_3+t_3^2)+ \\ qS_1t_1+\frac{\alpha S_3}{2}(t_3^2-2t_3t_\infty+t_\infty^2).\end{gathered} \tag{5.29}$$

In equation (5.29) C ratios were calculated as follows:

$$\begin{gathered}C^{(1)}=\frac{S^{(1)}k^{(1)}}{L^{(1)}}, \\ C^{(2)}=\frac{S^{(2)}k^{(2)}}{L^{(2)}}.\end{gathered} \tag{5.30}$$

Therefore, two options may be considered: the direct minimization of the function (5.29), by selecting value of nodal temperature or use extreme conditions of the function. The last method requires the derivative of formula (5.29) with respect to the nodal variables and the alignment derivatives to zero. As a result of this operation we obtain as many equations as there are unknowns. It was assumed that the $S^{(1)} = S^{(2)} = S$. considered system of equations:

$$\left.\begin{aligned}\frac{\partial J}{\partial t_1}&=C^{(1)}t_1-C^{(1)}t_2+qS=0\\ \frac{\partial J}{\partial t_2}&=-C^{(1)}t_1+\left(C^{(1)}+C^{(2)}\right)t_2-C^{(2)}t_3=0\\ \frac{\partial J}{\partial t_3}&=-C^{(2)}t_2+\left(C^{(2)}+\alpha S\right)t_3-\alpha St_\infty=0\end{aligned}\right\}. \tag{5.31}$$

The system of equations (5.31) can be written in matrix form:

$$\begin{bmatrix} C^{(1)} & -C^{(1)} & 0 \\ -C^{(1)} & C^{(1)}+C^{(2)} & -C^{(2)} \\ 0 & -C^{(2)} & C^{(2)}+\alpha S \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix}+\begin{Bmatrix} qS \\ 0 \\ -\alpha St_\infty \end{Bmatrix}=0, \tag{5.32}$$

or:

$$[H]\{t\}+\{P\}=0\,, \tag{5.33}$$

where:

$[H]$ – matrix of coefficients of the system of equations (5.31);
$\{P\}$ – vector of the right part of the system of equations (5.31).

It should be noted that the resulting matrix of coefficients of the system of equations is symmetric and band (pasmowa). Considered solution to the same problem is based on the received general solution (5.13). For the individual case, formulas (5.14) and (5.15) can be written as follows (for any finite element):

$$[H]=\int_V k\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T dV+\int_S \alpha\{N\}\{N\}^T dS\,, \tag{5.34}$$

$$\{P\}=-\int_S \alpha\{N\}t_\infty dS+\int_S q\{N\}dS\,. \tag{5.35}$$

Vectors, entering to the equations (5.34) and (5.35) can be written as follows:

$$\{N\}=\begin{Bmatrix} N_i \\ N_j \end{Bmatrix}=\begin{Bmatrix} \dfrac{x_j-x}{L} \\ \dfrac{x-x_i}{L} \end{Bmatrix},$$

$$\{N\}^T=\{N_i \quad N_j\}=\left\{\frac{x_j-x}{L};\quad \frac{x-x_i}{L}\right\},$$

$$\left\{\frac{\partial\{N\}}{\partial x}\right\}=\begin{Bmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{Bmatrix},$$

$$\left\{\frac{\partial\{N\}}{\partial x}\right\}^T=\left\{-\frac{1}{L}\quad \frac{1}{L}\right\}.$$

Then the matrix [H] can be define as follows:

$$[H]=\int_V k\begin{Bmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{Bmatrix}\left\{-\frac{1}{L}\quad \frac{1}{L}\right\}dV+\int_S \alpha\begin{Bmatrix} \dfrac{x_j-x}{L} \\ \dfrac{x-x_i}{L} \end{Bmatrix}\left\{\frac{x_j-x}{L};\quad \frac{x-x_i}{L}\right\}dS$$

and after integration we have:

$$[H]=k\begin{Bmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{Bmatrix}\left\{-\frac{1}{L}\quad \frac{1}{L}\right\}SL+\alpha\begin{Bmatrix} N_i \\ N_j \end{Bmatrix}\{N_i \quad N_j\}S.$$

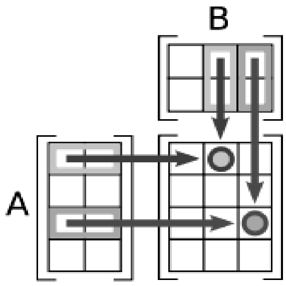

Fig. 5.4. Scheme for matrix multiplication

According to the rules of matrix multiplication A by B matrix C was obtained (for the first element fig. 5.2).

$$[H]^{(1)} = k\begin{bmatrix} \frac{1}{L^{(1)2}} & -\frac{1}{L^{(1)2}} \\ -\frac{1}{L^{(1)2}} & \frac{1}{L^{(1)2}} \end{bmatrix} SL^{(1)} = \begin{bmatrix} \frac{Sk}{L^{(1)}} & -\frac{Sk}{L^{(1)}} \\ -\frac{Sk}{L^{(1)}} & \frac{Sk}{L^{(1)}} \end{bmatrix},$$

and for finite element number 2 (Fig. 5.2):

$$[H]^{(2)} = k\begin{bmatrix} \frac{1}{L^{(2)2}} & -\frac{1}{L^{(2)2}} \\ -\frac{1}{L^{(2)2}} & \frac{1}{L^{(2)2}} \end{bmatrix} SL^{(2)} + \alpha\begin{bmatrix} N_i N_i & N_i N_j \\ N_i N_j & N_j N_j \end{bmatrix} S,$$

or:

$$[H]^{(2)} = \begin{bmatrix} \frac{Sk}{L^{(2)}} & -\frac{Sk}{L^{(2)}} \\ -\frac{Sk}{L^{(2)}} & \frac{Sk}{L^{(2)}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \alpha S \end{bmatrix} = \begin{bmatrix} \frac{Sk}{L^{(2)}} & -\frac{Sk}{L^{(2)}} \\ -\frac{Sk}{L^{(2)}} & \frac{Sk}{L^{(2)}} + \alpha S \end{bmatrix}.$$

The vector load (heat flux) expressed by the formula (5.35) can be converted to the form:

$$\{P\} = -\int_S \alpha \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} t_\infty dS + \int_S q \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} dS .$$

For the finite element 1 vector {P} is equal to:

$$\{P\}^{(1)} = q\begin{Bmatrix} N_i \\ N_j \end{Bmatrix} S = q\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} S = \begin{Bmatrix} qS \\ 0 \end{Bmatrix},$$

for the finite element 2 vector {P} is equal to:

$$\{P\}^{(2)} = -\alpha t_\infty S\begin{Bmatrix} N_i \\ N_j \end{Bmatrix} = -\alpha t_\infty S\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\alpha t_\infty S \end{Bmatrix}.$$

In order to obtain the system of equations for the area (domain) local matrix [H] must be connected to global matrix [H]:

$$[H]=\sum_{e=1}^{n_e}[H]^{(e)}.$$

The next step is the construction of the stiffness matrix. In the case of second finite element connection matrix can be describe as follows:

	1	2
1	2, 2	2, 3
2	3, 2	3, 3

The sum of matrix elements taking account of local elements of the matrix in the global matrix [H] is equal to:

$$[H]=\begin{bmatrix} \frac{Sk}{L^{(1)}} & -\frac{Sk}{L^{(1)}} & 0 \\ -\frac{Sk}{L^{(1)}} & \frac{Sk}{L^{(1)}} & 0 \\ 0 & 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{Sk}{L^{(2)}} & -\frac{Sk}{L^{(2)}} \\ 0 & -\frac{Sk}{L^{(2)}} & \frac{Sk}{L^{(2)}}+\alpha S \end{bmatrix}=$$

$$=\begin{bmatrix} \frac{Sk}{L^{(1)}} & -\frac{Sk}{L^{(1)}} & 0 \\ -\frac{Sk}{L^{(1)}} & Sk\left(\frac{1}{L^{(1)}}+\frac{1}{L^{(2)}}\right) & -\frac{Sk}{L^{(2)}} \\ 0 & -\frac{Sk}{L^{(2)}} & \frac{Sk}{L^{(2)}}+\alpha S \end{bmatrix}.$$

Load vector can be defined in the same way:

$$\{P\}=\begin{Bmatrix} qS \\ 0 \\ 0 \end{Bmatrix}+\begin{Bmatrix} 0 \\ 0 \\ -\alpha t_\infty S \end{Bmatrix}=\begin{Bmatrix} qS \\ 0 \\ -\alpha t_\infty S \end{Bmatrix}.$$

As can be seen the same result was obtained, as is shown in the equations (5.30)-(5.32).

1.1. ZADANIA RACHUNKOWE

Task 5.1. Calculate temperature in the nodes of finite element mesh for the problem of steady flow of heat in the rod (fig. 5.2).
Input data:

thermal conductivity - k=50 W/mK,
heat transfer coefficient - α=10 W/m^{2}K,
area - S=2 m^2,
element length - L=5 m, $L^{(1)}$=2.5 m, $L^{(2)}$=2,5 m,
heat flux - q= -300 W/m^2,
Ambient temperature - t_∞=400 K

Solution.
Calculation of the coefficients of the system of equations (5.32):

$$C^{(1)} = \frac{Sk}{L^{(1)}} = \frac{2m^2\ 50\frac{W}{mK}}{2{,}5m} = 40\frac{W}{K} = C^{(2)}\ ;$$

$$\alpha S = 10\,\tfrac{W}{m^2K}\cdot 2m^2 = 20\,\tfrac{W}{K}\ \ ;$$

$$qS = -150\,\tfrac{W}{m^2}\cdot 2m^2 = -300\,W\ \ ;$$

$$\alpha S t_\infty = 10\,\tfrac{W}{m^2K}\cdot 2\,m^2\cdot 40K = 8000\,W\ .$$

So we have following system of equations:

$$\begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 60 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} + \begin{Bmatrix} -300 \\ 0 \\ -8000 \end{Bmatrix} = 0$$

After its solution was with respect to *t* the following temperature were obtained: {t}={430, 422.5, 415}.

Task 5.2. Calculate the temperature at the nodes of the finite element mesh of the task 5.1 by direct minimization of the functional (5.29).
Solution.

$$J = \frac{C^{(1)}}{2}(t_1^2 - 2t_1t_2 + t_2^2) + \frac{C^{(2)}}{2}(t_2^2 - 2t_2t_3 + t_3^2) + qS_1t_1 + \frac{\alpha S_3}{2}(t_3^2 - 2t_3t_\infty + t_\infty^2)$$

C=	40
Alfa S=	20
qS=	-300
tsr=	400
J1=	800000
J2=	200000
J3=	-30000
J4=	0
J=	970000
t1=	100
t2=	300
t3=	400

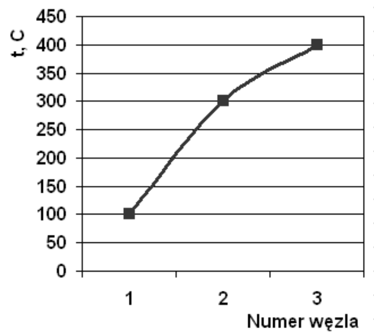

Fig. 5.4. EXCEL sheet with the given initial temperature distribution.

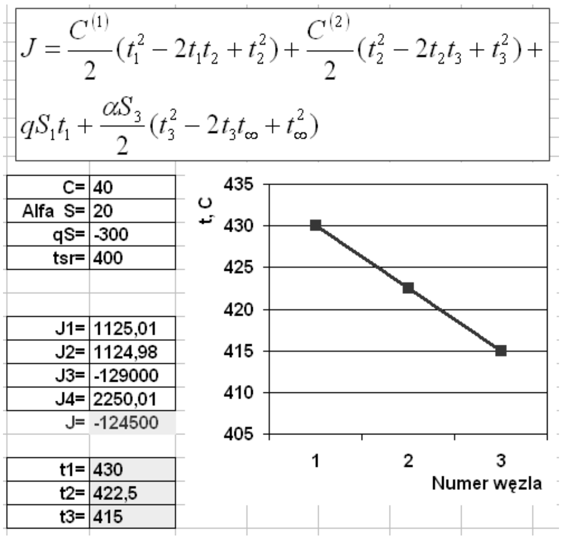

Fig. 5.5. EXCEL sheet after the minimization of functional.