Material for Functional Analysis classes in March 2024

23 marca 2024

`I will quote the references as follows:`
`[B]= B.Bollobas "Linear Analysis"; [C]= J.B.Conway, "Functional Analysis"; [A] D.Arnold's short text` symbol:= means "equals by definition". If $A, B \subset X, t \in \mathbb{K}$, then $tA := \{tx : x \in A\}$ and $A - B := \{x - y : x \in A, y \in B\}$.

The course started with the definition of "TVS" i.e. *Topological Vector Space* (say, X) over the scalar field $\mathbb{K}$, where either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We say that a set $E \subset X$ is *circled* (or balanced), if $\forall_{\alpha \in \mathbb{K}} |\alpha| \leqslant 1 \Rightarrow \alpha E \subset E$. Using the existence of a neighbourhood basis of 0 consisting of circled sets, we've proved first two theorems:

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Th. 1. Any two $n-$ dimensional, Hausdorff TVS, where $n < \infty$ are isomorphic (by a linear homeomorphism) and any linear map on such spaces are automatically continuous. (These two claims are contained in [B] Theorem2, p.60 and in Corollary 3., p,61)[1]

Th. 2. A linear functional $f : X \to \mathbb{K}$ is continuous $\Leftrightarrow \ker(f) := \{x \in X : f(x) = 0\}$ is closed $\Leftrightarrow f(U) \neq \mathbb{K}$ for some nonempty open set $U \subset X$. (the crucial part of the result is proved at the beginning of page 46 in [B])

Th. 3. If M is a finite-dimensional proper (i.e. $M \neq X$) subspace in a Hausdorff TVS X , then M is closed ([B], p.662, Corollary 7), has empty interior (otherwise- it would be a neighbourhood of 0, hence absorbing, which for subspaces implies $M = X$). Using Baire Category Theorem deduce that there is no Banach space algebraically spanned by z countable linearly independent sequence (e_n) (=[B], Exercise 22. p.84). —

Th.4. Equivalent conditions for continuity on a linear operator between normed spaces ([B],page 28/29)

Th.5. $\mathcal{B}(X, Y)$ is complete if Y is a complete normed space (i.e. a *Banach space*)- this is a part of Theorem 4. at p.30 in [B](which also verifies that $\|T\|$ as defined there - is a norm on $\mathcal{B}(X, Y)$).

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We define the Minkowski functional p_E for an absorbing set E. (Since we assume that E is convex, the absorbing property means that for any $x \in X\ \exists t > 0 x \in tE$. We show that $p_E(x) < s \Rightarrow x \in sE$, that P_E is a sublinear functional (called in [B] a convex functional) -i.e. p_E is subadditive and homogeneous with respect to positive scalars. Moreover $\{x : p_E(x) < 1\} \subset E \subset \{x : p_E(x) \leqslant 1\}$. The first inclusion is equality in the case of open set E. In [B] Minkowski functional is denoted $q(x)$ at p.28 -but only for absolutely convex sets that are not containing any line $\{tx : t \in \mathbb{R}\}$- then q is even a norm. A more detailed proof is at p. 106 in [C] (Chapter IV, Proposition 1.14 p.106 and Prop. 3.2 -at p.111)

— Th.6.(Hahn-Banach) Linear functional f on a subspace M of a real vector space X, dominated on M by a sub-linear (=convex) functional $p : X \to \mathbb{R}$ has a linear extension on X dominated on X by p. + the case when p is a semi-norm, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, $|f(x)| \leqslant p(x)$, $x \in M$. [B] Chapter 3: Theorem5,p.50, for complex scalars- Th.6 p.51.

(first group of) Corollaries:
(1) Norm- preserving extensions,([B],Theorem 6 p.51);
(2) X^* separates the points of X, ([B],Cor. 8);
(3) $\forall_x \exists_{f \in X^*} \|f\| = 1, f(x) = \|x\|$ ([B] Corollary 7.) (+ "Dual formula for the norm $\|x\|$"), (4) The canonical injection $\iota : X \to X^{**}$ is an isometry.([B] p.52 Th.10) Definition of reflexive spaces.

(Geometric corollaries): Separation of disjoint convex sets: (1) if one of them is open [C] p.113 (ChapterIV,Th.3.7 (2) if both are closed and one is compact [C] p.114 Th.3.9

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[1] The proofs are carried in [B] only in the normed spaces case, but it will suffice, if you learn this case. Start reading [B] with Chapter 2, since its first pages (19-22) contain basic definitions and notation.