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The Establishment of Functional Analysis

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This article surveys the evolution of functional analysis, from its origins to its establishment as an independent discipline around 1933. Its origins were closely connected with the calculus of variations, the operational calculus, and the theory of integral equations. Its rigorous development was made possible largely through the development of Cantor's "Mengenlehre," of set-theoretic topology, of precise definitions of function spaces, and of axiomatic mathematics and abstract structures. For a quarter of a century, various outstanding mathematicians and their students concentrated on special aspects of functional analysis, treating one or two of the above topics. This article emphasizes the dramatic developments of the decisive years 1928–1933, when functional analysis received its final unification.

Die vorliegende Arbeit gibt einen Überblick über die Entwicklung der Funktionalanalysis von ihren Anfängen bis zu ihrer Konsolidierung als ein selbständiges Gebiet um etwa 1933. Ihre Anfänge waren eng mit der Variationsrechnung, den Operatorenmethoden und der Integralgleichungstheorie verbunden. Ihre strenge Entwicklung wurde vor allem durch die Entwicklung der Cantorschen Mengenlehre, der mengentheoretischen Topologie, die präzise Definition der Funktionenräume sowie der axiomatischen Mathematik und der abstrakten Strukturen ermöglicht. Ein Vierteljahrhundert lang konzentrierten sich zahlreiche hervorragende Mathematiker und ihre Schüler auf spezielle Gesichtspunkte der Funktionalanalysis und bearbeiteten ein oder zwei der obengenannten Gebiete. Die vorliegende Arbeit betont besonders die dramatischen Entwicklungen der entscheidenden Jahre 1928–1933, in denen die Funktionalanalysis ihre endgültige Vereinheitlichung erfuhr.

Cet article porte sur l'évolution de l'analyse fonctionnelle, à partir de ses origines jusqu'à son établissement comme discipline indépendante vers 1933. Ses origines prennent racine dans le calcul des variations, le calcul opérationnel, et la théorie des équations intégrales. Son développement rigoureux est dû principalement au développement du "Mengenlehre" de Cantor, de la topologie, des définitions précises des espaces fonctionnels, de l'axiomatique, et des structures abstraites. Pendant un quart de siècle, des mathématiciens éminents et leurs élèves concentrèrent leurs efforts sur certains aspects de l'analyse fonctionnelle, en traitant un ou deux des sujets mentionnés. Cet article souligne l'importance du développement dramatique des années décisives 1928–1933, alors que l'analyse fonctionnelle se voyait définitivement unifiée.

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1. INTRODUCTION

The development of functional analysis, with its wide range of applications, was one of the major mathematical achievements of the first half of this century. In recent years, at least two books [Dieudonné 1981; Monna 1973] and several important articles have been devoted to the study of its origins and development [1].

Central to functional analysis is the concept of a *function space*. Loosely speaking, by a "function space" is meant a topological space, the "points" of which are functions. Many such spaces (for instance, all Banach spaces) are vector spaces having a "metric" d, often defined in terms of a *norm* $\|f\|$, which yields a distance $d(f, g) = \|f - g\|$ between any points f and g in the space.

The *idea* of a function space was already latent in the 19th century. However, the rigorous organization and systematization of much of analysis about the concept of a function space took nearly fifty years, roughly the first half of this century. It was made possible by the development of set theory and point-set topology (general topology), and by the general acceptance of axiomatic definitions and abstract structures. Conceptually and technically, this development owes much to the calculus of variations, the theories of differential and integral equations, and the evolution of "modern" algebra. Various complexes of unsolved practical problems and meaningful generalizations of classical analysis also had profound influence. Time was needed for the concept of an *operator* (as contrasted with a differential or integral *equation*) to evolve and become clarified. For these and other reasons, the first stages of this evolution were by no means uniform.

This article will survey the development of functional analysis from its beginnings to the time when it finally became established as a coherent branch of mathematics around 1933. It will emphasize the decisive events of the years 1928–1933, which constituted in some sense the final unifying period of this development.

To make precise the idea of a function space, one must first have clear definitions of the words "function" and "space." Accordingly, our first concern will be to recall how far these concepts had developed prior to the earliest studies of what are today called "functionals," say, prior to about 1880.

The concept of "function," taken for granted by most mathematicians today, evolved very slowly. In the work of Leonhard Euler (1707–1783) and in his time, interest concentrated on *real special functions* as they occurred in geometry, mechanics, astronomy, probability, and in other applications. Their study [Dieudonné 1978, Chap. I] constituted a wide and heterogeneous area of research, which soon included as well the classical orthogonal polynomials, model cases of general theories to come in a distant future.

To be sure, Euler thought of "arbitrary functions" as being given by their graphs, but he did nothing systematic to develop this idea. Somewhat differently, Joseph Louis Lagrange (1736–1813) based his *"calcul différentiel"* on the assumption that every function is *"analytique,"* and can be expanded locally into its Taylor series near every point.

Euler's interpretation of arbitrary (real) functions as being given by their graphs was remarkably corroborated by Joseph Fourier [2] (1768–1830), beginning around 1807 and culminating in his masterpiece *La Théorie Analytique de la Chaleur* [3] (1822), which became a landmark in the evolution of both classical analysis and mathematical physics. Fourier exploited Euler's discovery, made at age 70, of the *orthogonality* of the "trigonometric system" of functions appearing in trigonometric series

$$\frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

to show that even a *discontinuous* (periodic) function could be expanded in such a series. To honor Fourier's work, these series, with coefficients given by Euler's formulas

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx,$$

are called "Fourier series" today, even though they had been invented by Daniel Bernoulli in 1750 in connection with the vibrating string problem, and Fourier contributed none of the basic results of the *theory* of these series.

Fourier's demonstration that discontinuous functions could be represented by infinite series of continuous (even analytic) terms must have astonished his contemporaries. In particular, it fascinated Dirichlet at Berlin and later Riemann at Göttingen.

Gustav Lejeune-Dirichlet (1805–1859), who had studied in Paris and knew Fourier, gave in 1829 [*Journal für die reine und angewandte Mathematik* **4**, 157–169; *Werke*, Vol. 1, pp. 117–132] the first rigorous proof of the convergence of Fourier series for a wide class of periodic functions (those which are continuous, except for finitely many jumps, and have finitely many local maxima and minima in each period). In this paper he also defined the *Dirichlet function*, which equals c for rational and $d \neq c$ for irrational values of the argument, pointing out that "the various integrals [in the Fourier series] lose every meaning in this case."

Consequently, it hardly seems by chance that a few years later, in another article on Fourier series published in 1837, Dirichlet formulated the first "modern" definition of an "arbitrary function" on a real interval $[a, b]$: to each $x \in [a, b]$ is assigned a unique $y = f(x) \in \mathbb{R}$ [Dirichlet, *Werke*, Vol. 1, pp. 133–160].

Even before Dirichlet's efforts to rigorize Fourier's conclusions, Augustin-Louis Cauchy (1789–1857) had done much to clarify the notion of function. Not only did he provide a fairly plausible "proof" of the fact that every continuous function is integrable, but he also gave an example of a bounded, infinitely differentiable function (namely, $f(x) = \exp(-x^{-2})$ when $x \neq 0, f(0) = 0$) that cannot be expressed near $x = 0$ by a series in powers of x. By establishing the fact that for functions of a *complex* variable, continuous differentiability implies analyticity, he also went a long way toward giving complex function theory its modern form.

2. CONCEPT OF FUNCTION AROUND 1880

However, it was above all Bernhard Riemann (1826–1866) and Karl Weierstrass (1815–1897) whose ideas dominated function theory, real and complex, in 1880. Building on Dirichlet's work, Riemann's 1854 *Habilitationsschrift*, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe" [4] [Riemann 1892, 227–264] was the next great advance in the theory of Fourier series. Riemann was inspired to create an integration theory for *bounded* functions, far more rigorous and more general than Cauchy's earlier integration theory for *continuous* functions (see [Birkhoff 1973]).

Riemann's 1854 work was also the point of departure for many subsequent investigations on Fourier series and real functions, by Georg Cantor's colleague Eduard Heine (1821–1881) in 1870, the Italians Giulio Ascoli (1843–1896) in 1873 and Ulisse Dini (1845–1918) in 1874, and Weierstrass' former student Paul du Bois-Reymond (1831–1889) in the same year and subsequently. These papers concerned questions of convergence, termwise integrability, sets of discontinuity, etc., and are typical of the development of greater rigor and generality in dealing with functions [5].

The most influential exponent and promoter of rigor around 1880 was Weierstrass. Indeed, throughout his long life, Weierstrass emphasized the importance of rigorous analytic formulations, in contrast to Riemann, who also used geometrical and physical intuition. His emphasis on precise definitions and generality in complex analysis, as well as the spirit of his partially critical contributions to real analysis, made "*Weierstrassian rigor*" (a term coined by Felix Klein) proverbial (cf. [Dieudonné 1978 1, 370–373; Birkhoff 1973, 71–72]).

Essential for rigor is the concept of *uniform convergence*. This first appeared in papers by Stokes in 1847, von Seidel in 1848, and Cauchy in 1853, but it was Weierstrass who discovered it first (in 1841; cf. [*Werke*, Vol. 1, p. 67]), named it, and made its fundamental importance generally appreciated.

Many basic questions about functions were still unresolved in 1880 or had just been settled. For instance, whether nonuniform convergence of a series implies the discontinuity of the sum function remained open for many years until 1875, when Darboux and (independently) du Bois-Reymond answered it in the negative sense. Again, for decades it was believed that every continuous function has a Fourier series which converges to it everywhere, until du Bois-Reymond [*Göttinger Nachrichten*, p. 571] gave a counterexample in 1873. Many other instances are described in [Hawkins 1975, Chaps. 1–3]; see also [Birkhoff 1973, Selection 32].

A new perspective on functions was given by Weierstrass' idea of "approximately representing continuous functions by polynomials" [Weierstrass' approximation theorem, 1885; *Werke* 3, 1–37]. Since the theorem referred to uniform approximation over any closed bounded interval I, it gave new insight into the "space" (cf. Section 3) $C(I)$ by showing that the *polynomials* are dense in $C(I)$.

Finally, very important for the evolution of functional analysis in its early stages was the critical work of Weierstrass on the *calculus of variations*. Specifically, using the classical technique of setting up a one-parameter family of functions ("admissible functions," "admissible curves")

$$f_\varepsilon = f + \varepsilon g$$

with the parameter ε restricted to some finite interval, Weierstrass introduced a "distance" (actually, several such distances) between members of this family, thereby implicitly treating each such function as a *"point"* in a (very special) *"function space,"* an idea which is at the root of the functional analytic approach.

Variational problems are discussed again in Section 6. For the moment, it suffices to call attention to the fact that early "functional analysis" (a name first used in 1922 by Paul Lévy; see below, Section 13) had variational ideas among its main stimuli. The work of Arzelà (see Section 4) confirms this clearly.

3. CONCEPT OF "SPACE" AROUND 1880

If the concept of "function" was still evolving in 1880, that of "space" was even more rudimentary. Without doubt, the spectacular development of various geometries during the 19th century, beginning with non-Euclidean geometries (Gauss, Lobachevsky, Bolyai) and culminating in 1872 in Klein's *Erlanger Programm,* had profound influence on the idea of a general "space."

Curiously, the general concept of a space of arbitrary (finite) dimension seems to have been suggested by mechanics. Lagrange's *Mécanique Analytique* (1788) discusses dynamical systems whose configuration depends on arbitrarily many coordinates $q_1, \ldots, q_r$. For example, the n-body problem of celestial mechanics has a $3n$-dimensional "configuration space." Such configuration spaces, and later "phase spaces", were intensively studied in the 19th century by Liouville, Hamilton, Jacobi, Poincaré, and others.

In 1844, Arthur Cayley (1821–1895) wrote about "analytical geometry of n dimensions" [*Works,* Vol. 1, p. 55], and in the same year Hermann Grassmann published his very original *Ausdehnungslehre* [calculus of extension], which contains the concept of an *n-dimensional vector space*. The Preface of this earliest axiomatic discussion of multilinear algebra mentions Lagrange's *Mécanique Analytique* as a source of inspiration. But unfortunately, Grassmann's abstract approach was so obscurely worded that even a completely reorganized version published in 1862 was not widely appreciated for some time.

Riemann and topology. Far more influential was Bernhard Riemann. Actually, the *idea* of a "function space" already appeared in his famous doctoral thesis of 1851 "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse" [6], where he says [p. 30]:

> The totality of the functions forms a connected domain closed in itself [*ein zusammenhängendes in sich abgeschlossenes Gebiet*], since each of these functions can go over continuously into every other. . . . [Riemann 1892, 3–48]

Riemann has been called the initiator of topology [Bourbaki 1974, 175]. For instance, in his work on algebraic functions and their integrals he introduced the "Betti numbers." He did this first for surfaces [*Ibid.*, 92–93], and later [pp. 479–

482] for manifolds of any dimension, applying these numbers to the periods of Abelian integrals, hence to a problem in Analysis. A subtitle on page 91 reads "Theorems of Analysis Situs for the Theory of Integrals . . . ," and he says that this concerns "that part of the theory of continuous quantities which completely disregards metric properties [*Massverhältnisse*] . . . ."

In his famous 1854 *Habilitationsvortrag* "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" [Riemann 1892, 272–297], Riemann elaborated on the conceptual aspect and general role and character of space in geometry [with a corresponding outline on Riemannian metrics in a subsequent paper presented to the Paris Academy in 1861; *Ibid.*, 391–423]. Here he formulated in a nutshell the idea of function spaces of infinite dimension in the form:

> . . . But there also exist manifolds in which the determination of location [*die Ortsbestimmung*] requires not a finite number but either an infinite sequence or a continuum of determinations of quantities [. . . *sondern entweder eine unendliche Reihe oder eine stetige Mannigfaltigkeit von Grössenbestimmungen erfordert*]. Such a manifold, for instance, is formed by the possible determinations of a *function* for a given domain. . . . [Riemann 1892, 276]

This talk was published in 1868 (by Dedekind), two years after Riemann's early death. It attracted general attention, but there seems little doubt that these revolutionary ideas were understood and accepted only very slowly [7]. Indeed, it was only around 1870 that Richard Dedekind (1831–1916), Georg Cantor (1845–1918), and Charles Méray (1835–1911) showed how to construct the *real number system* rigorously from the integers. Their constructions provided solid foundations for the "arithmetization of Analysis" that took place (thanks to "Weierstrassian rigor") in the last quarter of the 19th century.

Dedekind, a pioneer of modern abstract algebra, recognized that to clarify Riemann's topological ideas, the nature of the real field $\mathbb{R}$ had to be analyzed in depth. He began to do this in 1858, but published his ideas in definite form only in 1872 (*Stetigkeit und irrationale Zahlen*) and 1888, in an even more fundamental study entitled *Was sind und was sollen die Zahlen?* [8]. Meanwhile, the first rigorous theory of irrational numbers, by C. Méray, had appeared in 1869.

Dedekind was also a precursor on metric spaces. Indeed, his paper "*Allgemeine Sätze über Räume*" [9] was an attempt to construct a theory of $\mathbb{R}^n$ *ab ovo*, without appeal to geometric intuition.

Cantor's "Mengenlehre." Functional analysis, as we know it today, depends crucially on *set theory* [*Mengenlehre*], founded by Georg Cantor (1845–1918), a pupil of Weierstrass, at Halle. Cantor was motivated by his study of Riemann's work on trigonometric series and, beyond mathematics, by ideas from Scholasticism. His first paper on sets, published in 1874, sharply distinguished, for the first time, between countable infinity and the power of the continuum c, by showing that the set of all real numbers is not denumerable, whereas the set of all algebraic real numbers is denumerable [10]. This gave as an immediate corollary the fact that almost all real numbers are transcendental. More important for us, it opened up totally new vistas in analysis as well, initiating a classification of infinite sets. Thus, it gave meaning to the concept of a *countably* additive measure, to be developed by Borel and extended by Lebesgue into a radically new theory of integration; see Section 5.

In 1877, Cantor made a second revolutionary discovery: that the cardinality of Euclidean n-space $\mathbb{R}^n$ is independent of n, its dimension [11]. This constituted a radical departure from accepted ideas such as the facile definition of the "dimension" of a "space" as the number of coordinates required to specify its "points," which had been standard before Cantor proved that $\mathbb{R}$ and $\mathbb{R}^n$ with any $n \in \mathbb{N}$ have the same cardinality. Cantor himself was shaken by this discovery of 1877, which was different from what he had hoped to find, and which seemed to undermine the concept of dimension itself. However, Dedekind reassured Cantor, pointing out that it should be possible to prove that $\mathbb{R}^n$ and $\mathbb{R}^m$ with $m \neq n$ are not homeomorphic (not his term, of course). The radicalism of Cantor's ideas perhaps explains their hostile rejection by Leopold Kronecker (1823–1891) and other mathematicians of an older generation, except for Weierstrass, who observed the efforts of his former student with interest.

Topology in 1900. Apart from defining the notion of the *derived* set S' of a given set S, and the associated notion of a "perfect" set (one satisfying $S = S'$), all of the above writings were rudimentary and largely intuitive insofar as the *topology* of the plane and higher-dimensional spaces are concerned. Indeed, it was not until about 1910 that the foundations of topology became rigorously formulated, even for *finite*-dimensional spaces. It is therefore not surprising that considerable vagueness surrounded the notion of *infinite*-dimensional function space throughout the 19th century, even after Cantor's work had gained wide recognition.

4. ITALIAN PIONEERS

It is generally agreed that functional analysis, properly speaking, originated in Italy. During the last four decades of the 19th century, there occurred a powerful resurgence [*risorgimento*] of Italian mathematical creativity. First came three great geometers, Betti, Beltrami, and Cremona, and not long after six notable analysts, each of whose contributions related to early functional analysis we will discuss individually: Giulio Ascoli (1843–1896), Cesare Arzelà (1847–1912), Ulisse Dini (1845–1918), Giuseppe Peano (1858–1932), Salvatore Pincherle (1853–1936), and Vito Volterra (1860–1940).

Ascoli's theorem. Ascoli and Arzelà proved what was probably the first substantial mathematical theorem about functional analysis, published in 1883–1884 [12]. If a sequence $\{f_m\}$ of real-valued functions on [0,1] is uniformly bounded and *equicontinuous* on [0,1], then $\{f_n\}$ contains a *uniformly* convergent subsequence. This theorem essentially generalizes the Bolzano–Weierstrass theorem to the infinite-dimensional function space $C[0,1]$. The latter asserts that any bounded sequence $\{x_n\}$ of real numbers contains a convergent subsequence—and more generally that the same is true in n-dimensional space $\mathbb{R}^n$. Actually, Ascoli's theorem continues to hold in more general settings. For instance, it holds with [0,1] replaced by any closed, bounded subset of $\mathbb{R}^n$.

Dini participated in the finer and more rigorous study of convergence problems centered around uniform convergence ("Dini's theorem") and variants of it, convergence of Fourier series ("Dini test"), generalizations of Fourier series foreshadowing eigenfunction expansions ("Dini series" being Fourier–Bessel series) and deeper aspects of differentiability ("Dini derivatives") [13]. He also played an influential role as a teacher of Volterra.

Peano was a creative and individualistic personality, very original and independent in his work [14]. He took a step forward in a book written in 1888 and intended to popularize the *Ausdehnungslehre* of Grassmann (see above) [15]. There, in Chapter IX, he gave examples of infinite-dimensional vector spaces, along with a rather modern axiomatic definition of a vector space.

Two years later, Peano published his "space-filling curve," a *continuous* surjection of an interval onto a square, whereas Cantor had obtained earlier a *discontinuous* bijection. This further discredited purely intuitive topology, and reinforced the Weierstrassian insistence on uncompromising logical rigor.

Peano's *Formulaire de Mathématiques* [16] was enormously influential for mathematical *logic and foundations*, and Peano's symbolism ["pasigraphy"] was (with modifications) adopted by A. N. Whitehead and B. Russell, E. H. Moore, and many others. Also, for the next ten years, Peano became one of the leaders in the field. Beyond all this, his book helped to promote the abstract approach to mathematics, including the idea that all mathematical deductions could be formalized. Peano's contributions to integration theory, made at about the same time, are discussed in [Hawkins 1975, Chap. 4].

Pincherle was a pioneer enthusiast for functional analysis. His influence pertains to the early phase of developments in the field. A distinguished and prolific scholar, he was fascinated by the operational calculus from 1885 on. In his book *Le Operazioni Distributive e le loro Applicazioni all'Analisi*, co-authored with his pupil Ugo Amaldi, he gave a systematic exposition of his ideas as of 1901 [17]. However, he had been writing about "function spaces" for some years earlier, suggesting the terms "spazio funzionale," "operazioni funzionali," and "*calcolo funzionale*," and concentrating on linear operators on complex sequence spaces, whose "points" he then regarded as coefficient sequences of Taylor series, thus relating his work to Weierstrassian complex analysis [18].

Although an influential proponent of the abstract point of view, his emphasis was primarily on *algebraic* formalisms. For example, in connection with the differential operator $D = d/dt$ he analyzed and generalized formulas such as $e^{hD} = I + \Delta_h$, where $\Delta_h u = u(x + h) - u(x)$. He paid little attention to questions about the *continuity* of operators or to convergence problems. His 1906 article "Funktionaloperationen und -gleichungen" in the German *Encyklopädie* [*EMW*], a translated and somewhat updated version of which was published in the French *Encyclopédie* of 1912, surveys much 18th- and 19th-century work. However, it still largely ignores Weierstrassian rigor, which had come to dominate the foundations of analysis during the preceding decades, and it had little effect on the later development of functional analysis [19].

Volterra. As will become apparent, Volterra influenced the development of functional analysis for a long time, and in many ways. A student of Dini's and later his colleague at Pisa, it was Dini who introduced him to the theory of real functions which was then developing, and who guided his early work. For instance, as a student of age 21, he proved two important conjectures of Dini's by constructing (i) a nowhere dense set of positive outer content, and (ii) a function the derivative of which is bounded but not Riemann-integrable [20].

His first paper on a topic truly belonging to "functional analysis" appeared in 1887 [21]. In this and several later papers (all of 1887), Volterra investigated (special classes of) functionals (this term being Hadamard's, suggested as a noun only in 1904 or 1905; cf. [Taylor 1982]). He first called them "functions of functions" and later, to avoid misunderstanding, *"functions of lines"* [*funzioni dipendenti da linee, fonctions de lignes*]. These were defined as continuous mappings $X \to \mathbb{R}$, where X is a set of continuous curves (continuous functions on $[a, b]$ with range in $\mathbb{R}$ or $\mathbb{R}^n$).

In these papers, Volterra's intention was "to clarify the concepts which I believe need to be introduced to extend Riemann's theory of functions of complex variables and which, I think, can recur usefully also in various other researches." This may reflect Betti's influence; Betti was Riemann's friend and Volterra's teacher at Pisa. Since this seems to be the earliest known study of functionals as such, 1887 is generally considered the birthyear of functional analysis.

Arzelà. Two years later, Arzelà made a first attempt to justify "direct" variational arguments like the Dirichlet principle by using sequential compactness concepts. A brief resumé of his efforts and related developments may be found in Volterra's Madrid lectures [Volterra 1930, Chap. VI, Sect. I, §1]. Actually, Arzelà's methods were much closer to what we think of as "functional analysis" today than were those used by Volterra.

Arzelà's interest in the foundations of the calculus of variations was presumably stimulated by Weierstrass' 1870 counterexample to the conjecture that all functionals that were bounded below could be minimized. Namely [*Werke*, Vol. 2, pp. 49–54] the integral

$$\int_{-1}^{1} [x\phi'(x)]^2 \, dx$$

is nonnegative, yet it is not minimized by any function in the set of all real-valued continuously differentiable functions satisfying $\phi(-1) = a$, $\phi(1) = b$, $a \neq b$ (cf. [Birkhoff 1973, 390]).

5. HADAMARD AND FRÉCHET: 1897–1906

Jacques Hadamard (1865–1963) and Maurice Fréchet (1878–1973) played major roles in the establishment of functional analysis. To appreciate their early contributions, one must recall the extent to which Paris was a center of brilliant mathematical activity around 1900. Camille Jordan (1838–1921) and Gaston Darboux (1842–1917) were still active, and Charles Hermite (1822–1902) was still alive.

Moreover, Hermite's student Henri Poincaré (1854–1912) was the world's leading mathematician; while Hermite's son-in-law Emile Picard (1856–1941), Edouard Goursat (1858–1936), Hadamard, and many others had achieved international fame or were on the way to it. Complex analysis and the differential equations of classical physics formed the main streams of mathematical interest.

But the scene was about to change, mainly due to the work of Emile Borel (1871–1956), René Baire (1874–1932), and Henri Lebesgue (1875–1941). These notable mathematicians were about 30 years younger than Ascoli, Arzelà, and Dini, and about 10 years younger than Volterra. Unlike their Italian predecessors, they were strongly influenced by Cantor's set theory, and used it to found new theories of measure and integration.

Early attempts to define a "measure" of sets (cf. [Hawkins 1975, Chap. 3]) were followed in 1887 by Peano's book *Applicazioni Geometriche del Calcolo Infinitesimale,* and in 1892 by Jordan's paper on "content," motivated by the conceptual difficulties in double integration. Although Jordan's content was not yet general enough, his idea of a measure-theoretic approach to the Riemann integral had great influence on Borel (and later on Lebesgue).

Borel. In his 1894 doctoral thesis (on a continuation problem in complex analysis considered earlier by Poincaré), and in more detail in his 1898 book *Leçons sur la Théorie des Fonctions,* Borel constructed the first *countably* additive measure. He also introduced what were later called "Borel sets" (obtained from open sets by iterating the processes of forming *countable* unions and differences). He then defined for Borel sets a "measure" with the key property that

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k) \tag{5.1}$$

for *disjoint* (Borel) sets. Borel's proof of the existence of this measure made essential use of the fact that, if a sequence of open intervals $I_k = (a_k,b_k)$ *covers* the unit interval $I = [0,1]$, then

$$\sum_{k=1}^{\infty} (b_k - a_k) > 1$$

[*Oeuvres*, p. 842]. This, in turn, is a corollary of the

HEINE–BOREL THEOREM. *If a family of open sets covers a closed, bounded set S in $\mathbb{R}^n$, then there is a finite subset of the family which already covers S.*

Baire. In 1899, Baire's doctoral thesis "Sur les fonctions de variables réelles" appeared in *Annali di Matematica Pura ed Applicata* **3**(3), 1–122 (by invitation of Dini). In order to characterize limits of convergent sequences of continuous functions (and their limits, etc.), Baire defined [p. 65] a subset of $\mathbb{R}$ to be of *first category in* $\mathbb{R}$ when it is the union of countably many nowhere dense sets in $\mathbb{R}$. As the result basic to functional analysis, he proved "Baire's theorem" that $\mathbb{R}$ is of the "second category" (i.e., not of the first) in itself, a result which he extended to $\mathbb{R}^n$ in 1904 [22].

Lebesgue. Borel's new techniques were developed much further in Lebesgue's 1902 doctoral thesis "Intégrale, longueur, aire," published in *Annali di Matematica* 7(3), 231–359. This inaugurated the "modern" theory of integration, involving the concepts of Lebesgue measure, measurable function, and integral [23]. In it, Lebesgue established the great power, generality, and elegance of his new integral, applying it to Fourier series and other problems. In particular, he demonstrated its flexibility in limit processes, such as taking limits under the integral sign,

$$\lim_{n\to\infty} \int f_n(x)\, dx = \int \left(\lim_{n\to\infty} f_n(x)\right) dx, \tag{5.2}$$

which now became valid under very general assumptions.

His lectures on the subject at the Collège de France in 1902–1903 were published in his 1904 book *Leçons sur l'Intégration et la Recherche des Fonctions Primitives* (2nd enlarged ed., 1928). Although Hermite and Poincaré were unenthusiastic about its generality, the Lebesgue integral was to prove fundamental for functional analysis, as we shall see.

By 1905, Borel, Lebesgue, and Baire had all written monographs for a new series initiated by Borel, in which the "théorie des ensembles" was applied to sets of *functions,* and especially to the topics treated in their theses. Moreover, Baire, Borel, Lebesgue and Hadamard published a sequence of letters in the *Bulletin de la Société Mathématique de France* **33** (1905), 261–273, which helped to clarify the foundations of Cantor's still new set theory.

Hadamard. Although Hadamard published comparatively little about functional analysis, he greatly influenced its evolution. His first paper on functional analysis was a short note presented at the First International Congress of Mathematicians, held at Zürich in 1897. From 1897 to 1906, Hadamard and his student Fréchet would develop set-theoretic ideas into a new tool of functional analysis.

At the time, Hadamard was best known for his work in complex analysis and on the distribution of primes (see [Birkhoff 1973, 98–103]), but he was soon to become famous for his work on partial differential equations. Hadamard's note called attention to the significance which an application of the ideas of Cantor's set theory to *sets of functions* might have, remarking [24]

> Mais c'est principalement dans la théorie des équations aux dérivées partielles de la physique mathématique que des études de cette espèce joueraient . . . un rôle fondamental.

On this note in the *Verhandlungen* of the Congress [1897, pp. 201–202], Pincherle and Borel commented critically.

Hadamard was soon to turn his attention to the theory of partial differential equations. Here his concept of a "well-posed problem" has become classic [25]; its requirement that "the solution must depend continuously on the initial and boundary conditions" obviously refers to an assumed topology on the space of functions considered, and should be regarded as an interpretation of his 1897 remark.

Hadamard continued to explore the ideas in his note, first in a short paper [*Bulletin de la Société Mathématique de France* **30** (1902), 40–43] on Volterra's derivatives of "fonctions de lignes," and in 1903 in an important note [*Comptes Rendus* (Paris) **136,** 351–354] in which he suggested considering "functionals" on *arbitrary* sets. In this note, he showed that every bounded linear functional U on the space $C[a,b]$ can be represented in the form

$$U(f) = \lim_{m\to\infty} \int_a^b f(x)H_m(x)\,dx; \tag{5.3}$$

here the H_m are also continuous on $[a,b]$, but are not uniquely determined by U.

Maurice Fréchet (1878–1973) had been Hadamard's student in a lycée in 1890–1893, and had been advised by him ever since (see [Taylor 1982]). He quickly developed Hadamard's ideas on functionals in two papers published in the recently founded *Transactions of the American Mathematical Society* [**5** (1904), 493–499; **6** (1905), 134–140]. In the first of these, Fréchet gave a new proof of Hadamard's representation (5.3) which, at the same time, yielded a series expansion of U (analog of the Taylor series). Near the end of this paper, he used the interchange of limit and Lebesgue integration similar to that in (5.2) to construct a sequence of continuous functions $H_n(x)$ whose limit $K(x)$ is Lebesgue- but not Riemann-integrable, so that $\int f(x)K(x)\,dx = U(f)$ is defined only in the Lebesgue sense. He observed that this shows the value of "not rejecting as too artificial any functions which are L- but not R-integrable."

In his second *Transactions* paper, Fréchet proved that any bounded linear functional U on $C^n[a,b]$ can be represented in the form

$$U(f) = \sum_{j=0}^{n-1} A_j f^{(j)}(a) + \lim_{p\to\infty} \int_a^b H_p(x) f^{(n)}(x)\,dx. \tag{5.4}$$

In a third paper in the same volume of the *Transactions* (*Ibid.*, 435–449), generalizing Weierstrass' idea of a "neighborhood" of a function, Fréchet defined a metric "distance" [*écart*] between pairs of curves parametrically represented by uniformly continuous functions, and looked for conditions on a family of such curves sufficient to imply *compactness* in the sense of the theorem of Ascoli and Arzelà.

Fréchet's thesis. Especially this last paper can be regarded as a partial prepublication of Fréchet's famous doctoral thesis of 1906, "Sur quelques points du calcul fonctionnel," which appeared in *Rendiconti del Circolo Matematico di Palermo* **22,** 1–74. This was a landmark that had enormous influence [26] on the development of both functional analysis and point-set topology. One can only speculate about how much it owes to Hadamard; Fréchet's necrology of 1963 [*Comptes Rendus* (Paris) **257,** 4081–4086] was surely far too modest!

In his thesis, Fréchet introduced the notion of a *metric space*, using Jordan's term "écart" [*Journal de Mathématiques* **8**(4) (1892), 71] for "metric" [p. 30 of the thesis]. The name "metric space" was later coined by Hausdorff. Fréchet's definition is amazingly modern (precisely that used now), and constituted a great advance over the techniques of Volterra, who always referred to sets of curves, surfaces, and functions, and never to "elements" of a space satisfying certain axioms.

Fréchet introduced the notions of *compactness, completeness,* and *separability* into point-set theory, in the context of infinite-dimensional function spaces, and clearly recognized and emphasized the importance of these concepts. This went far beyond Cantor's "perfect sets," or the concepts in the *Bericht* by Schoenflies on the topology of $\mathbb{R}^n$, or the techniques used by Volterra in treating "functions of lines."

Fréchet devoted a substantial part of his thesis to the discussion of special spaces, as opposed to general theory. In particular, he considered the space $C[a,b]$ (not his notation, of course) stating that it was "first used systematically by Weierstrass" [p. 36]. Fréchet's work, like that of Hadamard, incorporated ideas of many earlier mathematicians; thus his (sequential) compactness was inspired by the theorems of Arzelà and Ascoli as well as by the earlier Bolzano–Weierstrass theorem.

In his thesis, Fréchet also attempted to characterize *nonmetric* features which are common to both sets of points and sets of functions. His studies of special spaces, some of them intimately connected to problems of classical analysis, made obvious the great variety of infinite-dimensional topological spaces which arise naturally in analysis. Thus he discussed examples of what were later called *limit spaces* (his "classes (L)"). He realized that his limit concept was so general that in classes (L), derived sets may not even be closed [p. 17]. In order to obtain a richer theory, Fréchet also introduced more special spaces in which derived sets are closed. He called them "classes (V)" (V for *"voisinage,"* meaning a number axiomatically associated to pairs of points). However, in 1910 he conjectured that these are actually metric spaces, as was finally proved in 1917 by E. W. Chittenden [*Transactions of the American Mathematical Society* **18,** 161–166].

6. CALCULUS OF VARIATIONS

Much as nascent point-set topology provided the necessary conceptual foundation for the *theory* of functional analysis, the calculus of variations and—somewhat later—the theory of integral equations provided some basic *techniques* as well as many of the most impressive early *applications* of functional analysis. We will discuss this influence of the calculus of variations in the present section, and that of integral equations in Section 7.

Variational principles such as "a straight line segment is the shortest path between two points in space" and "of all the plane regions having a given perimeter, the circular disk has the greatest area" date from antiquity. And variational problems from mechanics sprang up almost immediately after the invention of the calculus. However, the question of the *existence* of a curve or surface minimizing some positive quantity (a "functional" on the "space" of all curves or all surfaces satisfying certain conditions) was not considered carefully until the second half of the 19th century.

Thus, why should there exist a "path of *shortest* time"

$$t = \int_P^Q ds/v(x)$$

joining two points P and Q? Physically, the existence of shortest paths seems almost obvious—this may help to explain why the existence problem was not discussed systematically before Weierstrass. Again, why should there exist a function ϕ minimizing the *Dirichlet integral*

$$D(\phi,\phi) = \int_R \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] dV \tag{6.1}$$

on a given region R in space and assuming specified values on the boundary of R? Or a surface of least area spanning a given simple closed space curve (the Plateau problem)?

It was in the calculus of variations that the idea of a *distance between functions* arose first, in the special context of a one-parameter family of functions defined by

$$y_\varepsilon(x) = y(x) + \varepsilon\eta(x) \tag{6.2}$$

("admissible functions"). Consider the problem of finding a function which minimizes the functional

$$J[y] = \int_a^b F(x, y(x), y'(x))\, dx \tag{6.2'}$$

on the set of all twice continuously differentiable admissible functions on $[a, b]$ having given values $y(a) = c$ and $y(b) = d$. To make all $y_\varepsilon(a) = c$ and $y_\varepsilon(b) = d$, we require $\eta(a) = \eta(b) = 0$. If y minimizes J, then $\partial J/\partial\varepsilon = 0$ when $\varepsilon = 0$ for all such η. This implies Euler's famous equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}. \tag{6.3}$$

Similar ideas were used in a more "functional-analytic" spirit by Volterra in his first papers on "functions of lines" of 1887 [Volterra 1954–1962 1, 294–328]. However, it was first in Fréchet's thesis that the interpretation in terms of distance [*écart*] in an infinite-dimensional function space was given. A Weierstrassian metric $\max|f(x) - g(x)|$ [Fréchet 1906, 36] shows some of the initial inspiration and association of ideas.

Four years later, the Foreword of Hadamard's *Leçons sur le Calcul des Variations* began:

> The calculus of variations is nothing else than a first chapter of the doctrine called today the Functional Calculus, and whose development will doubtlessly be one of the first tasks of the Analysis of the future.

This statement was followed by a chapter [pp. 281–312] entitled "Generalizations. The Functional Calculus," which concluded with an analysis of the variation of

Green's and Neumann's functions with the domain and points concerned. (See also C. Carathéodory's review [*Bulletin de la Société Mathématique de France* **35** (2) (1911), 124–141], which is quite enthusiastic about the new functional-analytic spirit of the book, calling it a landmark in the history of the field.)

Dirichlet's principle. The problem of finding a function ϕ which assumes given boundary values on the boundary of a domain Ω, and satisfies $\nabla^2\phi = 0$ in Ω, is called the *Dirichlet problem* (for the Laplace equation). The assertion that such a solution ϕ can be constructed as the function that minimizes the Dirichlet integral (6.1) subject to the boundary conditions—including the claim that such a function exists—was called the *Dirichlet principle* by Riemann, who had attended Dirichlet's lectures in Berlin for two years [27].

The claim that this minimum exists was based on the fact that the integral (6.1) is bounded below (by zero). Indeed, in lectures given at Göttingen in 1856–1857, but not published until 1876, Dirichlet had claimed that "it is immediate [*es liegt auf der Hand*] that the integral (6.1) . . . has a minimum because it cannot be negative" [28].

Riemann used the principle in his doctoral thesis of 1851. There, on page 30, denoting by L the Dirichlet integral (in two dimensions) and by Ω the integral

$$\int \left[\left(\frac{\partial\omega}{\partial x} - \frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\omega}{\partial y} + \frac{\partial\beta}{\partial x}\right)^2\right] dT,$$

he says (in extension of the quotation in Section 3 above; see [29]):

> Die Gesammtheit der Functionen λ bildet ein zusammenhängendes in sich abgeschlossenes Gebiet, indem jede dieser Functionen stetig in jede andere übergehen, sich aber nicht einer längs einer Linie unstetigen unendlich annähern kann, ohne dass L unendlich wird (Art. 17); für jedes λ erhält nun, $\omega = \alpha + \lambda$ gesetzt, Ω einen endlichen Werth, der mit L zugleich unendlich wird, sich mit der Gestalt von λ stetig ändert, aber nie unter Null herabsinken kann; folglich hat Ω wenigstens für Eine Gestalt der Function ω ein Minimum.

Actually, the Dirichlet principle had been suggested by Gauss in 1839, and stated clearly by Kelvin in 1847 [Birkhoff 1973, 379]. Weierstrass had criticized it as a method of proof for some time, but Felix Klein states [*Werke,* Vol. 3, pp. 492] that Riemann "attached no special importance to the derivation of his existence theorems," and was unimpressed by these criticisms.

After Riemann's death in 1866, and especially after Weierstrass had constructed the counterexample discussed in Section 4, the criticisms of Weierstrass bore fruit. Although his lectures on the calculus of variations were only available through notes by his students, they helped to spark great activity in the field, by du Bois-Reymond, Poincaré, A. Kneser, Hilbert, Hadamard, and others.

Concerning the Dirichlet principle, H. A. Schwarz [*Gesammelte mathematische Abhandlungen* **2,** 175–190] had already published his alternating method [*alternierendes Verfahren*], which enabled him to prove the existence of a solution of the Dirichlet problem in any plane domain bounded by piecewise analytic curves. (In the same year, Carl Neumann proposed solving the Dirichlet problem with the help of integral equations; see the next section.)

Next, in a magnificent paper published in 1890 (preceded by a note of 1887 [*Comptes Rendus* (Paris) **104,** 44], Poincaré [1950–1956 9, 28–113; Birkhoff 1973, 395–399] showed that the Dirichlet problem for the Laplace equation has a solution under very mild restrictions. He proved this by a very ingenious "*méthode du balayage*" (sweeping-out process [30]) that foreshadowed the "relaxation methods" to be developed by R. V. Southwell 40 years later. With the aim of justifying generalizations of Fourier's method of orthogonal expansions to the Helmholtz equation in general domains, Poincaré then made effective use of the Rayleigh quotient

$$\rho(\phi) = D(\phi,\phi)/(\phi,\phi) = \left[\int_R |\nabla\phi|^2 \, dV\right] \Big/ \left[\int_R \phi^2 \, dV\right], \tag{6.4}$$

showing that each eigenfunction is characterized by a "minimax" property [31].

In two papers published in 1900 and 1901 [reprinted in 1905; see Hilbert 1932–1935 III, 10–37], Hilbert brilliantly revived and generalized the Dirichlet principle as a "guiding star for finding rigorous and simple existence proofs". He worked out two cases in detail: (a) shortest curves on a surface; and (b) the Dirichlet problem for a plane domain bounded by a curve with continuous curvature, and continuously differentiable boundary values. His "direct method" involved (i) first constructing a "minimizing sequence" of approximate solutions ϕ_n, with the property that $J[\phi_n] \downarrow \operatorname{Inf}\{J[\phi]\}$; (ii) next making restrictions on the class of admissible ϕ sufficient to guarantee the existence of a convergent subsequence tending uniformly to some ϕ_0; and (iii) finally, showing that $J[\lim \phi_n] \leqq \lim J[\phi_n]$.

The 19th and 20th Problems in Hilbert's famous 1900 list of 23 unsolved problems are concerned with applying his new "direct method" to other problems (e.g., involving variable coefficients, $n \geqq 3$ independent variables, or even nonlinear), and showing that the solution obtained is necessarily *analytic*. Although S. N. Bernstein and others were able to handle the quasilinear variable-coefficient case for $n = 2$ by 1910, Plateau's problem was not successfully treated until the 1930s, and the case $n \geqq 3$ was not satisfactorily resolved until after 1950 (see Serrin and Bombieri in [Browder 1976, 507–535]).

In the meantime, critical publications by Hadamard in 1906 [*Oeuvres*, Vol. 3, pp. 1245–1248] and Lebesgue in 1907 [*Oeuvres*, Vol. 4, pp. 91–122] showed that Hilbert's "direct methods" were by no means adequate for all cases. Also, an interesting example of nonexistence was provided by Lebesgue in 1913 [*Oeuvres*, Vol. 4, p. 131]: he constructed a region with a very sharp reentrant spine, on which the Dirichlet problem is *not solvable* for general continuous boundary values.

7. INTEGRAL EQUATIONS AROUND 1903

The *integral operator* J: $u \mapsto v = J[u]$, where the "image" v of u is defined by

$$v(x) = \int_0^x u(t) \, dt, \tag{7.1}$$

is much easier to interpret in most function space contexts than its inverse, the *derivative operator* D: $u \mapsto u'$. This is because, as was essentially proved by Cauchy and Riemann [Birkhoff 1973, Part IB], J is defined for *all* functions u in $C[a,b]$ (similarly in $L[a, b]$, etc.), whereas D is only defined on a *dense subset*. In short, it is much easier to interpret the limiting processes of analysis for J, in most function spaces, than for D. Similar remarks apply to other integral operators, like K: $u \mapsto v = K[u]$, where

$$v(x) = \int_a^b k(x,y)u(y)\,dy, \tag{7.2}$$

as contrasted with partial differential operators. It may have been for this reason that Pincherle's scholarly study of "*operazioni distributive,*" which emphasized *differential* (and difference) operators, had little influence on later developments in functional analysis, whereas his work on the Laplace transform (an integral transform) was quite fruitful.

The systematic study of integral operators of the form (7.2) began relatively late. In 1823, Abel had solved a *special* integral equation associated with the tautochrone [*Werke,* Vol. 1, pp. 11–27] (cf. [Birkhoff 1973, 437–442]). Abel's integral equation was

$$f(x) = \int_0^x \frac{\phi(y)}{\sqrt{x-y}}\,dy; \tag{7.3}$$

it is called an "integral equation of the *first kind*" (Hilbert's term), because the unknown function ϕ occurs only under the integral sign.

The earliest known integral equations in which the unknown function also appears outside the integral ("integral equations of the *second kind*") were used in 1837 by Liouville to generalize Fourier series expansions from solutions of $u'' + k^2u = 0$ to eigenfunctions of so-called Sturm-Liouville problems, defined by a "Sturm-Liouville differential equation"

$$L[u] + \lambda\rho(x)u = 0$$

with self-adjoint

$$L[u] = (pu')' + qu, \qquad p > 0,$$

and homogeneous boundary conditions

$$k_1u(a) + k_2u'(a) = 0, \qquad l_1u(b) + l_2u'(b) = 0$$

referring to the endpoints of an interval $[a, b]$.

As with Fourier series (the case $L[u] = \alpha u_{xx}$), more general *heat conduction equations* $u_t = L[u]$, and *wave equations* $v_{tt} = L[v]$ can be easily solved by such expansions. For, given the initial values $u(x,0) = \Sigma c_j\phi_j(x)$, solutions satisfying the specified boundary conditions are

$$u(x,t) = \Sigma c_je^{-\lambda_jt}\phi_j(x), \qquad v(x,t) = \Sigma c_je^{\pm ik_jt}\phi_j(x),$$

where $k_j^2 = \lambda_j$, the eigenvalue to which ϕ_j corresponds.

The connection with integral equations is that $f = L[u]$ is equivalent to

$$u(x) = \int_a^b G(x,y)f(y)\,dy, \tag{7.4}$$

where G is the *Green's function*. Likewise, when $\rho \equiv 1$ (Liouville normal form), the eigenproblem for a Sturm–Liouville system is to solve

$$\lambda\phi(x) = -\int_a^b G(x,y)\phi(y)\,dy, \tag{7.4'}$$

and so the solution of $L[u] + \lambda u = f$ satisfies

$$u(x) + \frac{1}{\lambda}\int_a^b G(x,y)\phi(y)\,dy = \frac{1}{\lambda}f(x). \tag{7.4''}$$

Another important class of integral equations arose in connection with the so-called *Neumann method* for solving the Dirichlet problem. The idea of this method was proposed by A. Beer in 1856 and again in 1865; see [Dieudonné 1981, 41–46]. It consisted in assuming the solution to be the potential of a layer of "dipoles" normal to the bounding surface Γ, of unknown density $\sigma(Q)$, $Q \in \Gamma$. Specifically, writing this potential in the form

$$u(P) = \frac{1}{2\pi}\int_\Gamma \sigma(Q)\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS(Q), \qquad r = d(P, Q),\ Q \in \Gamma, \tag{7.5}$$

one obtains for the unknown density σ the integral equation

$$\sigma(Q) + \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial\nu}\left(\frac{1}{r^*}\right)\sigma(Q^*)\,dS(Q^*) = \phi(Q), \qquad \phi = u \text{ on } \Gamma,\ r^* = d(Q, Q^*). \tag{7.6}$$

Because of the term $\sigma(Q)$, the operator form of this "integral equation of the second kind" is

$$(I + K)\sigma = \phi. \tag{7.7}$$

Its solution should obviously be

$$\sigma = (I + K)^{-1}\phi = \phi - K\phi + K^2\phi - K^3\phi + \cdots. \tag{7.7'}$$

Using this idea, *Carl Neumann* (1832–1925) proved in 1870 a mathematical existence theorem for the Dirichlet problem (valid for convex bounded domains) by integral equation methods. He solved (7.6) iteratively (by "successive approximation"), defining $\sigma_0 = \phi$ and

$$\sigma_n = (-K)\sigma_{n-1} = -\frac{1}{2\pi}\int_\Gamma \sigma_{n-1}\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)dS = (-K)^n\phi.$$

The result was the "Neumann series" (7.7′), first published in *Berichte und Verhandlungen der Sächsischen Akademie der Wissenschaften* (1870), and in more detail in *Untersuchungen über das logarithmische und Newton'sche Potential* [(Leipzig: Teubner, 1877)].

It was not until 1888 that Paul du Bois-Reymond (1831–1889) suggested the name *"integral equation"* and pointed out that, since these equations arise from partial differential equations, the latter could probably be handled successfully if one only had a *general theory* of integral equations. (See *Journal für die reine und angewandte Mathematik* **102,** 204–229; also [Birkhoff 1973, Chap. 13].)

A related equation then in the center of interest was the *Helmholtz equation*

$$\nabla^2 u + k^2 u = 0 \tag{7.8}$$

and its *spectral theory*. Already in 1869, using variational ideas, H. Weber [*Mathematische Annalen* **1,** 1–36] had tried to prove the existence of a basis of eigenfunctions in a general domain. Essentially, Weber's variational approach was based on a variant of Riemann's "Dirichlet principle," and so was subject to the same Weierstrassian criticism.

We have already mentioned Poincaré's solution in 1890 of the Dirichlet problem by his *"méthode du balayage."* In that paper, he also discussed the eigenproblem for the Laplace operator, but only solved the analogous matrix eigenproblem. In 1894 he published a second major paper [Poincaré 1950–1956 9, 123–196], in which he established a spectral theory for the Helmholtz equation (7.8) in a general domain $\Omega \subset \mathbb{R}^3$ and $u|_{\partial\Omega} = 1$, by proving (with as much rigor as was customary and possible at that time) the existence of a basis of orthogonal eigenfunctions. He did this by using Picard's results of 1893 on a modified Neumann method, an extension of an "a priori inequality" from his own 1890 paper, and a method used by H. A. Schwarz in his fundamental paper on minimal surfaces, published in 1885 [32].

Le Roux and Volterra. The first mathematicians to establish existence and uniqueness theorems for *general* classes of integral equations were J. M. Le Roux [33] and Vito Volterra [34, 35]. Although their methods were very similar, and Le Roux's paper was published first, Volterra's work was far more influential because it included an explicit solution formula and emphasized a guiding principle: the analogy with systems of linear *algebraic* equations.

Indeed, in 1896 Volterra created a general theory of integral equations of the form

$$\phi(s) - \int_a^s k(s, t)\phi(t)\, dt = f(s) \tag{7.9}$$

with unknown ϕ, given f, and given *kernel* k, which can be any continuous function. Since we may assume that k vanishes when $t > s$, it has triangular support and thus corresponds to a *triangular* coefficient matrix. Although one can no longer obtain the solution in finitely many steps, simple iteration still converges exponentially, so that existence and uniqueness are relatively easy to prove.

Volterra used an expansion in terms of iterates and the idea of the *resolvent kernel* (which in special cases had been employed before by J. Caqué in 1864 [*Journal de Mathématiques* **9** (2), 185–222] and by E. Beltrami [*Rendiconti dell' Istituto Lombardo di Scienze e Lettere* (Milan) **13**(2) (1880), 327–337; and *Me-*

morie della R. Accademia delle Scienze dell'Istituto di Bologna **8**(4) (1887), 291–326]), in order to express the solution in terms of an integral equation of the second kind. He proved that the series involving the iterated kernels converges uniformly and the solution thus obtained is unique.

Fredholm. The decisive papers on integral equations were written by Ivar Fredholm (1866–1927), who received his Ph.D. at Uppsala in 1898 and then became an assistant of Mittag-Leffler and later his colleague at Stockholm. After his visit in Paris, where he got in touch with Poincaré, Fredholm developed his famous theory of "Fredholm equations of the second kind" (a name given later by Hilbert):

$$\phi(x) - \lambda \int_a^b k(x,y)\phi(y)\, dy = f(x). \tag{7.10}$$

He announced it in 1900 [Birkhoff 1973, 437] [36] and published it in full in 1903 [*Acta Mathematica* **27**, 365–390; Birkhoff 1973, 449–465] [37]. Using Poincaré's work as a starting point, but avoiding any function-theoretic arguments, he employed as the basic idea of his approach the replacement of the integral by Riemann sums, the solution of the resulting system of n linear algebraic equations by determinants and passage to the limit as $n \to \infty$. In this last step, Fredholm expanded his determinant in a series of principal minors, as had been done earlier by H. von Koch (1896). He defined his "determinant" of the kernel

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_a^b \cdots \int_a^b k \begin{pmatrix} s_1, \ldots, s_n \\ s_1, \ldots, s_n \end{pmatrix} ds_1 \cdots ds_n$$

and his "first minor"

$$D(x,y,\lambda) = k(x,y) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_a^b \cdots \int_a^b k \begin{pmatrix} x, s_1, \ldots, s_n \\ y, s_1, \ldots, s_n \end{pmatrix} ds_1 \cdots ds_n,$$

where

$$k \begin{pmatrix} x_1, \ldots, x_n \\ y_1, \ldots, y_n \end{pmatrix} = \det(k_{ij}), \qquad k_{ij} = k(x_i, y_j),\ i, j = 1, \ldots, n.$$

In the convergence proof, Fredholm used Hadamard's famous determinant inequality of 1893 [*Bulletin de la Société mathématiques* **17**(2), 390–398], which states that the "volume" of an "n-dimensional parallelepiped" cannot exceed the product of the lengths of the n edge vectors. Fredholm proved that $D(\lambda)$ and $D(x, y, \lambda)$ are entire functions of λ, as had been conjectured by Poincaré.

In full analogy to the theory of finite systems of linear algebraic equations, he then answered all questions concerning the solvability of (7.10) with continuous kernel, by establishing what became later known as the "*Fredholm alternative,*" that is, for any λ, either (A) or (B) holds:

Case (A). If λ is not a zero of $D(\lambda)$, then (7.10) has precisely one solution, which, in terms of Fredholm's "*noyau résolvant*" [resolvent kernel]

$$R(x, y, \lambda) = \frac{D(x, y, \lambda)}{D(\lambda)},$$

can be written

$$\phi(x) = f(x) + \lambda \int_a^b R(x, y, \lambda) f(y)\, dy. \qquad (7.11)$$

In this case, the homogeneous equation

$$\phi(x) - \lambda \int_a^b k(x, y)\phi(y)\, dy = 0 \qquad (7.12)$$

has only the trivial solution $\phi = 0$.

Case (B). If λ is a zero of $D(\lambda)$ of order m, then (7.12) has at least one nontrivial solution and at most m linearly independent ones. In this case (7.10) is not always solvable, but only for those f which satisfy the "orthogonality conditions"

$$\int_a^b f(x)\psi(x)\, dx = 0$$

for every solution ψ of the "transposed" homogeneous equation

$$\psi(x) - \lambda \int_a^b k(y, x)\psi(y)\, dy = 0.$$

The remarkable simplicity of Fredholm's methods contrasted with the methods used in earlier work on integral equations. His papers had the effect of moving these equations suddenly into the center of interest of contemporary mathematics. Fredholm's work has become very significant in mathematical physics and as a starting point of general spectral theory. Last but not least, Neumann's results were now obtained by a simple application of Fredholm's theory, without further difficulty.

8. HILBERT'S "INTEGRALGLEICHUNGEN"

Fredholm's sensational results quickly spread to Göttingen:

> In the winter of 1900–1901, the Swedish mathematician E. Holmgren reported in Hilbert's seminar on Fredholm's first publications on integral equations, and it seems that Hilbert caught fire at once. [Weyl, *Bulletin of the American Mathematical Society* **50** (1944), 645]

Just a year earlier, David Hilbert (1862–1943) had published his famous *Grundlagen der Geometrie,* reprinted in eight editions during his lifetime, and very influential in helping to popularize the axiomatic method. A few months earlier he had given his celebrated Paris talk on unsolved problems (see [Browder 1976]) and had sketched his vindication of the Dirichlet principle (cf. Section 6). His main work during the next decade would concern the theory of *integral equations* (IEs) and developments resulting from it. These achievements, together with his earlier brilliant work on invariant theory and algebraic number theory, would establish this reputation as the foremost mathematician of his generation after the death of Poincaré in 1912.

Hilbert published his major contributions to IEs in the *Göttinger Nachrichten* of 1904–1910 in six articles. These papers were republished in book form [Hilbert 1912], with a 20-page summary and an additional chapter on the theory of gases. Their contents will be the theme of this section. The work of Hilbert's students and collaborators will be taken up in Section 9.

Hilbert drew his intuitive inspiration directly from Carl Neumann, Poincaré, Picard, and Fredholm, and indirectly from Fourier, Liouville, Gauss, Green, Dirichlet, Riemann, and Weber. Presumably, having in mind Sturm–Liouville theory and the Helmholtz equation as well as the generalized Dirichlet principle which he had formulated by 1900 (see Section 6), Hilbert developed *spectral theory*. He did this first for Fredholm IEs of the second kind

$$\phi(s) - \lambda \int_0^1 k(s, t)\phi(t)\, dt = f(s) \qquad (8.1)$$

with continuous and symmetric kernel k (and continuous f) and later in much greater generality.

Actually, Hilbert had already lectured on partial differential equations in 1895–1896, and his student Ch. A. Noble had published a paper on Neumann's method (Section 7) based on Hilbert's ideas in the *Göttinger Nachrichten* (1896), 191–198. Starting in 1901, Hilbert lectured systematically on ideas about IEs, from which soon resulted three doctoral theses, by O. D. Kellogg [38] in 1902, by his fellow-American Max Mason, and by A. Andrae in 1903.

In 1904, when Hilbert began to publish his new theory, he was ready to announce that he had entirely recast the spectral theory of self-adjoint differential equations:

> My investigation will show that the theory of the expansion of arbitrary functions by no means requires the use of ordinary or partial differential equations, but that it is the *integral equation* which forms the necessary foundation and the natural starting point of a theory of series expansion, and that those . . . developments in terms of orthogonal functions are merely special cases of a *general integral theorem* . . . which can be regarded as a direct extension of the known algebraic theorem of the orthogonal transformation of a quadratic form into a sum of squares, ...
>
> By applying my theorems there follows not only the existence of eigenfunctions in the most general case, but my theory also yields, in a simple form, a necessary and sufficient condition for the existence of infinitely many eigenfunctions. [Hilbert 1912, 2–3]

Most important of Hilbert's six papers on IEs are the first (1904) and the fourth (1906), and we shall concentrate on these (see also [Hilbert 1932–1935], as well as [Hellinger & Toeplitz 1927]).

Hilbert's first paper concerned the IE (8.1) with continuous kernel k. Replacing the integral with Riemann sums, Hilbert obtained from (8.1) the finite system of algebraic equations [p. 4]

$$\phi_s - \frac{\lambda}{n} \sum_{t=1}^{n} k_{st}\phi_t = f_s, \qquad s = 1, \ldots, n. \qquad (8.2)$$

He began by reproving some of Fredholm's results and of his method of solution [pp. 10–13].

In his further work he made the essential assumption that the kernel be *symmetric*, $k(s, t) = k(t, s)$. He also assumed that the Fredholm determinant, $\delta(\lambda)$ in his notation, has no multiple zeros. [This he later removed; cf. pp. 36–38.]

Along with (8.2), he considered the quadratic forms [p. 4]

$$Q_n(x) = \sum_{s=1}^{n} \sum_{t=1}^{n} k_{st} x_s x_t, \tag{8.3}$$

which he later (in his fourth paper; p. 110) called "*Abschnitte*" [sections]. Emphasizing principal axes reduction rather than determinants (as in von Koch's work), Hilbert developed the "passage to the limit" as $n \to \infty$ from the heuristic guiding principle it had been to Volterra and Fredholm into a method of proof. This limit process "worked": it gave Hilbert the existence of at least one eigenvalue of the kernel (in modern terms: *reciprocal* eigenvalue of the homogeneous IE), the orthogonality of eigenfunctions $\psi_n(s)$ [p. 17] and, by switching from [0, 1] to $[a, b]$, the generalization of the principal axes theorem, namely [pp. 19, 20],

$$\int_a^b \int_a^b k(s, t) x(s) y(t)\, ds\, dt = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} (x, \psi_n)(y, \psi_n), \tag{8.4}$$

where the "Fourier coefficients" of x and y with respect to the normalized "eigenfunctions" ψ_n corresponding to λ_n are given by

$$(x, \psi_n) = \int_a^b x(s)\psi_n(s)\, ds, \qquad (y, \psi_n) = \int_a^b y(s)\psi_n(s)\, ds$$

and the series converges absolutely and uniformly for all continuous and square-integrable x and y.

Since Hilbert established (8.4) without presupposing the existence of eigenvalues, he made (8.4) the key formula of this theory. He immediately concluded from it the existence of finitely many eigenvalues for kernels which are finite sums of continuous products of the form $k_j(s)l_j(t)$, and for any other continuous symmetric kernel the existence of countably infinitely many eigenvalues without accumulation point [p. 22; in present terms: which accumulate at zero].

Hilbert also showed [p. 24] that any function f which can be represented in the form [39]

$$f(s) = \int_a^b k(s, t) g(t)\, dt \qquad (g \text{ continuous})$$

has an *eigenfunction expansion*

$$f(s) = \sum_{n=1}^{\infty} c_n \psi_n(s), \qquad c_n = (f, \psi_n), \tag{8.5}$$

which converges absolutely and uniformly.

On pages 30–35, he extended his theory beyond continuous kernels to those which have singularities "of order less than $\frac{1}{2}$." However, in order to obtain *for IEs* the *full* analogy to algebra, in which the eigenvalue ∞ (i.e., 0 in modern terms) plays no exceptional role, one would have to admit Lebesgue square-integrable eigenfunctions, as became apparent after the discovery of the Riesz–Fischer theorem (Section 10) in 1907.

Hilbert's second paper, which is not very relevant for our purpose, discusses applications to boundary value problems for self-adjoint ordinary and for elliptic differential equations. There, Hilbert relied on the existence of Green's functions [pp. 42, 61] to act as kernels for his IEs. These functions are easy to construct for *ordinary* differential equations, but their construction may cause serious difficulties in the case of *partial* differential equations.

Hilbert's fourth paper on IEs (published in 1906) marks the beginning of spectral theory in the modern *functional analytic* spirit and of the functional analytic approach to IEs as well. There, Hilbert created a general theory of "continuous" bilinear and quadratic forms *independently* of IEs, but *applicable* to large classes of them. His bridge [*Bindeglied;* p. 177] between the two theories was an *"orthogonales vollständiges Funktionensystem"* (a complete orthogonal system of functions, an orthogonal basis of functions), such as the "trigonometric system" in a Fourier series, the completeness of such a system $\{\phi_n\}$ being defined by the requirement that the "completeness relation" [p. 177]

$$(u, v) = \sum_{n=1}^{\infty} (u, \phi_n)(v, \phi_n)$$

be valid for any continuous u and v. This concept generalizes the idea of Cartesian coordinates to infinite-dimensional "function spaces."

Hilbert showed that from any continuous $f \neq 0$ and continuous (not necessarily symmetric) kernel k in (8.1) with $\lambda = 1$, one obtains an infinite system [p. 165]

$$x_p - \sum_{q=1}^{\infty} k_{pq} x_q = f_p, \qquad p = 1, 2, \ldots, \tag{8.6}$$

such that $\Sigma\Sigma k_{pq}^2$ and Σf_p^2 converge and each solution $\{x_p\}$ with convergent Σx_p^2 yields a continuous solution ϕ of (8.1); cf. pages 180–185. In this connection the real sequence space l^2 appeared [pp. 125–126] for the first time (not in this terminology or notation!) [40].

His search for the most general conditions under which the analog of the principal axes theorem still holds in the infinite-dimensional case led Hilbert to the discovery of "complete continuity." He called the infinite quadratic form

$$Q(x) = \sum_{p=1} \sum_{q=1} k_{pq} x_p x_q \qquad (k_{pq} = k_{qp}) \tag{8.7}$$

vollstetig [completely continuous] when

$$\lim_{n\to\infty} Q_n(x) = Q(x), \qquad \text{where } Q_n(x) = \sum_{p=1}^{n}\sum_{q=1}^{n} k_{pq}x_px_q \tag{8.8}$$

uniformly for all $x = \{x_p\}$ such that $\Sigma x_p^2 \leqq 1$. Complete continuity of a symmetric bilinear form he defined similarly. (This corresponds to what would later be called "weak topology" on abstract Hilbert space.)

Generalizing orthogonal transformations to infinite dimension, that is [p. 129],

$$y_p = \sum_{q=1}^{\infty} \omega_{pq}x_q \qquad \left(\sum_{q=1}^{\infty} x_q^2 < \infty\right) \qquad (p = 1, 2, \ldots),$$

where

$$\sum_{r=1}^{\infty} \omega_{pr}\omega_{qr} = \delta_{pq}, \qquad \sum_{r=1}^{\infty} \omega_{rp}\omega_{rq} = \delta_{pq},$$

he showed [pp. 148–150] that any completely continuous quadratic form can be transformed to principal axes,

$$Q(x) = \sum_{j=1}^{\infty} k_jy_j^2 \qquad (k_j \text{ the eigenvalues}),$$

by an orthogonal transformation, thus establishing the existence of (generally) countably many eigenvalues of Q and of a corresponding eigenbasis.

Furthermore, he proved [pp. 165, 170] that in the completely continuous case, the system (8.6) has all the essential properties of linear equations in finitely many unknowns. That is, either (8.6) has precisely one solution $x = \{x_q\} \in l^2$ for every $f = \{f_p\} \in l^2$ (in particular $x = 0$ when $f = 0$); or the homogeneous system ($f = 0$) has finitely many linearly independent solutions, in which case the "transposed system" has the same number of linearly independent solutions, and (8.6) is solvable if and only if f satisfies the same number of linear relations.

Bounded forms and their spectral theory make up the major part of Hilbert's fourth paper on IEs. Hilbert defined [p. 125] a quadratic form Q to be *bounded* if $|Q_n(x)| \leqq c\ (< \infty)$ in (8.8) for all n and all $x = \{x_p\}$ such that $\Sigma x_p^2 \leqq 1$. On page 127 he stated that these forms are continuous (in the l^2-metric).

In contrast to the completely continuous case, for bounded forms not completely continuous the situation is radically different from that in the finite-dimensional case, mainly because of the possible occurrence of a "continuous spectrum," as was known from work by Stieltjes, Wirtinger, and others. Probably Hilbert's most significant accomplishment in spectral theory was the discovery that an essential part of the theory of orthogonal transformations can be extended to infinite *bounded* forms, despite the complications caused by the continuous spectrum. Using ingenious new methods, by a limit process, Hilbert obtained from the finite-dimensional case the following results [pp. 111–147]. The vector $x = \{x_p\}$ in (8.7) can be transformed orthogonally to $x' = \{x_p'\}$, $\xi = \{\xi_p\}$ so that [p. 145]

$$Q(x) = \sum_{(p)} k_p x_p'^2 + \int_{(s)} \frac{d\sigma(\mu, \xi)}{\mu}. \tag{8.9}$$

Here we sum over the point spectrum (the eigenvalues) and extend the (Stieltjes) integral over the *"Streckenspektrum"* s [the continuous spectrum] on the μ-axis. The "spectral form"

$$\sigma(\mu, \xi) = \sum_{p,q} \sigma_{pq}(\mu)\xi_p\xi_q$$

is a positive definite quadratic form which depends on the parameter μ and whose value, for every fixed ξ, increases with μ in a monotone fashion from 0 to $\Sigma\xi_p^2$, and which, for all continuous functions $u(\mu)$, satisfies the relations

$$\sum_r \int_{(s)} u(\mu)\, d\sigma_{pr}(\mu) \int_{(s)} u(\mu)\, d\sigma_{rq}(\mu) = \int_{(s)} u(\mu)^2\, d\sigma_{pq}(\mu),$$

$$\int_{(s)} d\sigma(\mu, \xi) = \sum_p \xi_p^2.$$

These suffice for a complete characterization of this form.

As a simple consequence, Hilbert proved [p. 147] that for a bounded form, the corresponding infinite system

$$x_p - \lambda \sum_{q=1}^{\infty} k_{pq}x_q = 0 \qquad (p = 1, 2, \ldots)$$

is solvable in l^2 if and only if λ is an eigenvalue. Similarly, if λ is not in the spectrum, the corresponding nonhomogeneous system (with the right-hand side in l^2) has a unique solution [p. 139], which is representable in terms of the "resolvent" [p. 123], which in the present case is a bounded quadratic form depending analytically on λ.

In his summary paper [*Werke*, Vol. III, No. 6], prepared in connection with the 1908 International Congress of Mathematicians in Rome, Hilbert characterized his approach to spectral theory via forms as "providing a unified methodology for treating Algebra and Analysis." For the further evolution of functional analysis, complete continuity and continuity of forms and functionals were probably most significant among all of Hilbert's new ideas arising from IEs.

To appreciate fully the progress achieved by Hilbert's work, recall that only twenty years earlier, the existence proof of the fundamental frequency of a membrane (of a general shape) had required the greatest effort (H. A. Schwarz, 1885), and now one had constructive proofs for all eigenvalues and functions under rather general assumptions, with applications to differential equations, as given in Hilbert's second and third papers on IEs. Hilbert's work had also shown that essential to the spectral theory of an "integral operator" is its complete continuity rather than its representations in terms of integrals. We discuss some basic consequences of this discovery in Section 12.

9. THE HILBERT SCHOOL

Hilbert's theory of IEs and general spectral theory, which had been developed with "algebraization" as the guiding principle, was soon simplified, explored in greater detail, and extended by the many outstanding graduate students who were exposed and reacted to Hilbert's ideas. The 69 doctoral theses (over 40 of them between 1900 and 1910) supervised by Hilbert included those of E. R. Hedrick (published in 1901), G. Hamel (1901), O. D. Kellogg (1902), R. Füeter (1903), Max Mason (1903), E. Schmidt (1905), E. Hellinger (1907), H. Weyl (1908), A. Speiser (1909), A. Haar (1909), R. Courant (1910), E. Hecke (1910), H. Steinhaus (1911), and H. Bolza (1913). Hilbert's active disciples also included O. Toeplitz and E. Hilb.

Erhard Schmidt. Of all those just named, Erhard Schmidt (1876–1959) probably made the greatest individual contribution to functional analysis, by giving a new, simplified approach to Hilbert's theory of IEs with continuous kernel (in his doctoral thesis of 1905, published in 1907), and by developing a "geometrized" Hilbert space theory in 1908 [1907, 1908].

In his thesis he gave a direct proof of the existence of the eigenvalues and eigenfunctions of the IE (8.1) with continuous symmetric kernel k. He abandoned completely the reduction to principal axes and used no special properties of the integral beyond its linearity. The key for his existence proof of the eigenvalues was:

> a method . . . which, modeled after a famous proof of H. A. Schwarz [of 1885], in terms of Fredholm's formulas would amount to solving the equation $D(\lambda) = 0$ by Bernoulli's method [41]. From the existence theorem follow the expansion theorems . . . without Hilbert's assumption [42]. . . . The complications in Hilbert's original approach caused by multiple zeros of . . . $D(\lambda)$ do not occur here. . . . [Schmidt 1907, 435]

Schmidt is of course referring to multiple eigenvalues. His method does not depend on any results by Fredholm or Hilbert. Instead, he used successive approximation, the Schwarz and Bessel inequalities being his only tools in proving convergence. On page 442 he introduced the "Gram–Schmidt orthogonalization" and used orthonormal systems throughout. Schmidt's method remained unsurpassed in simplicity and lucidity, and had great influence on further developments. Results similar to some of Schmidt's were obtained independently by W. Stekloff [*Annales de la Faculté des Sciences de Toulouse* **6**(2) (1905), 351–475].

Perhaps even more important was Schmidt's paper of [1908]. In it, following a suggestion of G. Kowalewski of Bonn (later Dresden), Schmidt introduced the geometric language now commonly used to describe Hilbert space. Whereas Hilbert had never used the word "space," Schmidt spoke of "vectors in infinite-dimensional space" [p. 56] and "Geometrie in einem Funktionenraum" (also referring to Fréchet's thesis and emphasizing the importance of the recently discovered Riesz–Fischer theorem).

In terms of Hilbert sequence space l^2 he gave a thorough discussion of the basic concepts of Hilbert space theory, notably inner product, norm, orthogonality, orthogonal projection on closed subspaces [called by him "*lineare Funktionengebilde*"; p. 63], basis, and the projection theorem [p. 64]. In Chapter II [pp. 69–77], Schmidt applied his theory to infinite systems

$$(a_n; Z) = \sum_{m=0}^{\infty} a_{nm} Z_m = c_n \qquad (n = 1, 2, \ldots) \tag{9.1}$$

with arbitrary $a_n = (a_{nm}) \in l^2$, and gave necessary and sufficient conditions for the existence of a solution $Z = (Z_m) \in l^2$. These included earlier results by Hilbert ("Fourth paper," 1906) and Toeplitz [*Göttinger Nachrichten* (1907)] on bounded quadratic forms.

Hellinger and Toeplitz. Hilbert's theory left unanswered the question for the analog of the eigen*vectors* in the case of the *continuous* spectrum. In 1909, J. Hellinger (1883–1950), in *Journal für die reine und angewandte Mathematik* **136,** 210–271, solved this problem by introducing his so-called *eigendifferentials*. In 1912, Hans Hahn [*Monatshefte für Mathematik und Physik* **23,** 161–224] simplified Hellinger's results and [on p. 216] used them to obtain necessary and sufficient conditions for the orthogonal equivalence of bounded quadratic forms (hence of bounded linear operators on l^2) in terms of the eigenvalues and the eigendifferentials of the forms. (We discuss Hahn's further work in Section 15.)

As a simple alternative to Fredholm's solution theory and Hilbert's spectral theory of IEs based on bilinear and quadratic forms, E. Hellinger and O. Toeplitz [1927] developed a theory of infinite bounded matrices, extending ideas by Cayley and Frobenius to infinite dimensions. Starting with a note in the *Göttinger Nachrichten* (1906), they presented the details in *Mathematische Annalen* **69** (1910), 289–330. On pages 318–321 they proved a *uniform boundedness theorem* for linear forms (functionals) and on pages 321–326 the famous

HELLINGER–TOEPLITZ THEOREM [43]. *If a bilinear form*

$$A(x, y) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{pq} x_p y_q \qquad (a_{pq} = a_{qp}) \tag{9.2}$$

converges (say, under rowwise summation) for every pair $x = (x_p), y = (y_q)$ *satisfying*

$$\sum_{p=1}^{\infty} x_p^2 \leqq 1, \qquad \sum_{q=1}^{\infty} y_q^2 \leqq 1,$$

then A is "bounded," that is there is an M such that $|A(x, y)| \leqq M$ *for all those x and y.*

Hilb. In his second paper on IEs, after recalling that Sturm–Liouville theory can be related to IEs, Hilbert had observed that the same is possible for other boundary value problems and problems involving infinite intervals. Regarding the latter, in his 1908 *Habilitationsschrift*, published in *Mathematische Annalen* **66,** 1–66, E. Hilb (1882–1929) broke ground by studying two such *singular problems* in relation to Hilbert's theory.

Weyl. A general theory of those singular problems was soon created by Hermann Weyl (1885–1955), the most notable of Hilbert's students. In his doctoral thesis of 1908 [Weyl 1968 1, 1–87] he began as follows:

> . . . Prof. Hilbert [5. Mitteilung on IEs] has proved . . . that if $K(s,t)$ is any continuous symmetric function in $a = s$, $t = b$, the "homogeneous IE"
>
> $$0 = \phi(s) - \lambda \int_a^b K(s,t)\phi(t)\, dt$$
>
> [has infinitely many eigenvalues]. An analogous theorem will also still hold if we choose as the interval of integration not a . . . b but, say, 0 . . . ∞ provided then $K(s,t)$ is sufficiently regular at infinity. . . . On the other hand, examples of "kernels" $K(s,t)$ can easily be found for which *Hilbert's theory is no longer valid.* As Professor Hilbert emphasized in one of his lectures, the kernel $\cos(st)$ deserves particular interest. [By] the Fourier integral theorem . . . this kernel has at most the two eigenvalues $+\sqrt{2/\pi}$, $-\sqrt{2/\pi}$ . . . $\cos(st)$ has . . . at infinity a high singularity, hence it is not surprising that also the IE [with this function as the kernel] is "*singular,*" i.e., does not satisfy Hilbert's theorems on eigenvalues. . . . In order to make the . . . singular behavior of IEs understood . . . we shall have to use . . . the more general tool of *bounded quadratic forms* about whose nature, information will be given by the 4. Mitteilung . . . [*Göttinger Nachrichten* 1906, pp. 157–209] and recently by Mr. Hellinger's Dissertation [of 1907].

Here, "more general" means "more general than completely continuous." In this thesis and in another paper of 1908 entitled "Singuläre Integralgleichungen" [*Mathematische Annalen* **66**, 273–324; Weyl 1968 1, 102–153], Weyl gave conditions on K of a singular IE in order that the corresponding form be bounded, so that Hilbert's spectral theory applies.

In 1910, in his *Habilitationsschrift* on ordinary differential equations, Weyl [*Mathematische Annalen* **68**, 220–269; Weyl 1968 1, 248–297] then extended Hilb's results and discovered that the general *singular* Sturm–Liouville problem for an equation

$$L[u] + \lambda u = 0, \qquad L[u] = (p(t)u')' - q(t)u, \tag{9.3}$$

with p, $q > 0$ on $[0, +\infty)$ (and a suitable condition at infinity) could be treated by results from his own thesis. Using two solutions, he constructed a Green's function on a *finite* interval $[0, a]$. He then let $a \to +\infty$ and showed that Hilbert's theory applied to the resulting singular IE, and that, moreover, the solution was in $L^2[0, +\infty)$. Furthermore, using Hellinger's eigendifferentials, he generalized Fourier's integral formula in a way that had been hoped for by Wirtinger in 1897 and established in special cases by Hilb in 1908. Weyl's *Habilitationsschrift* is also the earliest work on *unbounded* operators, which were to play a central role in quantum mechanics about twenty years later (cf. Section 18).

10. FRÉDÉRIC RIESZ (RIESZ FRIGYES)

Of all the creators of functional analysis, the famous Hungarian mathematician Frédéric Riesz (1880–1956) made perhaps the most many-sided and seminal contributions. Educated in Zürich, Budapest, and Göttingen, and older brother of another notable mathematician (Marcel Riesz, 1886–1969), he had a unique flair for establishing profound and original connections. In particular, he coordinated work of the Paris and Göttingen schools. His total activity in functional analysis spanned a 35-year period (1905–1939), followed by an impressive summary in his 1952 book, co-authored with B. Szökefalvi-Nagy (now at Szeged) and later translated into English [Riesz & Sz.-Nagy 1955].

At the beginning of this period, we find Riesz as a high school teacher [*Oberschullehrer*] in a small country town (Leutschau, Löcse) of about 7000 inhabitants; he obtained his first university position only in 1912 (at Klausenburg, Kolozsvár). In this section we review his pre-1912 contributions.

Riesz received his doctoral degree in 1902 with a thesis on geometry [44] written in Hungarian (see [Riesz 1960, 1529–1557] for a French translation), the same year in which Lebesgue published his thesis on measure and integration (Section 5). Four years later, in his fourth paper on integral equations, Hilbert created his spectral theory of bounded quadratic forms in his "Hilbert space" model l^2. He did this in greater generality than was needed for IEs with symmetric kernel and completely independent of the latter. The following year (still at Leutschau), Riesz discovered the famous *Riesz–Fischer theorem* and made it public [45] just four days after E. Fischer (at Brünn) had presented practically the same result in his seminar.

RIESZ–FISCHER THEOREM. *Given any sequence $\{a_i\}$ of real numbers and any orthonormal system $\{\phi_i\}$ in $L^2[a, b]$, there exists a function $f \in L^2[a, b]$ which has these real numbers as its "Fourier coefficients" with respect to $\{\phi_i\}$, that is,*

$$\int_a^b f(x)\phi_i(x)\,dx = a_i, \qquad i = 1, 2, \ldots,$$

if and only if $\Sigma a_i^2 < \infty$.

From this theorem it follows that the *metric space* $L^2[a,b]$ of all such functions is complete and separable, and isomorphic to the "Hilbert sequence space" l^2.

The Riesz–Fischer theorem provided a completely unexpected and enormously fruitful application of Lebesgue's still new theory within developing "functional analysis," and Riesz was to become second only after Lebesgue himself in showing the power of these new ideas and tools. As another consequence, the theorem paved the way for extending much of the theory of IEs from continuous to (Lebesgue) square-integrable kernels and eigenfunctions.

In the same year, Fréchet [*Transactions of the American Mathematical Society* **8** (1907), 433–446] and Riesz [*Comptes Rendus* (Paris) **144** (1907), 1409–1411; Riesz 1960, 386–388] obtained independently the representation

$$U(f) = (f,g)$$

for any bounded linear functional U on the Hilbert space $L^2(\Omega)$, where Ω is the unit circle in Fréchet's work and is left unspecified in Riesz' (but Riesz most likely had $[a,b]$ in mind). It is of course easy to establish the analogous result in an (axiomatically defined) abstract Hilbert space, as was done by Riesz in 1934–1935 [*Acta Szeged* **7**, 34–38; Riesz 1960, 1150–1154].

Two years later, in 1909, Riesz made a major advance in *duality theory* by tackling a substantially more difficult problem: the representation of bounded linear functionals A on the space $C[a, b]$ by a Stieltjes integral in the form

$$A(f) = \int_a^b f(x)\, d\alpha(x) \tag{10.1}$$

[*Comptes Rendus* (Paris) **149**, 974–977; Riesz 1960, 400–402]. Here α is a function of bounded variation on $[a, b]$, with total variation equal to $\|A\|$, and can easily be made unique, as Riesz indicated (indirectly) on page 402, *ibid.* For example, a functional which cannot be represented by a Riemann integral is given by $A(f) = f(x_0)$ with fixed $x_0 \in [a,b]$; it is, however, represented by (10.1) with

$$\alpha(x) = \begin{cases} 0 & \text{if } x < x_0 \\ 1 & \text{if } x \geqq x_0. \end{cases}$$

In this way, Riesz solved definitively the problem attacked by Hadamard and Fréchet in 1903–1904 (Section 5), indicating connections to Haar's Göttingen thesis of 1909 and to Hadamard's work of 1903 on page 402, *ibid.* [46]. Riesz was surely aware that the present problem was basically different in nature from that in a Hilbert space: α is not from the same space as f, since it may have discontinuities, so that one can no longer relate functionals A to elements in that space $C[a, b]$.

It is curious that Stieltjes' idea [47] of integrating with respect to a general "mass distribution," without requiring a density, remained practically unnoticed for 15 years until Riesz used it. A close observer of the Göttingen scene, Riesz may have got the first impetus in this direction from a short remark by Hilbert [1912, 109]. Riesz' elegant work of 1909 sparked further development basic to the theory of integration, by J. Radon [*Sitzungsberichte der Akademie der Wissenschaften, Wien,* IIa **122** (1913), 1295–1438] and many others.

Spaces L^p and l^p and their duals. In his 1908 paper [Schmidt 1908] (cf. Section 9), E. Schmidt had discussed systems of countably many linear equations

$$(f_j,\xi) = \sum_{n=1}^{\infty} f_{jn}\xi_n = c_j \qquad (j = 1, 2, \ldots) \tag{10.2}$$

with any given $f_j = (f_{jn}) \in l^2$ and given scalars c_j. He gave necessary and sufficient conditions for the existence of solutions $\xi \in l^2$ of (10.2).

After a preliminary announcement in 1909 [*Comptes Rendus* (Paris) **148**, 1303–1305] [48], Riesz initiated and developed L^p-space theory in 1910. He extended Schmidt's results from $p = 2$ to general p, $1 < p < +\infty$, hence from Hilbert space to other Banach spaces. His 1910 paper "Untersuchungen über Systeme integrierbarer Funktionen" [*Mathematische Annalen* **69**, 449–497; Riesz 1960, 441–489] concerned the solution of countably (or even uncountably) many equations of the form

$$\int_a^b f_j(x)\xi(x)\,dx = c_j \qquad (j \in J) \tag{10.3}$$

with given $f_j \in L^p[a,b]$ and given scalars c_j. Riesz was looking for a solution $\xi \in L^q[a,b]$; here $1/p + 1/q = 1$, $p > 1$. For $p = 2$, he noted that his results could be obtained from Schmidt's by the Riesz–Fischer theorem. More importantly, Riesz clearly recognized that "in very general cases, decisive criteria can be developed only . . . since the concept of an integral underwent Lebesgue's ingenious and felicitous [*geistreiche und glückliche*] extension."

Although Riesz did not use the words "dual" or "conjugate" space, employing his theory of solutions of (10.3), on page 475 he showed that for $1 < p < +\infty$, the spaces $L^p[a, b]$ and $L^q[a, b]$ with q as above are dual. Of course, he stated [p. 455] and used both Hölder's and Minkowski's inequalities, referring in this connection [pp. 452, 455] to a short note by E. Landau [*Göttinger Nachrichten* (1907), 25–27] containing the only results on linear forms on l^p with arbitrary p (>1) known at that time. On page 452 he indicated that the sequence spaces l^p could be treated similarly (as he demonstrated later in his book of 1913, to which we turn in Section 12).

On pages 464–466, Riesz defined strong convergence of a sequence $\{f_i\}$ to f in $L^p[a, b]$ by

$$\lim_{i\to\infty} \int_a^b |f(x) - f_i(x)|^p\,dx = 0$$

(as has since become standard), and weak convergence of $\{f_i\}$ to f in a fashion which he showed to be equivalent to the nowadays familiar

$$\lim_{i\to\infty} \int_a^b f_i(x)\gamma(x)\,dx = \int_a^b f(x)\gamma(x)\,dx \qquad \text{for all } \gamma \in L^q[a, b].$$

Riesz defined a transformation to be "*vollstetig*" [completely continuous] if it transforms every weakly convergent sequence into a strongly convergent sequence [p. 487] and noted that this is equivalent to Hilbert's definition of complete continuity (cf. Section 8).

Riesz was well aware of the general significance of his results, and put them into perspective by saying [p. 452]:

> In this paper the assumption of square integrability is replaced by that of the integrability of $|f(x)|^p$. . . . [For each $p > 1$] the role of the class [L^2] is here taken over by two classes [L^p] and [$L^{p/(p-1)}$]. . . . The investigation of these classes of functions will shed particular light on the real and seeming advantages of the exponent $p = 2$; and one can also claim that it will yield useful material for an *axiomatic study* of these function spaces.

11. INTEGRAL EQUATIONS IN 1914

Hilbert's *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* appeared in 1912. So did two other books on the same subject, by Heywood and Fréchet (with a preface by Hadamard) and Lalesco (with a preface by Picard). Bôcher's introductory Cambridge tract, and a concise survey by Korn, had appeared 2–3 years earlier. So it seems that by 1912 the theory of IEs had matured considerably. Because of the continuing influence of this theory on functional analysis, we will next review some opinions about its status as of that time.

We subscribe to A. Korn's statement, made in 1914, that "the young field of linear integral equations, although some of its leading developers [*Hauptvertreter*] have moved in strongly divergent directions, was born and grew under the star of the method of successive approximations, in particular under the star of Schwarz' *Weierstrass-Festschrift* [1885, on minimal surfaces; *Gesammelte mathematische Abhandlungen* **1,** 223–269] and the creations of Picard, Poincaré, and Volterra" [49]. (Korn then proceeded to solve two problems by that method. One was the conformal mapping of a smooth portion of a surface into the plane, previously solved by E. E. Levi (1907) and L. Lichtenstein (1911) in terms of a linear integral equation. The other was the Dirichlet problem for the Laplace equation in an elliptical disk "which can also be reduced to a linear integral equation, but of a kind that *cannot* be solved immediately by Fredholm–Hilbert methods, whereas the application of the method of successive approximations presents no essential difficulties.")

To establish conclusions like those of Korn (and Hilbert) with "Weierstrassian rigor" is an extremely tedious, if necessary, task. To prove rigorously the existence of a *Green's function*, assumed by Green, Dirichlet, and Riemann on intuitive grounds, is easy in one dimension, but requires very sharp thinking in the plane, and is extraordinarily difficult in three or more dimensions. Thus in $n \geqq 4$ dimensions, the Green's function of the Dirichlet problem is not even square-integrable. The temptation to rely on "well-known" results is almost irresistible!

Yet Hadamard and Lebesgue had constructed striking examples of boundary conditions for the Dirichlet problem, for the treatment of which Hilbert's methods were inadequate (see Section 6). To an *analyst*, moreover, Hilbert's purely *algebraic* reformulation of Sturm–Liouville theory and other eigenfunction expansion problems (in terms of infinite quadratic forms) may have appeared to emphasize the wrong ideas. Finally, his methods were at first limited to *linear, self-adjoint* problems, a restriction that seemed undesirable (if not unnatural) to many functional analysts [50].

Volterra and Hadamard. While Hilbert and his followers were making dramatic progress in developing the spectral theory of *symmetric integral operators* (associated with self-adjoint differential equations and boundary conditions), by "algebraic" methods, a number of French and Italian mathematicians were extending the basic concepts of analysis in very different directions. Under the leadership of Volterra and Hadamard, they were considering general operators on all kinds of function spaces, while Fréchet continued to experiment with a host of *topologies* on these spaces.

Volterra's ideas, around 1913, are clearly explained in his two monographs, "Leçons sur les Fonctions de Lignes" (lectures of 1910 at Rome) and "Leçons sur les Équations Intégrales et les Équations Intégro-différentielles" (lectures of 1912 at the Sorbonne). In the first of these (written up by Joseph Pérès), Chapter

III relates "functions of lines" to the calculus of variations, citing a remark of Fréchet (*Annali di Matematica* **11,** 187) and a "remarkable Thesis of Paul Lévy," besides Volterra's own works. The latter concerns "equations with functional derivatives" even more general than the *integro-differential* equations which were the central theme of Volterra's two monographs.

Characteristically, the main thrust of Volterra's books was to subject new *natural phenomena* to mathematical formulation and analysis. These phenomena included deformations of elastic materials and heredity in population biology (see Volterra's Chapter VIII). One can imagine applications in mechanical engineering to "work-hardening" and in solid mechanics to "creep" as subjects which could be treated qualitatively by formulas like Volterra's. Whereas Volterra was doing and stimulating much work on IEs and especially integro-differential equations during this period, Hadamard was more interested in applying the "calcul fonctionnel" to variational problems. We will discuss this in Section 13.

Moore, Bôcher, and Evans [51]. In the United States, Eliakim Hastings Moore (1862–1932) and Maxime Bôcher (1867–1918) were the two leading experts in the theory of integral equations. Both had studied European work attentively, but from very different standpoints.

Moore stated his opinions forcefully in a 1912 survey article [*Bulletin of the American Mathematical Society* **18,** 334–362]. Like Hilbert, he thought that "the theory of linear integral equations . . . has its tap root in the classical analogies between an algebraic sum, the sum of an infinite series, and a definite integral" [pp. 334–335]. However, like Volterra and Hadamard, he had more grandiose ambitions, asserting that "the general theory of linear integral equations is merely a division in . . . a certain form of general analysis" [p. 340]. He had explained what he meant by general analysis in his Colloquium Lectures of 1906 (published in 1910). There he had mentioned [p. 3] Hilbert's 1906 "theory of functions of denumerably many variables" as "another step in this direction" (of general analysis), then citing Fréchet's "more general theory," also of 1906:

> M. Fréchet has given, with extensive applications, an abstract generalization of a considerable part of Cantor's theory . . . and of the theory of continuous functions.

He also paid especial tribute [p. 343] to papers by Pincherle, which he interpreted as applicable to "Fredholm's integral equation in General Analysis."

Bôcher was primarily interested in integral equations because of their relevance to Sturm–Liouville problems. He had written the [*EMW*] article about these in 1900, eight years after completing a Prize Dissertation at Göttingen on "The series expansions of potential theory."

Perspective on the status of the theory of integral equations in 1912 is provided by the papers of Bôcher, E. H. Moore, and Bôcher's student G. C. Evans in the *Proceedings* of the International Congress of Mathematicians held in Cambridge, England [Vol. I, pp. 163–195, 230–255, 387–396]. Whereas the Riesz brothers attended this Congress, neither Hilbert, Schmidt, Felix Klein, nor Weyl was there.

Bôcher's invited address treated only *one*-dimensional boundary value (i.e., two-endpoint) problems. For these, he attributed the "method of successive approximations" to Liouville (1840). He also discussed variational methods (in his Section 8), before taking up "the method of integral equations" (Section 9), with emphasis on "Hilbert's beautiful theory of integral equations with real symmetric kernels." He left it to "Dr. Toeplitz's forthcoming book on integral equations" to discuss "linear boundary problems" in more than one dimension.

For Sturm–Liouville systems, he observes that "the mere fact of an infinite number of . . . eigenvalues ("characteristic numbers") (proved for instance under certain restrictions in Hilbert's *5th Mitteilung*) is an even more obvious corollary of Sturm's work." Later [p. 190], he pays tribute to A. Kneser [*Mathematische Annalen* **58** (1903), 81–147; **60** (1905), 402–423] as having "completely and satisfactorily settled . . . all the more fundamental questions concerning the development of an arbitrary function in a Sturm–Liouville series."

Bôcher considers the 1908 paper of G. D. Birkhoff [52] as constituting "the essential advance," because it covers the *nth order* case, observing somewhat caustically that "the method was rediscovered by Blumenthal" (in 1912), and that Hilb had obtained a "very special case of Birkhoff's result . . . by essentially the same method" in *Mathematische Annalen* **71** (1911), 76–87.

Only then does he acknowledge Hilbert's "remarkable application of integral equations to this development problem," under "extremely restrictive" conditions, weakened by Kneser [*Mathematische Annalen* **63** (1907), 477–524]. He then pays tribute to Haar's Göttingen Thesis of 1909 [*Ibid.* **69** (1910), 331–371; **71** (1911), 38–53], which covered the expansion of arbitrary *continuous* functions.

In his conclusion, Bôcher states:

> Of the methods invented during the last few years undoubtedly that of integral equations is the most far-reaching and powerful. This method would seem however to be chiefly valuable in . . . two or more dimensions where many of the simplest questions are still to be treated. In . . . one dimension where we now have to deal with finer . . . questions . . . older methods have . . . proved to be more serviceable,

and he emphasizes the "present vitality of these [older] methods."

Subsequent to his talk of 1912 at the International Congress in Cambridge (see above), G. C. Evans gave a survey on "Functionals and Their Applications" at the 1916 Cambridge Colloquium of the AMS which is concerned with functionals and integral equations and documents the extent to which the main ideas of Volterra, Hadamard, Fréchet, Riesz, Bôcher and Moore were slowly gaining general recognition in the United States at that time. Interestingly, the companion article in this volume, by O. Veblen, deals with combinatorial (not point-set!) topology arising from Poincaré's work of 1895–1900.

12. RIESZ' SPECTRAL THEORY AND COMPACT OPERATORS

F. Riesz' next major contribution of interest to us here is his book of 1913, entitled *Les Systèmes d'Équations Linéaires à une Infinité d'Inconnues* (Paris: Gauthier–Villars; in [Riesz 1960, 829–1016]. Its Preface states that "our subject is not part of the *Theory of functions* properly speaking. It should rather be considered as . . . a first stage in the theory of functions of infinitely many variables." Motivated by interest in orthogonal functions, integral equations, and operators, Riesz developed a conceptually different approach to Hilbert's spectral theory of 1906, replacing Hilbert's continuous forms by bounded linear operators [*substitutions linéaires*] on the "espace hilbertien" l^2 [Riesz 1960, 912], a setting and method that were to become standard. In this revision, "continuity" and "complete continuity" are given more prominent roles. On page 913, Riesz defines strong convergence of a sequence $\{x_k^{(n)}\}$ $(n = 1, 2, \ldots)$ in l^2 by $\Sigma|x_k - x_k^{(n)}|^2 \to 0$ and "*convergence au sens ordinaire*" by $x_k^{(n)} \to x_k$ for every k. He calls [p. 930] a bounded linear operator A "completely continuous" if A maps any convergent sequence onto a strongly convergent one, and shows that this is equivalent to Hilbert's "complete continuity" [53].

Next he introduces basic concepts and facts from spectral theory, such as convergence of sequences of operators [p. 941], spectral value [p. 948], resolvent, holomorphic character of the resolvent [p. 951], etc. For continuous real-valued f and self-adjoint bounded linear A he defines $f(A)$ and obtains [p. 971] a *spectral representation,* written in the now usual form

$$f(A) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda, \qquad (12.1)$$

where $[m, M] \subset \mathbb{R}$ is the shortest interval containing the spectrum of A, and (E_λ) is the spectral family associated with A.

Compact operators. In a basic paper on "linear functional equations," written and submitted in 1916, but not published until 1918 [*Acta Mathematica* **41,** 71–98; in Riesz 1960, 1053–1080] [54], Riesz created his famous *theory of compact operators* [55]. Since he developed this theory on general Banach spaces, just as in his L^p-space theory (Section 10) he no longer had available the powerful machinery connected with orthogonality. In the Introduction, he stated:

> The present paper treats the inverse problem for a certain class of linear transformations of continuous functions. . . . The most important concept applied in this connection is that of a compact set (here, especially, a compact sequence), introduced by Fréchet into general topology [*in die allgemeine Mengenlehre*]. . . . This concept permits an especially simple and felicitous definition of a completely continuous [*vollstetigen*] transformation, which is essentially modeled after a similar definition of Hilbert. . . .
>
> The restriction to continuous functions made in this paper is not essential. The reader familiar with the more recent investigations on various function spaces will recognize immediately the more general applicability of the method; he will also notice that certain among those, such as the square integrable functions and Hilbert space of infinitely many dimensions, still admit simplifications, whereas the seemingly simpler case treated here may be regarded as a test case [*Prüfstein*] for the general applicability [of the method].

That "seemingly simpler case" is $C[a,b]$, but Riesz developed everything in terms of the *norm* concept, and $C[a,b]$ hardly occurs in the formulas. Moreover, on the next page [p. 72], Riesz introduced (in 1916!) what were to become axioms for a Banach space six years later, saying:

We call the totality [of continuous functions on $[a,b]$] to be considered a *function space* [*Funktionalraum*]. We call the maximum of $|f(x)|$ the *norm* of $f(x)$ and denote it by $\|f\|$; hence $\|f\|$ is generally positive, and is zero only when $f(x)$ vanishes identically. Furthermore . . .

$$\|cf(x)\| = |c|\,\|f(x)\|; \qquad \|f_1 + f_2\| \leq \|f_1\| + \|f_2\|$$

By the *distance* of f_1, f_2 we understand the norm $\|f_1 - f_2\| = \|f_2 - f_1\|$.

On page 74 he defined a continuous linear operator to be *compact* [vollstetig] if it transforms every bounded sequence into a compact sequence. He essentially derived a general spectral theory of compact linear operators on Banach spaces, obtaining Fredholm's general theorems as special cases. Indeed, he showed that the set of the eigenvalues of a compact operator A is at most countable, that $\lambda = 0$ is the only possible point of accumulation of the eigenvalues, and that every $\lambda \neq 0$ in the spectrum of A is an eigenvalue, with finite-dimensional eigenspace [56]. Hence, the null space of A_λ^j, $j = 1, 2, \ldots$ ($A_\lambda = A - \lambda I$, $\lambda \neq 0$) is finite-dimensional; furthermore, the range of A_λ^j is closed, and so on. (For the "Fredholm alternative," see p. 86.) Riesz' results on the exceptional role of $\lambda = 0$ also explained puzzling earlier results on integral equations of the *first* kind in terms of the facts that then one may encounter an infinite-dimensional null space or a range which is not closed or is not of finite codimension.

Riesz' results and proofs underwent no substantial revision or extension in the further evolution of the theory of bounded linear operators; only some results on adjoint operators were added in 1928 by T. H. Hildebrandt [*Acta Mathematica* **51,** 311–318] and in 1930 by J. Schauder [*Studia Mathematica* **2,** 183–196]. For this reason, this theory is often called the Riesz–Schauder theory.

13. "FUNCTIONAL ANALYSIS" IS NAMED

Hadamard and Fréchet continued to promote functional analysis for at least two decades after Fréchet's thesis appeared. Thus Hadamard published an influential article on "le calcul fonctionnel" in *L'Enseignement Mathématique* in 1912 [Hadamard 1968, **4,** 2253–2266]. As was usual in his pronouncements about this subject, what he said was *philosophical* and nontechnical. Characteristic of his opinions were the following statements:

The functional continuum . . . offers no simple visualization [*image*] to our spirit, ... Geometric intuition teaches us nothing *a priori*. We are forced to remedy this ignorance, and we can only do so analytically, in creating for applications to the functional continuum a chapter of set theory.

If one wants to continue to follow, in regard to functions, the same path that was traversed for numbers, it will remain:

1° To think of functions themselves, not as defined specifically, but as being continuously varied;

2° To subject them, not just to two or three of certain operations but to more or less arbitrary ones.

The branch of Mathematics whose aim is thus defined is what is today called *functional calculus* [*calcul fonctionnel*].

It results from the preceding considerations that one should see in it the sequel and natural consequence of the infinitesimal Calculus itself and of the current of ideas to which it gave birth.

Somewhat later in this article, he emphasized what had become his own special interest in functional analysis:

> The calculus of variations is for functional operations what differential calculus is for functions.

Fréchet. Pursuing Hadamard's ideas, Fréchet introduced the *Fréchet differential* in a series of papers, of which the first was published in 1909 and the last in 1925 under the title "On the notion of differential in General Analysis" [57]. His steady stream of papers, expatiating on the general ideas of Hadamard, included two in *Transactions of the American Mathematical Society* **15** (1914), 135–161, and **16** (1915), 215–234, and one on "the dimension of an abstract set" in *Mathematische Annalen* **68** (1910), 145–168. In this paper, a set of topological spaces (each being a subset of a "classe (*L*)") is quasi-ordered by the relation "*S* is homeomorphic to a subset of *T*." This paper illustrates the extreme generality of many of Fréchet's ideas, which unfortunately often had the *weakness* of extreme generality: that of having few useful properties. Fréchet's very enthusiasm for generalization may have been a liability: his prolific writings often obscured his long-range objectives.

Another promising young functional analyst pursuing Hadamard's ideas was R. Gâteaux. He wrote three notes and three papers on functionals and abstract differentiation in 1913 and 1914. He was killed in the first months of World War I.

Lévy. Hadamard had encouraged his students Paul Lévy (1886–1971) and Joseph Pérès (1890–1963) to assist Volterra just before World War I. After the war, Lévy edited the three papers by Gâteaux mentioned above, which were published in the *Bulletin de la Société Mathématique de France* **47** (1919), 70–96, and **50** (1922), 1–21. The "Gâteaux differential," defined under even weaker assumptions than the "Fréchet derivative," had considerable influence on later work in this area.

More substantial was Lévy's book *"Leçons d'Analyse Fonctionnelle"* (Paris: Gauthier–Villars, 1922), prefaced by Hadamard. The first part of the book, entitled (in French) "The foundations of the functional calculus," divides the *"calcul fonctionnel"* [on p. 5] into *"algèbre fonctionnelle"* and *"analyse fonctionnelle"* (functional analysis, here named for the first time). The first of these includes "problems whose unknowns are ordinary functions," and the second "problems whose unknowns are functionals." As a typical problem of "functional analysis," he considers the variation δG in the Green's function $G_\Omega(x;\xi)$ of the Dirichlet problem for a domain Ω with variations δx, $\delta\xi$, and $\delta\Omega$ in x, ξ, and the boundary $\Gamma = \partial\Omega$ of Ω. Lévy soon turned his attention to probability theory, where his book *Calcul des Probabilités* [1925] contains an appendix on "probability in abstract spaces," written in the spirit of Fréchet and foreshadowing Kolmogoroff's *Grundbegriffe der Wahrscheinlichkeitsrechnung* of 1933. Much later, in 1951, he issued a revised edition of his book. Part IV of this edition, dealing with "analytic functionals," the theory of which had been created by L. Fantappiè (beginning around 1925) and F. Pellegrino, may be regarded as the final expression of the "Volterra–Hadamard school."

Pérès. In 1924, Pérès co-authored with Volterra a book, *Leçons sur la Composition et les Fonctions Permutables*, also published by Gauthier–Villars. This somewhat meandering monograph applied operational methods to convolution formulas, with special reference to methods of E. T. Whittaker and N. E. Nörlund.

G. C. Evans. During the years 1910–1926 the "*calcul fonctionnel*" of Hadamard and Fréchet as well as Volterra's theories of integral operators was also being actively developed in the United States, where the connections with E. H. Moore's *General Analysis* were appreciated. We have already mentioned (in Section 11) Evans' Colloquium Lectures of 1916. In addition, Evans and P. J. Daniell translated Volterra's Rice Institute Lectures of 1917. Daniell also constructed a very original theory of integration in infinite-dimensional space, which influenced the thinking of Norbert Wiener among others [58].

Norbert Wiener (1894–1964) made at least three significant contributions to functional analysis in the 1920s. Among these, most lasting was his construction of a countably additive "Wiener measure" in the space of all continuous functions on ℝ. He sketched this construction, which was to become the cornerstone of his "generalized harmonic analysis" *and* of the theory of Gaussian stochastic processes, in three notes of 1920–1921. These notes published in the *Annals of Mathematics* and the *Proceedings of the National Academy of Science* (USA) **7,** 253–260; 294–298 (see also [Wiener 1976– , 1, 435–454]), and the fuller accounts in the *Journal of Mathematics and Physics* [**2,** 131–174], *Proceedings of the London Mathematical Society* **22,** 454–467 (see also [Wiener 1976– , 1, 455–512]), refer copiously to Fréchet, Gâteaux, Lévy, and E. H. Moore.

14. POINT-SET TOPOLOGY ADVANCES

Fréchet's attempt of 1906 to characterize nonmetric properties of convergence common to sequences of points and sequences of functions (cf. Section 5) led him to a very general concept of *limit space* ["*classe (L)*"; p. 5 of his thesis]. This concept made possible simple and useful definitions of *separability* and (sequential) *compactness,* as well as of the concept of a *derived set,* basic for the studies of Cantor and Baire. However, in the context of function spaces, the notion of a derived set leads to a tangled web of possibilities with no good way out. This in spite of the ingenuity of Baire's thesis and the fruitful concept of Baire *category*.

Almost simultaneously, a more promising axiomatic approach to point-set topology (general topology) was sketched by F. Riesz in 1906 (published in 1907; [Riesz 1960, 110–154]) [59] and summarized in 1908 in the *Atti* of the International Congress, Rome, Vol. 2, pp. 18–24 (see also [Riesz 1960, 155–161]).

As his central concept, Riesz axiomatized the notion of "*Verdichtungsstelle*" or *accumulation point* (*not* condensation point!) in modern terms, as follows. For each subset M and point p, it is defined whether p is an acccumulation point of M or not (is "isolated"), and this association satisfies the four axioms [Riesz 1960, 119]:

1. Finite sets have no accumulation points.
2. An accumulation point p of a set M is also an accumulation point of any set containing M.

3. If M is partitioned into M_1 and M_2, then any accumulation point of M is also an accumulation point of M_1 or M_2 (or both).
4. For an accumulation point p of M and a point $q \neq p$ there is a subset M^* of M such that p is an accumulation point of M^* but q is not.

In terms of accumulation points, Riesz then defined [pp. 120–121] the notions of neighborhood, interior point, boundary point, open set, derived set, etc., and showed that his axioms implied the expected properties. Furthermore [p. 122], he gave a formulation for the concept of *connectedness* previously discussed by C. Jordan [*Cours*, 1893] and A. Schoenflies [*Mathematische Annalen* **58** (1904), 195–234]. Increasing general interest in point-set topology and its rapid development since 1900 is in part reflected in the second edition [1913] of the Schoenflies *Bericht* on point-set theory, which profited greatly from critical remarks by Brouwer and from the close collaboration between Schoenflies and Hahn.

L. E. J. Brouwer (1881–1966), at about this same time, had just made several famous discoveries: his fixed point theorem (cf. Section 16), the invariance of dimension of $\mathbb{R}^n$ (i.e., $\mathbb{R}^n$ and $\mathbb{R}^m$ with $m \neq n$ are not homeomorphic; *Mathematische Annalen* **70** (1911), 161–165), the invariance of domain and a proof of the extension of Jordan's curve theorem to $\mathbb{R}^n$ [*Mathematische Annalen* **71** (1912), 305–313].

Hans Hahn (1879–1934) of Vienna, later the founder of the Viennese school of topology and functional analysis of the 1920s and 1930s, was originally interested in the calculus of variations (in 1904 he wrote an article on this subject for the *EMW*, jointly with E. Zermelo, both of whom were then at Göttingen), but soon turned his interest to Fréchet's investigations and real functions in the spirit of Borel, Baire, and Lebesgue.

Another approach to nonmetric topology is obtained by utilizing the concept of a *neighborhood* of a point p, that is, an open set U that contains p. Foreshadowed in special contexts, such as complex analysis [H. Weyl, *Die Idee der Riemannschen Fläche,* Leipzig: Teubner, 1913], the neighborhood concept took definite form in 1914 in *Grundzüge der Mengenlehre* (New York: Chelsea, 1965) by Felix Hausdorff (1868–1942), dedicated to Georg Cantor. The appearance of this book marks the emergence of point-set topology as a separate discipline.

In his book [p. 213], Hausdorff defined a topological space to be a set E in which to each element x there are associated subsets U_x [*Umgebungen von x*] such that:

U1. Each x has at least one neighborhood U_x and belongs to each of its neighborhoods.

U2. The intersection $U_x \cap V_x$ of any two neighborhoods of x contains a neighborhood W_x of x.

U3. For any $y \in U_x$ there is a neighborhood $U_y \subseteq U_x$.

U4. For any two distinct points there exist disjoint neighborhoods.

He defined the concepts of an open set (calling it a "*Gebiet*," thus deviating from our standard [Weierstrassian] terminology), boundary, closed set, and reinterpreted the (sequential) limit concept in terms of neighborhoods; his axiom U4 assured the uniqueness of the limit of convergent sequences. To honor his contributions, spaces satisfying U1–U4 are today called *Hausdorff spaces*.

Hausdorff also introduced the name *"metrischer Raum"* as a substitute for Fréchet's *"espace distancié,"* and the words "metric" and "metric space" gradually became standard in English.

Fréchet was unenthusiastic about, even critical of, Hausdorff's approach to general topology. Generalizing Hausdorff's ideas in 1917, Fréchet [*Comptes Rendus* (Paris) **165,** 359–360] proposed a notion of *"espace (V)"* [*V* for "voisinage"]. In his book *Les Espaces Abstraits* of 1928 (Paris: Gauthier-Villars), he argued further that one could equally well define a "neighborhood" of a point p as *any* set containing p in its interior.

Still other ways of topologizing a set without using a metric were soon discovered. In 1922, C. Kuratowski [*Fundamenta Mathematicae* **3,** 182–199] published his doctoral thesis "Sur l'opération $\bar{A}$ de l'Analysis Situs," in which he introduced the concept of closure. Pursuing ideas related to those of Riesz, he defined a topological space to be a set X together with a *closure operation* $C: S \mapsto \bar{S}$ on the subsets of X satisfying the axioms [p. 182].

C1. $\overline{S \cup T} = \bar{S} \cup \bar{T}$ (Distributivity)
C2. $S \subset \bar{S}$ (Inclusion)
C3. $\bar{\emptyset} = \emptyset$
C4. $\bar{\bar{S}} = \bar{S}$ (Idempotence)

He called S a *closed set* if and only if $S = \bar{S}$.

By 1912, the need to assume some form of C4 in topology had become recognized, and E. R. Hedrick [*Transactions of the American Mathematical Society* **12** (1911), 285–294] demonstrated that in the *compact* case, the assumption of this condition gives a quite satisfactory theory in the limit space context.

By dropping C3 and weakening C2 to

C2′. If $S \subset T$, then $\bar{S} \subset \bar{T}$ (Monotonicity),

we get the closure axioms proposed by E. H. Moore in 1906 (published in 1910 in the New Haven Colloquium Publication of the AMS) as a way to characterize "extensionally attainable" properties.

Dual to the notion of a closed set is that of an open set, whose complement is closed. Axioms for open sets as the central concept for defining a topology were proposed in 1923 by H. Tietze [*Mathematische Annalen* **88,** 290–312] and in the present-day form in 1925 by P. Alexandroff [*Ibid.* **94,** 296–308]. As two further possibilities, the use of derived sets (satisfying a certain "monotonicity condition") or of closed sets as the primitive concept in defining a topology were explored in 1927 by W. Sierpiński [*Mathematische Annalen* **97,** 321–337).

These various approaches to the definition of a topological space are summarized in a new section [Section 40] of Hausdorff's classic in the second edition of 1927, simply entitled *Mengenlehre.* To the disappointment of his readers, Hausdorff had to shorten the original version by about 200 pages and accomplished this by concentrating mostly on metric spaces. Time has vindicated the approach to topology of F. Riesz, E. H. Moore, Kuratowski, and Hausdorff.

Separation axioms. In the early 1920s, Hausdorff's axiom U4, today designated as (T_2), became recognized as only one of a series of "separation axioms"; cf. the classic "Topologie I" by P. Alexandroff and H. Hopf (Springer, 1935). In particular, two other separation axioms (T_3) and (T_4) were found to be basic.

(T_3) Any closed set and single point not in the set have disjoint neighborhoods.

(T_4) Any two disjoint closed sets have disjoint neighborhoods.

Axiom (T_3) was introduced by L. Vietoris (then at Graz) in 1921, and (T_4) by H. Tietze of Munich in 1923. Tietze also coined the term "*Trennungsaxiom*" [separation axiom]. A space satisfying (T_1) and (T_3) became known as *regular*, and one satisfying (T_1) and (T_4) as *normal* [60].

The power of these axioms was first demonstrated by P. Alexandroff and P. Urysohn [61]. It is easy to show that every metric space is normal; Alexandroff and Urysohn proved two converses to this result in 1924. First, they showed that a *compact* Hausdorff space is *metrizable* (i.e., homeomorphic to a metric space) if and only if it has a countable basis of neighborhoods. And second, they showed that any *normal* space having a countable neighborhood basis is homeomorphic to a subset of the Hilbert space l^2.

The Polish school. No discussion of advances in point-set topology during the 1920s would be complete without mention of the Polish school, created by Z. Janiszewski (1888–1920), W. Sierpiński (1882–1969), and S. Mazurkiewicz (1888–1945) shortly after the reconstitution of Poland in 1918. The work of these mathematicians centered around logic and foundations, set theory and measure and, in particular, point-set topology, whose remarkable development since 1920 owes much to this school. Many of their publications appeared in the newly founded *Fundamenta Mathematicae,* which became *the* leading journal on point-set topology almost overnight.

We can here give only three samples of their many accomplishments [62]. First, continuing Peano's famous work on curves (Section 4), Hahn of Vienna (1914) and Mazurkiewicz [*Fundamenta Mathematicae* **1** (1920), 166–209] proved that a Hausdorff space is a continuous image of [0,1] if and only if it is a "Peano space" (i.e., a compact, connected, locally connected metric space). (In a similar vein, Alexandroff and Urysohn proved in 1929 [*Verhandelingen K. Akademie van Wetenschappen, Amsterdam* **14,** 1–96] that every compact metric space is a *continuous* image of the "Cantor set.") Furthermore, in his thesis of 1922 (see above), Kuratowski showed that from a given set S at most 13 other (in general different) sets could be constructed by the repeated use of complementation and closure.

Finally, Hausdorff's penetrating analysis of measure was developed into the celebrated Banach–Tarski paradox. This states that by properly reassembling the pieces of a finite decomposition of the sun, one can fit them all (without overlap) into a pea! [For an exposition, see K. Stromberg, *American Mathematical Monthly* **86** (1979), 151–161].

American contributions. We have already mentioned Hedrick's contribution to the foundations of point-set topology in 1912; a related study was made in 1914 by E. H. Moore's student R. E. Root [*American Journal of Mathematics* **36,** 79–133]. Because of the congeniality between his "general analysis" and Fréchet's "*calcul fonctionnel,*" E. H. Moore also encouraged T. H. Hildebrandt and E. W. Chittenden to study related questions; cf. [*American Journal of Mathematics* **34** (1912), 237–290; *Transactions of the American Mathematical Society* **18** (1917), 161–166]. A review article of 1918 by A. D. Pitcher and Chittenden [*Ibid.* **19,** 66–78], which slightly generalized Hahn's 1908 theorem on the existence of continuous functions "separating" arbitrary point-pairs in regular spaces, gives a reliable impression of the confused state of the subject at that time. Chittenden's later survey of 1927 [*Bulletin of the American Mathematical Society* **33,** 20–34] relates the solution of the metrization problem by Alexandroff and Urysohn to earlier American results, including those of R. L. Moore and his students about *plane* sets of points.

15. BANACH AND FRÉCHET SPACES

Much as Warsaw, with its *Fundamenta Mathematicae,* became a major center of point-set topology, so Lwów became one in functional analysis, under the leadership of Hilbert's student Hugo Steinhaus, and with *Studia Mathematica* (founded in 1929) as its journal. Appropriately, since Lwów had been part of the old cosmopolitan Austro-Hungarian empire (under the name of Lemberg), the influence of the Austrian mathematician Hans Hahn and the Hungarian Friedrich Riesz was very evident.

The main interest of Hahn and Riesz in functional analysis was theoretical; they were not especially interested in applications to the partial differential equations of mathematical physics. The same was true of Stefan Banach (1892–1945), whom Steinhaus "discovered" in 1916. Thus in his book [Banach 1932], the chapter on spectral theory refers to Volterra, Fredholm, and Riesz, rather than to Hilbert's *Integralgleichungen,* and Hilbert's space l^2 gets no special emphasis [63].

Although Hahn had contributed to point-set topology as early as 1908, it was his 1921 *Theorie der reellen Funktionen* and his subsequent papers that were most significant for functional analysis. These were stimulated by Riesz' work and by Banach's initiation of a theory of *Banach spaces* to which we shall now turn.

Banach's first *magnum opus* was his Ph.D. thesis, submitted in 1920 and published in *Fundamenta Mathematicae* **3** (1922), 133–181. At the beginning of the thesis, after acknowledging the contributions of Volterra, Pincherle, Hadamard, Fréchet, and others, Banach credited Hilbert with emancipating functional analysis from a special concern for "continuous functions having derivatives of higher order." He then gave careful axiomatic definitions of *real* vector spaces and norms on them. As is done today, he defined a "norm" on a vector space X as a functional $X \to \mathbb{R}$, $x \mapsto \|x\|$, that satisfies

$$\begin{aligned} &\|0\| = 0, \quad \text{while} \quad \|x\| > 0 \qquad \text{if } x \neq 0, \\ &\|x + y\| \leqq \|x\| + \|y\|, \\ &\|\alpha x\| = |\alpha|\,\|x\| \qquad \text{for any scalar } \alpha. \end{aligned}$$

Banach also assumed X to be "complete" in the sense of Cauchy and Fréchet.

In itself, Banach's thesis was not earth-shaking, although it did prove the uniform boundedness principle (see below) for linear operators on Banach spaces, previously used in special cases by Lebesgue and others. In the special case of linear functionals, this theorem was simultaneously obtained by Hahn [*Monatshefte für Mathematik und Physik* **32** (1922), 3–88]. In this paper, Hahn also defined *norm* (without giving it a name) and Banach space, and he used various results from his book *Theorie der reellen Funktionen*. His *Reelle Funktionen* of 1932 (Leipzig: Akademische Verlagsgesellschaft; reprinted, New York: Chelsea, 1948) is an extended version of this book which was widely read after functional analysis became popular.

Indeed, the norm concept seems to have been "in the air" in 1920. Riesz had used the term "*norm*" (for the maximum norm, $\|f\| = \sup_x |f(x)|$) already in 1916, on page 72 of his paper in *Acta Mathematica* **41** [54]. In 1921, the Austrian Eduard Helly (1884–1943) used an axiomatically defined norm (which he called "*Abstandsfunktion*") in general sequence spaces [*Monatshefte für Mathematik und Physik* **31,** 60–91]. Norbert Wiener (1894–1964), who had sojourned for some time in France in 1920, following Fréchet around, independently defined Banach spaces in 1922 [*Bulletin de la Société Mathématique de France* **150,** 124–134] [64]. A year later, in a note on Banach's thesis [*Fundamenta Mathematicae* **4** (1923), 136–143], Wiener pointed out that by using *complex* vector spaces one obtains a complex analysis for functions of a complex argument with values in a normed space.

Banach continued to develop the theory of "Banach spaces" [*espaces* (*B*)] actively for another decade, first with the encouragement of Steinhaus and later with the collaboration of S. Mazur. Banach's famous book [1932], which resulted from these efforts, will be discussed in a separate section (Section 21).

Fréchet spaces. Stimulated by the "wider horizons" for functional analysis opened up by the axiomatization of Banach spaces, Fréchet introduced in 1926 the "more general" concept of what he called a "topologically affine space" [65], but we will call an *F-space* or *Fréchet space,* following [Banach 1932; Dunford & Schwartz 1958], and others [66]. By definition, this is a complete, metrizable topological vector space.

For example, the real numbers with $d(x,y) = |x - y|/(1 + |x - y|)$ form an *F*-space which is not a Banach space (since $d(\alpha x,\alpha y) \neq |\alpha| d(x,y)$ in general), and the same holds for the set of all real sequences with metric defined by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

The following two basic principles, a special case of the first for Banach spaces being contained in Banach's thesis, hold generally in any *F*-space:

UNIFORM BOUNDEDNESS PRINCIPLE. *Let* $\{T_\alpha\}_{\alpha\in A}$ *be a family of continuous linear operators on an F-space X into an F-space Y such that for each* $x \in X$ *the set* $\{T_\alpha x\}_{\alpha\in A}$ *is bounded. Then* $\lim_{x\to 0} T_\alpha x = 0$ *uniformly in* $\alpha \in A$.

This was proved for *F*-spaces by S. Mazur and W. Orlicz in 1933; cf. [Dunford & Schwartz 1958, 81].

INTERIOR MAPPING PRINCIPLE. *Under a continuous linear mapping T from an F-space* onto *another, the image of every open set is open.* (Also known as the *open mapping theorem,* this was proved by Banach in 1929 for Banach spaces and in [Banach 1932] for *F*-spaces.) *Hence if T is bijective, its inverse is also continuous ("Bounded inverse theorem").*

The following theorem results from the Interior Mapping Principle:

CLOSED GRAPH THEOREM. *If the* graph *(set of all pairs* (x,Tx)*) of a linear operator T from an F-space X into an F-space Y is* closed *in* $X \times Y$ *(with the usual topology), then T is continuous.*

Hahn's 1922 proof of the Uniform Boundedness Principle (see above) used a "method of the gliding hump," a device already applied earlier, in 1906, by Lebesgue (in his book on Fourier series) and by Hellinger and Toeplitz [*Göttinger Nachrichten,* 351–355]. In 1927, Banach and Steinhaus [*Fundamenta Mathematicae* **9,** 50–61] discovered a proof of the theorem based on Baire's category theorem (extended to general complete metric spaces); cf. Section 5 [67]. This demonstrated the importance of Baire's category concept, and proofs based on Baire's category theorem were soon discovered for the other two results mentioned above.

For these fundamental results, *completeness* of spaces is essential. In contrast, the following theorem holds in any normed space, regardless of its completeness.

HAHN–BANACH THEOREM. *Any continuous linear functional f on a subspace S of a real normed space X can be extended to a continuous linear functional on all of X having the same norm as f.*

Hahn proved this theorem in 1927 [*Journal für die reine und angewandte Mathematik* **157,** 214–219], acknowledging the stimulus of earlier work by Helly [*Sitzungsberichte der Math.-Nat. Klasse der Akademie der Wissenschaften Wien* **121** (1912), 265–297; *Monatshefte für Mathematik und Physik* **31** (1921), 60–91], and giving an interesting motivation in terms of integral equations of the second kind. In 1929, Banach [*Studia Mathematica* **1,** 223–239] rediscovered Hahn's result and method of proof, which he used to prove a more general form of the theorem (cf. our Section 21).

Duality in normed spaces. The Hahn–Banach theorem guarantees that every normed space is richly supplied with continuous linear functionals, thus permitting a satisfactory general duality theory. The continuous linear functionals on any normed space *X* constitute a Banach space, the *dual space* X^* of *X*, with norm $\|f\| = \sup_{\|x\|=1}|f(x)|$.

The duality of Banach spaces became clear soon after Hahn introduced the abstract notion of a dual space [*polarer Raum*] on page 219 of his above paper of 1927. He noted, as a corollary of the Hahn–Banach theorem, that for any nonzero

$x \in X$ there is an $f \in X^*$ of norm 1 such that $f(x) = \|x\|$, so that X^* is nontrivial. He also established an isomorphism of a normed space X onto a subspace of its second dual $X^{**} = (X^*)^*$, calling X "regular" (now called "reflexive") if that subspace is all of X^{**}. (In this case, X must be complete. Also, $X^* \cong X^{***}$, that is, X^* is isomorphic to its second dual.)

16. FIXED POINT THEOREMS TO 1926

By definition, "fixed point theorems" assert the existence of solutions of equations of the form $T(f) = f$, where T is a transformation of some "space" into itself. If T is "contractive" in some neighborhood of f, then a Cauchy sequence of approximate solutions f_n can often be constructed by simple *iteration:* choose an initial f_0 (*perhaps* $f_0 = 0$), set $f_{n+1} = T(f_n)$ and iterate. Newton's method for solving $F(x) = 0$ is a classic example; in this case, $x_{n+1} = x_n - F(x_n)/F'(x_n)$.

More relevant to us is Neumann's method for solving linear integral equations of the "second kind,"

$$f + Kf = \phi, \qquad \text{where } Kf(x) = \int_a^b k(x, y)f(y)\,dy, \tag{16.1}$$

Taking $f_0 = 0$, and setting

$$f_{n+1}(x) = \phi(x) - \int_a^b k(x, y)f_n(y)\,dy \tag{16.2}$$

often gives in $C[a,b]$ a Cauchy sequence of approximate solutions $f_n(x)$; the limit of these is then a solution.

Likewise, in 1890, E. Picard (1856–1941) used an iteration method to prove his existence and uniqueness theorem for first-order ordinary differential equations, $dy/dx = F(x, y)$, $y(a) = y_0$. He set $y_0(x) \equiv y_0$ and

$$y_{n+1}(x) = y_0 + \int_a^x F(t, y_n(t))\,dt.$$

Taking Picard iteration as a model, Banach proved in his thesis [p. 160] a fixed point theorem for *contraction* mappings T satisfying

$$d(Tx,Ty) \leqq \alpha d(x,y), \qquad \alpha < 1, \tag{16.3}$$

on any *complete* metric space (e.g., any Fréchet space). He proved that T then has a unique fixed point, which is the limit of any iterative sequence. The proof is very simple, essentially a repeated application of the triangle inequality and the use of the sum formula for the geometric series. This "Banach contraction theorem" has since been extended and greatly refined by careful analysis. Thus, in 1927, T. H. Hildebrandt and L. M. Graves [*Transactions of the American Mathematical Society* **29,** 127–153, 514–552] proved an implicit function theorem in any complete metric space.

More sophisticated than such *metric* fixed point theorems, and requiring much more ingenuity, are *topological* fixed point theorems. The earliest substantial result of the latter type was *Brouwer's fixed point theorem* of 1912 [*Mathematische Annalen* **71,** 97–115], which states that any continuous mapping of a closed ball in $\mathbb{R}^n$ into itself has a fixed point. Brouwer's proof made essential use of the concept of a *polyhedral complex*. This had been invented by Poincaré only 13 years before, and provides the foundation for *combinatorial topology*.

Also in 1912, Poincaré "enunciated a theorem of great importance . . . for the restricted problem of three bodies" [68]. This theorem, often called "Poincaré's last geometric theorem," can be stated as follows:

POINCARÉ–BIRKHOFF FIXED POINT THEOREM. *Let T be an area-preserving homeomorphism of an annulus which advances the points of the inner boundary circle in one sense and those of the outer boundary in the opposite sense. Then T has at least two fixed points.*

Dynamical systems. Poincaré's conjecture was part of his campaign to introduce "new methods into celestial mechanics," centering around the *n-body problem*. Hamilton and Jacobi had shown that the motion of *any* system of n bodies, under the action of *any* universal law of gravitation with $F_{ij} = gm_im_j/r^2$, was governed by the *Hamilton–Jacobi equations*

$$\dot{q}_i = \partial H/\partial p_i, \qquad \dot{p}_i = -\partial H/\partial q_i, \tag{16.4}$$

and Liouville had shown that the *flow* in $(p_1,q_1, \ldots ,p_{3n},q_{3n})$-space (the so-called *phase space*) is *volume-conserving*. Among other things, Poincaré showed that a careful consideration of this *geometrical* fact, and of the global *topology* of the phase space, provided a powerful new tool for treating the problems of celestial mechanics.

During the decade following Poincaré's death, G. D. Birkhoff concentrated on the development of "new methods in celestial mechanics", in the spirit of Poincaré. His AMS Colloquium Lectures of 1922 on "Dynamical Systems" [AMS 1927] were devoted to these; the existence of the "central motions" emphasized in it depends essentially on transfinite induction (Cantor), and refers to ideas of Hadamard. The Ph.D. thesis of Marston Morse (1892–1976), which constituted the beginning of "Morse theory," was written under Birkhoff's guidance during this period.

Of course, the main interest in fixed point theorems for analysis lies in the *infinite-dimensional* case. This was emphasized by G. D. Birkhoff and O. D. Kellogg in a path-breaking paper of 1922 [*Transactions of the American Mathematical Society* **23,** 96–115):

> Existence theorems in analysis deal with functional transformations. This suggests that such existence theorems may be obtained from known theorems on point transformations in space of two or three dimensions by generalization, first to space of n dimensions, and then to *function space* by a limiting process.

Then the authors described several methods for proving the existence of fixed points, making essential use of compactness hypotheses (Ascoli's theorem) to prove the existence of fixed points as well as of "invariant directions" (such that $Tx = \lambda x$). One of their basic ideas was to approximate infinite-dimensional compact convex sets and their mappings by finite-dimensional ones, and to apply Brouwer's theorem to the latter. The paper mentions "Hilbert space" [on p. 102], and stimulated later definitive work by Schauder and Leray (see Section 20).

17. FUNCTIONAL ANALYSIS IN 1928

The pioneers of functional analysis in 1910–1914—Volterra, Hadamard, Fréchet, and Hilbert—all gave invited addresses at the 1928 International Congress of Mathematicians in Bologna, of which Pincherle (who also gave a paper) was Honorary President. It is fascinating to read, in retrospect, what each of them said.

Volterra, although 68, gave the liveliest talk. Always more concerned with *applications* than with foundations ("pure" functional analysis), his Bologna lecture gave new applications to "hereditary phenomena," some of the general problems of which he reformulated (via integro-differential equations) in terms of spaces of functions and functionals. In his recently published Madrid Lectures, his concern (already in 1925) with the "new quantum mechanics of Heisenberg, Born and Jordan" was especially timely, as we shall see in Section 18 [69]. Even his remarks on Fréchet's "abstract spaces" [1930, 205] seem to the point, although he had yet to make noticeable use of Fréchet's general topology.

Hadamard subordinated technical results to philosophical remarks and graceful (sometimes fulsome) tributes to Volterra, Fréchet, P. Lévy, Gâteaux, and others. He observed [1968 1, 145] that functional analysis is on a higher level of abstraction than the theory of functions (of a real or complex variable), because: (1) it regards functions themselves as variables, and (2) it "subjects functions to the most varied and most general operations." He praised Fréchet and E. H. Moore for realizing that the simplest and clearest path was to adopt extreme generality from the outset [p. 150]. On page 151, he noted that Banach and Wiener gave a "deeper" meaning to vector spaces of functions, but did not mention their key tool, the norm concept. Although he credited Hilbert (along with Arzelà) for having rigorized the Dirichlet principle [p. 158], he never mentioned Hilbert's *Integralgleichungen* or his spectral theory.

Fréchet, speaking on "general analysis and abstract spaces," was also philosophical, referring to his book *Espaces Abstraits* for technical details. Thus [p. 269] he observed (citing his 1921 Calcutta Lectures) that:

> A difficulty arises in passing from Arithmetic to Analysis. One must give meanings to the notions of neighborhood, of limit, of distance, and of continuity for abstract elements. Within Functional Analysis, the difficulty is not excessive; classical Analysis gives [a guide]. For example, for continuous functions, measurable functions, and square integrable functions, the [natural] definitions of convergence will, in most cases, be uniform convergence in the first case, convergence in measure in the second, and mean square convergence [*convergence en moyenne*] in the third. . . .

And a few lines later he says that in general analysis, since the nature of the elements referred to is unknown, one can only hope for "descriptive and incomplete definitions."

Operational calculus. In England and the United States, moreover, the "operational calculus" was still being pursued with emphasis on Laplace transform techniques, in the spirit of Pincherle. The main emphasis was on trying to rigorize the unconventional ideas of Oliver Heaviside, with his "delta function" reminiscent of the Stieltjes integral. These had proved especially fruitful for electrical circuit theory, to the rigorization of which Thorton C. Fry had applied "generalized integrals" in 1920, citing Hildebrandt's survey article [70]. Then in 1925 Norbert Wiener [*Mathematische Annalen* **95**, 557–584] applied the Fourier integral (Plancherel's theorem) to the same end. Although he emphasized that the techniques of Pincherle and Volterra were inapplicable, he referred only to their work and that of Doetsch, not to Hilbert's spectral theory.

Alone of the four great leaders during the pioneer years of functional analysis, Hilbert had lost interest in the subject. His invited paper at Bologna concerned the foundations of mathematics and mathematical logic. Apparently, he considered his ideas about these fields to hold more promise for the future than spectral theory at that time. Curiously, by 1931 his ambitious hopes for formal logic would be shattered by Gödel, while the spectral theory stemming from his ideas of 1906 would be attractive again.

Section I-C in Volume III of the *Proceedings* of the Bologna Congress contains several other relevant papers. Most notable is F. Riesz' paper on "the decomposition of linear functional operations" [pp. 143–148 of Vol. III] into positive and negative parts. This was the seed of the modern theory of vector lattices ("Riesz spaces"), that is, vector spaces which form a lattice under an appropriate notion of positivity. Besides this and a paper by Pincherle, relevant communications were given by Tonelli (on the semicontinuity of double integrals) and Steinhaus, Kaczmarz, and Fantappiè (on quantum mechanics and on analytic functionals).

To round out the picture, one should recall that a year before the Bologna Congress, in 1927, Hellinger and Toeplitz completed their masterful (if retrospective) 250-page *Encyklopädie* article on integral equations and equations in infinitely many unknowns. This review gave Hilbert a dominant role, devoting just a few pages to the general analysis of E. H. Moore [Hellinger & Toeplitz, 1927, 1471–1476, 1495–1497]. The contrast between this strong emphasis expressed here and the meager references to Hilbert's [1912] in those invited talks at Bologna suggests that few mathematicians in 1928 regarded Hilbert's spectral analysis as part of the mainstream of "functional analysis."

Most surprising, the Bologna *Proceedings* give no sense of the creative ferment generated in Lwów by the systematic study of Banach spaces, or of the stimulus already being provided by the foundations of quantum mechanics. How misleading this turned out to be! Within five years, a new and dynamic generation of mathematicians would revolutionize the subject. This revolution was made possible by *combining* a concern for rigorous foundations with an interest in physical applications, *and* by coordinating the relevant literature in depth. By doing all this, a handful of outstanding young mathematicians was about to make functional analysis the dominant branch of analysis for at least the next two decades. It is with their achievements that the rest of this article will be concerned.

18. JOHN VON NEUMANN

Already in 1900, Hilbert had proposed, as the sixth in his celebrated list of problems, the axiomatization of "those physical sciences in which mathematics plays an important part," in the style of his own *Grundlagen der Geometrie* of 1899 [71]. During the years 1910–1920, Hilbert made progress on this problem his own primary concern, and continued "to lecture and conduct seminars on topics in physics" through the 1920s [Weyl, *Bulletin of the American Mathematical Society* **50** (1944), 653].

Most important for functional analysis, by 1927 Hilbert had already revived interest in the spectral theory of linear operators, as a by-product of his lectures in 1926–1927 on the then brand new (1925–1926) quantum mechanics of Heisenberg, Schrödinger, Dirac, Born, and Jordan. This impulse came in a joint paper with L. Nordheim and J. von Neumann [*Mathematische Annalen* **98** (1927), 1–30; Neumann 1961–1963 1, 104–133]. John von Neumann (1903–1957) had just arrived from Budapest and joined the Göttingen Institute as Hilbert's assistant in 1926. The paper represents Nordheim's formulation of a proposal by Hilbert for axiomatizing quantum mechanics, thereby making "previously vague concepts such as probability, lose their mystical character." It cites papers by Schrödinger and by Born and Wiener as demonstrating that *operator theory* provides "the connection between Schrödinger's theory and Heisenberg's matrix mechanics," [*Ibid.*, 6].

An algebra of *"complete operators"* [*vollständige Operatoren*] is postulated, which includes the Heaviside–Dirac δ-symbol, and *probability amplitudes* are determined by "the kernel of the associated integral operator." Von Neumann's role in this paper consisted in supplying "some important derivations."

At the same time, von Neumann was working on another approach to quantum mechanics, which he published in three papers in 1927 [*Göttinger Nachrichten*, 1–57, 245–272, 273–291; Neumann 1961–1963 1, 151–255]. In these papers, he emphasized [p. 157] that Hilbert's 1906 spectral theory of bounded operators, even as extended by his students, was inadequate for quantum mechanics, since even the simplest problems required *unbounded* operators. In this connection, as an essential step forward, he gave [p. 165] the earliest *axiomatic* definition of Hilbert space, pointing out that the sequence space l^2 and the function space $L^2(\Omega)$ are models of this same abstract separable space.

Little was known about unbounded operators or forms in 1927. Practically the only results were Weyl's 1910 paper (Section 9), the Hellinger–Toeplitz theorem of 1910 (Section 9), a spectral representation of unbounded "Jacobi forms" by Hellinger [*Mathematische Annalen* **86** (1922), 18–29], and T. Carleman's 1923 Uppsala thesis on singular integral equations. M. H. Stone [on p. 155 of his book] (cf. Section 19) called the latter "a first substantial advance into the theory of unbounded operators." In fact, some of Carleman's results were quite surprising when they first appeared, and became fully understood only later through the work of Stone and von Neumann. (See Section 19.)

Von Neumann started his spectral theory for unbounded operators in the first of those three 1927 papers. There [p. 175] he introduced *"Einzeloperatoren"* (being projection operators, analogs of Hilbert's *"Einzelformen"* in the bounded case) and used them [p. 183] to define a spectral representation for real self-adjoint operators on Hilbert space.

The core of von Neumann's spectral theory is contained in two very substantial papers of 1929 [*Mathematische Annalen* **102,** 49–131, 370–427; [Neumann 1961–1963 2, 3–143]. In the first of these, von Neumann developed a spectral theory of *symmetric linear operators T* on (separable) Hilbert space [72]. Basic to this theory was the concept of a *maximal* symmetric operator, having no proper symmetric extensions. His setting of the spectral problem was suggested by Riesz' form of Hilbert's spectral theory in the bounded case (Section 12). His key new idea was the reduction of the possibly unbounded case to the bounded one by the "Cayley transform" U of the given operator T, defined by

$$T \mapsto U = (T + iI)(T - iI)^{-1}.$$

But von Neumann still had great difficulties in treating the *general* symmetric operator until E. Schmidt introduced the concept of *self-adjointness* of T (called "hypermaximality" by von Neumann) and showed that this is necessary and sufficient for the existence of a family of projections [*"Zerlegung der Einheit,"* or "resolution of the identity"] needed for a spectral representation of T [*Ibid.*, 16, 26]. Von Neumann then obtained that representation. Among other reults, he showed that every symmetric operator has a maximal symmetric extension, which may still not be self-adjoint. All these new facts are symptomatic of the drastic change of the entire situation in the transition from the bounded to the unbounded case.

His second paper [*Ibid.*, 86–143] includes further completely new and original ideas. Its first part concerns algebraic properties of the ring of bounded linear operators on a Hilbert space H. Here, von Neumann defined *weak topology* in terms of neighborhoods in H as well as the three now familiar types of convergence (*uniform, strong, weak*) for sequences of operators.

In the second part of the paper he defined (possibly unbounded) *normal operators T* in H (closed, densely defined linear operators T which commute with their Hilbert adjoint T^*). He established spectral representations (Theorem 17) for T and T^*, using the same spectral family, which he showed to be unique. This completed von Neumann's work on the spectral theory of unbounded operators, except for some late simplifications [*Ibid.*, 242–258].

In 1932, von Neumann published his well-known book *Mathematische Grundlagen der Quantenmechanik* [73]. In this book he described the connection of his ideas with quantum mechanics in much greater detail than in his papers. Thus his book included extensive studies of the probabilistic aspects of the process of physical measurement and uncertainty relations, of Dirac's theory of light, and so on. Von Neumann also contrasted his Hilbert space model with the formalisms used by Born, Heisenberg, and Jordan, and Dirac. In the opinion of A. S. Wightman, von Neumann's approach in his 1932 book constitutes "the most important axiomatization of a physical theory up to this time" [Browder 1976, 157].

A clearly written contemporary discussion of Hilbert space theory, from the standpoint of mathematical physicists, may be found in *Quantum Mechanics* by E. U. Condon and P. M. Morse (New York: McGraw–Hill, 1929, Sects. 12, 61). However, the goals of physicists (such as the prediction of the positron and other elementary particles) are very different from those of mathematicians, and they continued to use freely and fruitfully the Heaviside operational calculus and other formalisms. It seems fair to say that the greatest importance of von Neumann's work of 1926–1933 consisted in its convincing demonstration of the relevance of functional analysis (in particular, operators in Hilbert space) for the most exciting and active field of contemporary physics.

19. MARSHALL H. STONE

In 1932, Stone published a book *Linear Transformations in Hilbert Space,* on which he had been working since 1928. This book is extremely well organized and clearly written, and when it discusses results by Carleman, von Neumann, and Weyl, in most cases it is much more lucid than the original papers to which it refers. All this accounts for the popularity of the book and made it the standard reference on Hilbert space theory for at least two decades.

The "inside story" behind the writing of this book (and of von Neumann's as well) has kindly been given to us in a letter from Stone himself, from which we quote:

> There is some "secret" history concerning the relation of my work to von Neumann's without which a proper understanding is not possible. In 1927–28, Carathéodory was visiting professor at Harvard. When he left at the end of the year, he gave me proofsheets of articles to appear in *Math. Zeitschrift*, of which he was editor. There I found von Neumann's *original* treatment of the spectral theorem for symmetric operators. This paper made considerable use of Carleman's work, appealed to transfinite induction, and was written in ignorance of the concept "self-adjoint operator." I immediately realized that this concept had an essential role to play in the spectral theory for non-bounded operators (Hilbert, Riesz, etc. had already taken care of the bounded case pretty thoroughly) and that my previous work with differential operators (which went back to G. D. Birkhoff's Chicago thesis and J. D. Tamarkin's later generalizations) would yield a successful pattern of attack on the abstract problem. This proved very quickly to be the case. I was able to work out the necessary proofs and to write an article that was submitted to the *Transactions A.M.S.* for publication. At about that time, von Neumann published his *second* treatment, using the concept of self-adjoincy and the idea of the Cayley transform. von Neumann withdrew his *first* treatment and never published it (obtaining the publisher's permission only by agreeing to write a book on quantum mechanics eventually appearing as his celebrated monograph). The appearance of the *second* treatment in print led me to withdraw my paper, but Dunham Jackson and J. D. Tamarkin (then editors of the Transactions, I believe) counseled me to write a book on Hilbert space in which my independent results could be included. This was done by the end of 1932.

With this background, it is interesting to summarize the actual content of Stone's book. In it, Stone first defined Hilbert space axiomatically and—like von Neumann—assumed it to be separable. In defining bounded, closed, symmetric, and other classes of operators needed, he emphasized aspects that were essential for treating the unbounded case. On page 50, he stated as the main problems for symmetric linear operators *the determination of all:*

(1) *maximal symmetric extensions of a given operator T,*

(2) *maximal symmetric operators,*

(3) *self-adjoint linear operators.*

Deferring the solution of problems (1) and (2) to Chapter IX, Stone took up the study of problem (3) in Chapter V of his book. He obtained the spectral theorem for self-adjoint operators [p. 180] in a very original way, quite differently from that of von Neumann and Schmidt. His approach used the Stieltjes integral and was somewhat modeled after the function-theoretic method in Carleman's above work, but also utilized Hilbert's method of "sections" in the bounded case (see (8.3) in Section 8) [74].

In Chap. VII on the unitary equivalence of self-adjoint operators, Stone extended Hellinger's theory of eigendifferentials, as well as Hahn's 1912 results on the orthogonal similarity of operators on l^2 (cf. Section 9).

Only then does Stone's book take up problems (1) and (2), stating [p. 334] that "the results reported here are due to J. v. Neumann, whose exposition we shall follow with only occasional modifications and additions." Actually, the chapter contains interesting new results, for instance [on p. 387], proof of a conjecture by von Neumann on the extension of semibounded linear operators without increase of their norm.

In Chapter X on applications, Stone showed that his theory of Chapter IX can be used to handle "Carleman integral operators" arising in Carleman's work on singular integral equations (cf. Section 18); these are operators with kernel k such that

$$\int_a^b |k(s, t)|^2 \, dt < +\infty \qquad \text{a.e.} \tag{19.1}$$

(that is, except on a set of s-values of zero measure).

Chapter X also contains an extensive treatment of *ordinary differential equations*. Here the strange situation was that Weyl's work of 1909–1910 (cf. Section 9) had few successors until 1928 when Stone's paper [*Mathematische Zeitschrift* **28,** 654–676] appeared. The intention in Chapter X was to give a detailed discussion of "some special cases of recognized importance" in order to illustrate the abstract results in Chapter IX.

Unitary groups. Besides reshaping and applying to special problems many known results, Stone created a now classic theory of one-parameter groups of unitary operators. Almost from the beginning, the relevance of group theory to quantum mechanics was recognized, by Wigner (1927), von Neumann and Wigner (1928), and Weyl [75]. Thus Weyl wrote in the Preface to the second edition of his classic *Gruppentheorie und Quantenmechanik* that "the importance of . . . the theory of groups for . . . quantum theory has of late become more and more apparent" [76]. In this book, the famous Peter-Weyl theory of (compact) group representations was applied, with special emphasis on the rotation group and the symmetric group of all permutations of n symbols. Its Chapter I, entitled "Unitary Geometry," gives a set of axioms for (complex) Hilbert space, similar to those of von Neumann.

In 1930, Stone [*Proceedings of the National Academy of Sciences (USA)* **16,** 172–175] started to investigate the possible actions of *unitary* operators on a complex Hilbert space H. For any one-parameter *group* of unitary operators U_t acting on H, and depending weakly continuously on t, he gave a spectral representation of the form

$$U_t = \int_0^1 e^{2\pi it\lambda}\, dE_\lambda. \tag{19.2}$$

He also proved the existence of a (generally unbounded) self-adjoint linear operator A, called the *infinitesimal generator* of the group and defined by

$$Af = i \lim_{s\to 0} \frac{U_s - I}{s} f, \tag{19.3}$$

such that

$$U_t = e^{-itA}. \tag{19.4}$$

This was one of the first results on *infinite*-dimensional group representations.

F. Riesz. Besides von Neumann's [1929] and Stone's [1932] complete characterization of self-adjoint operators, there is a third of 1930, by F. Riesz [*Acta Szeged* **5,** 23–54; Riesz 1960, 1103–1134]. Riesz started from his version of spectral theory [§12] and observed that for self-adjoint operators it could be extended to the unbounded case, and that the same is true for Hellinger's work of 1909 [*Journal für die reine und angewandte Mathematik,* **136,** 210–271]. He then presented two solutions, one in his Szeged Seminar near the end of 1929, in which he replaced the polynomials used in his book of 1913 (cf. Section 12) by sums of partial fractions. In the second solution of 1930 [Riesz 1960, 1103–1134], Riesz based his method on a local decomposition theorem, namely, on the operator analog of the fact that a bounded quadratic form can be written locally as the difference of two positive definite orthogonal forms, and he emphasized [*Ibid.,* 1106] that this approach "presented the clearest insight into the close relation between the old and the new results [corresponding to the bounded and the unbounded case, respectively]."

20. FIXED POINT THEOREMS, ERGODIC THEORY

The creation of a spectral theory for unbounded self-adjoint linear operators, sparked by the search for a rigorous foundation of quantum mechanics, was by no means the only significant development in functional analysis during 1927–1933. A second major advance concerned new *fixed point theorems* in function spaces, going far beyond the theorems of Banach and Birkhoff-Kellogg discussed in Section 16.

Several of these were formulated and proved by J. P. Schauder (1899–1943), who had become interested in partial differential equations partly through his personal relationship with Leon Lichtenstein of Leipzig. First, in 1927 and 1930, Schauder established

SCHAUDER'S FIXED POINT THEOREM. *Any continuous mapping ϕ: $K \to K$ of a convex compact subset K of a Banach space V into itself has at least one fixed point.*

In his first paper on this theorem [*Mathematische Zeitschrift* **26** (1927), 47–65], Schauder proved the result under the assumption that V has a so-called *Schauder basis* (which he called a "linear basis"). By this is meant a sequence $\{e_n\}$ in V with the property that for every $f \in V$ there is a *unique* sequence $\{\alpha_n\}$ of scalars such that

$$\lim_{n\to\infty} \|f - (\alpha_1 e_1 + \ldots + \alpha_n e_n)\| = 0.$$

He showed that various function spaces of analysis have such a basis.

In a second paper of 1927 [*Ibid.* **26,** 417–431], Schauder weakened the assumptions in the original version of his theorem, so that it applied to boundary value problems for quasilinear elliptic equations

$$\nabla^2 z = f(x, y, z, z_x, z_y) \tag{20.1}$$

having merely continuous f.

In a third paper of 1930 [*Studia Mathematica* **2,** 171–180], Schauder extended his fixed point theorem to Fréchet spaces.

Schauder's theorem opened up another large area of applied functional analysis. Indeed, the theorem yielded existence theorems for ordinary differential equations, such as the simple Peano theorem or theorems on periodic solutions, as well as existence theorems for solutions of partial differential equations and complicated integral and integro-differential equations [77]. This marked the beginning of a development of topological fixed point theorems into one of the most important tools of *nonlinear functional analysis*. Especially innovative was work by Leray and Schauder [*Annales Scientifiques de l'École Normale Supérieure* **51** (3), 45–78], published in 1934, just after the end of the period considered here.

Ergodic theory [78]. Another important class of fixed point theorems arose in ergodic theory. This theory originated from the so-called Ergodic Hypothesis of Boltzmann (1871) and Maxwell (1879), later corrected to the Quasi-Ergodic Hypothesis of P. and T. Ehrenfest (cf. [*EMW* 4, Art. 32, footnotes 89a, 90, 93; published in 1911]). This hypothesis underlies classical statistical mechanics, and concerns Hamiltonian systems like those defined by the n-body problem in Section 16. As was mentioned there, the evolution of any such system in time defines a measure-preserving flow in phase space (Liouville's theorem).

The idea of applying Hilbert space theory to the study of classical dynamical systems first occurred in a 1931 note by Bernard Osgood Koopman [*Proceedings of the National Academy of Sciences* (*USA*) **17,** 315–318]. Koopman, a nephew of W. F. Osgood, had completed his doctoral thesis at Harvard in 1926; Stone had finished his a year earlier; both theses were supervised by G. D. Birkhoff. Stone's thesis had treated generalized Sturm–Liouville expansions in the tradition of Bôcher; Koopman's had concerned the three-body problem in the tradition of Poincaré.

Koopman pointed out that if the total measure of each "surface" $\Sigma(E)$ of "states" of energy E in phase space is finite (e.g., if each $\Sigma(E)$ is compact), then the flow mentioned above induces a *unitary group* on the Hilbert space $L^2(\Omega)$, to which Stone's theory of unitary groups acting on Hilbert space (cf. Section 19) applies.

In 1929, von Neumann had already given a *quantum*-mechanical derivation of the Quasi-Ergodic Hypothesis and the H-theorem [*Zeitschrift für Physik* **57,** 30–70]. In 1932 [*Proceedings of the National Academy of Sciences* (*USA*) **18,** 70–82] he proved a mean ergodic theorem in the context of classical mechanics, building on the result by Koopman (who had pointed out this possibility to von Neumann in the spring of 1930).

VON NEUMANN'S MEAN ERGODIC THEOREM. *Let U be a unitary operator on $L^2(E)$. Then for every $f \in L^2(E)$ the "average of the iterates" of U at f defined by*

$$A_n f = \frac{1}{n} \sum_{k=0}^{n-1} U^k f \tag{20.2}$$

converges in the mean (i.e., in the L^2-norm) to Pf, where P is the projection on the subspace consisting of all the fixed points of U, i.e., $Uf = f$.

Conversing at scientific meetings with von Neumann and with his own former students, G. D. Birkhoff instantly became interested in their work. As it happened, he had defined the related concept of *metric transitivity* not long before, in a paper with Paul Smith [*Journal de Mathématiques* **7** (9) (1928), 345–379]. Within weeks he had proved the following related result [*Proceedings of the National Academy of Sciences* (*USA*) **17** (1931), 650–660].

G. D. BIRKHOFF'S POINTWISE ERGODIC THEOREM. *Let Ω be the phase space of any Hamiltonian system, let E be an invariant subset of Ω having finite Lebesgue measure, and let V be a measurable subset of E. Finally, let $p_T(a;V)$ be the fraction of the time interval* $(0, T)$ *that a system initially in the state a will spend in V. Then the* time probability $\lim_{T\uparrow\infty} p_T(a;V)$ *exists for almost all states (points) $a \in E$.*

Whereas von Neumann's proof of his Mean Ergodic Theorem was in the Hilbert space $L^2(E)$, G. D. Birkhoff's proof of the sharper Pointwise Ergodic Theorem was in $L^1(E)$ and used entirely different methods of Lebesgue measure theory. Following the publication of these results, ergodic theory exploded into a recognized new branch of functional analysis, appreciated as an amalgamation of

Hilbert's spectral theory and the Poincaré–G. D. Birkhoff theory of dynamical systems [79]. Eberhard Hopf had just come to Cambridge; he and von Neumann each published four notes and papers on ergodic theory in 1932.

21. BANACH'S BOOK

Functional analysis, which had received its name only ten years before, became established as a major branch of analysis in the early 1930s. Its scope and power were demonstrated by three books, all of which appeared in 1932: von Neumann's *Mathematische Grundlagen der Quantenmechanik*, Stone's *Linear Transformations in Hilbert Space*, and Banach's *Théorie des Opérations Linéaires* [1932] [80]. We have already discussed two of these, as well as the role of fixed point theorems, and we now take up Banach's book.

Whereas von Neumann's book was primarily concerned with quantum mechanics and Stone's book presented the spectral theory of symmetric linear operators on Hilbert space, with applications to classical analysis (concluding with a 220-page chapter on differential and integral equations), Banach's emphasized intriguing theoretical questions involving linear operators and functionals on a wide range of "*espaces (B)*."

Here we can only sketch some of the main themes of Banach's book. It treats only *bounded linear* operators [called "*opérations linéaires*"; cf. pp. 23, 36], mostly on *real* Banach spaces. (Its Preface promised a second volume, employing "topological methods," and so presumably intended to contain Schauder's fixed point theorems [cf. p. 227].) Banach acknowledges substantial help from Auerbach and especially Mazur in preparing the book; this was probably essential since Banach's writing habits were very unsystematic [81].

The book begins with a brief introduction to metric spaces and real vector spaces. Here, on page 27, one finds Banach's form of the

HAHN–BANACH THEOREM. *Let p be defined on a real vector space E and satisfy*

$$p(x + y) \leqq p(x) + p(y), \qquad p(tx) = tp(x) \qquad \text{for } t \geqq 0.$$

Let f be a linear functional on a subspace $G \subseteq E$ such that $f(x) \leqq p(x)$ on G. Then there exists a linear functional F on E such that $F(x) \leqq p(x)$ on E and $F(x) = f(x)$ on G.

The next chapters concern "*espaces (F)*" [Fréchet spaces], normed spaces, and Banach spaces [*espaces (B)*]. They contain the other "big" theorems we have mentioned in Section 15, and are followed by a 10-page chapter on compact operators [*opérations totalement continues*]. Their adjoint operators [*opérations "associées"* or "*conjuguées*"] are also treated, after which comes another chapter on biorthogonal sequences. These are sequences $\{x_i\}$, $\{f_i\}$, with x_i in a Banach space E and $f_i \in E^*$ (the dual space of E), such that $f_i(x_j) = \delta_{ij}$. Here, Schauder bases occur [p. 110] and one can discern a recognition of the key relation $E \subseteq (E^*)^*$.

Next come two chapters on weak (sequential) convergence and compactness; Banach still adheres to *sequences*, referring only to the 1927 edition of Hausdorff. This forces him to introduce "transfinite closure" [p. 119] and entails other complications. Banach mentions neither Stone nor von Neumann, and Hilbert only once [p. 239], as inventor of the "complete continuity" concept in Hilbert space (which Banach identifies with $L^2[0, 1]$).

Referring to Hausdorff's paper on linear spaces of 1932 [*Journal für die reine und angewandte Mathematik* **167,** 294–311], Banach then presents F. Riesz' theory of compact operators of 1918 (cf. Section 12, above), adding on page 155 the contributions by Hildebrandt (1928) and by Schauder (1930) involving the adjoint operator. Applications to Fredholm and Volterra equations are given at the end of the chapter [pp. 161–164]. Isometries and homeomorphisms of metric spaces and isomorphisms of Fréchet and Banach spaces are then considered, and the Fréchet theory of linear dimension is reviewed with examples [Chap. XII].

Banach's book concludes with a detailed review of a great number of unsolved and recently solved problems. Thus on page 245 one finds a matrix of nearly 200 possible properties of important Banach spaces. This brought out what a fertile soil for pure research this newly organized subject provided.

One can hardly exaggerate the influence that Banach's book has had on the development of functional analysis. Embracing a much wider field of mathematical questions than is provided by Hilbert space theory, it has probably stimulated a greater volume of published mathematical papers than Stone's and von Neumann's books combined. Because of its greater generality, moreover, the theory of Banach spaces has retained much more of the original flavor of functional analysis (as anticipated or interpreted by Volterra, Fréchet, and F. Riesz) than the theory of linear operators on Hilbert space.

The book became quickly accepted as the climax of a long series of works initiated by Volterra, Hadamard, Fréchet, and F. Riesz. For those linking generality, it could be said to contain much of Hilbert space theory, including the spectral theory of compact operators, as a special case. Thus it acquired an impressive and substantial theory of its own. This went far toward establishing functional analysis as a broad and independent field of research [82].

Since Stone's book had demonstrated the applicability of functional analysis to several major areas of classical analysis, while von Neumann's book had shown its applicability to a new area of mathematical physics, the central role of functional analysis in modern mathematics became generally recognized. Hilbert space, the theory of which (often in the context of infinite quadratic forms) had been developed before 1927 with little explicit reference to the general ideas of Fréchet and E. H. Moore, was finally seen to fit neatly into a much more general framework.

Functional analysis had become established.

NOTES

1. See M. Bernkopf, The development of function spaces with particular reference to their origins in integral equation theory, *Archive for History of Exact Sciences* **4** (1966–1967), 1–96; F. E. Browder, The relation of functional analysis to concrete analysis in 20th-century mathematics, *Historia Mathematica* **2** (1975), 577–590; A. E. Taylor, Historical notes on analyticity as a concept in functional analysis, in *Problems in Analysis*, R. C. Gunning, ed., pp. 325–348 (Princeton, N.J.: Princeton Univ. Press, 1970).

2. See I. Grattan-Guinness, *Joseph Fourier 1768–1830* (Cambridge, Mass.: MIT Press, 1972).

3. We capitalize key words in book titles of *all* languages.

4. "On the representability of a function by a trigonometric series."

5. For details, see [Hawkins 1975], who describes the development until about 1910, with emphasis on preparatory work by Dirichlet, Jordan, Darboux, and others. See also E. Knobloch, Von Riemann zu Lebesgue—Zur Entwicklung der Integrationstheorie, *Historia Mathematica* **10** (1983), 318–343.

6. "Foundations for a general theory of functions of a variable complex quantity."

7. F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert* (Berlin: Springer, 1926: reprinted, New York: Chelsea, 1967). See Part I, 173.

8. "Continuity and Irrational Numbers," and "What Are Numbers and What Are They Good for?" (A translated edition bears the title *The Nature and Meaning of Numbers* (Chicago: Open Court, 1948).)

9. General theorems on spaces, posthumously published in *Werke*, Vol. 2, pp. 353–355. For Dedekind's great contributions to set theory, see also the Preface by J. Dieudonné to P. Dugac, *Richard Dedekind et les Fondements de l'Analyse* (Paris: Vrin, 1976); also [Dieudonné 1978 1, 373–375, 380–381].

10. Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen [On a property of the concept of all real algebraic numbers], *Journal für die reine und angewandte Mathematik* **77** (1874), 258–262.

11. Published in *Journal für die reine und angewandte Mathematik* **84** (1878), 242–258.

12. G. Ascoli, *Atti della R. Accademia dei Lincei* **18** (3), 521–586. See [Dunford & Schwartz 1958, 382] for historical details.

13. For more details, see [Hawkins 1975]. A characterization of Dini's role as the initiator of "modern" real analysis in Italy is given by Volterra [*Atti del IV Congresso Internazionale dei Matematici* (Rome, 1908), Vol. 1, pp. 61–62]. Dini's home town Pisa has named a street after him and erected a monument to honor him, both not far from the Leaning Tower.

14. See H. C. Kennedy, *Peano* (Dordrecht: Reidel, 1980).

15. *Calcolo Geometrico Secondo l'Ausdehnungslehre di H. Grassmann, Preceduto dalle Operazioni della Logica Deduttiva,* [*Geometrical Calculus According to the Calculus of Extension by H. Grassmann, Preceded by Operations of Deductive Logic*] (Turin: Bocca, 1888). For a discussion of the passages of Chapter IX relevant in the present context, see [Monna 1973, 117–121]. For an outline of the main ideas in the book, see also Peano, *Opere,* Vol. III, pp. 167–186 ("dimension" not being mentioned there, however).

16. Turin: Bocca, 1895–1901. Facsimile reproduction of the Italian version, *Formulario Mathematico*, Vol. V (Rome: Edizioni Cremonese, 1960).

17. Also extending his memoir in *Mathematische Annalen* **49** (1897), 325–382.

18. Pincherle attended Weierstrass' lectures in 1877–1878. The relation just mentioned is implicit in Laplace's book on probability (1812), in connection with the idea of a "generating function." Pincherle's work on integral operators is significant for the Laplace transform. For a summary of his work (written by himself), see *Acta Mathematica* **46** (1925), 341–362.

19. See also the remarks on Pincherle's work in [Diendonné 1981].

20. See [Hawkins 1975]. Biographies of Volterra are included in [Volterra 1954–1962 1; 1930].

21. Sopra le funzioni che dipendono da altre funzioni [On the functions which depend on other functions], *Rendiconti della Accademia dei Lincei* **3,** (IV), 97–105; [Volterra 1954–1962 1, 294–302].

22. For an outstanding exposition, see J. C. Oxtoby's *Measure and Category* (Berlin: Springer, 1971). Some of Baire's results for ℝ had been obtained before, by W. F. Osgood [*American Journal of Mathematics* **19** (1897), 155–190].

23. For a survey of W. H. Young's independent contemporary work on integration, see [Hawkins 1975, Sect. 5.2].

24. "But it is mainly in the theory of the partial differential equations of mathematical physics that studies of this type should play a fundamental role."

25. Hadamard's *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (New Haven, Conn.: Yale Univ. Press, 1923; reprinted, New York: Dover, 1952) gives a good introduction to his ideas and main results.

26. Interestingly, in Austria-Hungary, Germany, Italy, and the United States sooner than in France itself.

27. For historical accounts, see A. F. Monna, *Dirichlet's Principle* (Utrecht: Oosthoek et al., 1975); and L. Gårding, The Dirichlet problem, *Mathematical Intelligencer* **2** (1979), 43–53.

28. *Vorlesungen über die im umgekehrten Verhältniss des Quadrats der Entfernung wirkenden Kräfte* (Leipzig: Teubner, 1876); 2nd ed., 1887, Sect. 32. For Riemann's allusion to Dirichlet, see [Riemann 1892, 97].

29. The totality of the functions λ forms a connected, in itself closed domain, since each of these functions can go over continuously into every other, but cannot infinitely [closely] approach one which is discontinuous along a curve, without L becoming infinite (Art. 17); if we set $\omega = \alpha + \lambda$, [then] for every λ, Ω obtains a finite value which becomes infinite simultaneously with L, varies continuously with the form ("Gestalt") of λ, but can never decrease below zero; accordingly, Ω has a minimum for at least *one* form of the function ω.

30. For the history of related ideas, see A. F. Monna, *Dirichlet's Principle*, note 27 above, p. 20; for other applications, refer to [*EMW* II.1.1, 528].

31. For a general introduction to this area, see [Courant & Hilbert 1924] or S. H. Gould, *Variational Methods for Eigenvalue Problems,* 2nd ed. (Toronto: Univ. of Toronto Press, 1966).

32. *Gesammelte mathematische Abhandlungen* **1,** 223–269; the "Buniakowski-Cauchy-Schwarz inequality" appears on p. 251.

33. *Annales Scientifiques de l'École Normale Supérieure* **12** (8) (1895), 227–316; this summarizes a thesis of 1894.

34. [Volterra 1954–1962 2, 216–275], comprising four articles from *Atti Torino* and two from *Rendiconti della Accademia dei Lincei,* all published in 1896.

35. For Liouville's early contributions, see J. Lützen, *Historia Mathematica* **9** (1982), 373–391.

36. Sur une nouvelle méthode pour la résolution du problème de Dirichlet, *Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar* **57** (1900), 39–46, followed by two notes of 1902 in the *Comptes Rendus* (Paris) **134,** 219–222, 1561–1564.

37. "Sur une classe d'équations fonctionnelles."

38. *Mathematische Annalen* **58** (1904), 441–456; **60** (1905), 423–433. For Mason's work, see *Transactions of the American Mathematical Society* **7** (1906), 337–360, and his *Colloquium Lectures,* Vol. 14, 1906, published by the AMS in 1910.

39. Here, Hilbert originally imposed a condition on k which E. Schmidt later recognized as superfluous; cf. [Hilbert 1912, 190, footnote].

40. Indeed, nowhere in this work did Hilbert use the word "space."

41. Schmidt denoted the Fredholm determinant by $\delta(\lambda)$, instead of $D(\lambda)$.

42. This refers to the condition mentioned in note 39.

43. The paper is entitled Grundlagen für eine Theorie der unendlichen Matrizen. For operators, the theorem was proved by J. von Neumann, *Mathematische Annalen* **102** (1929–1930), 49–131 (p. 107), who referred to Toeplitz; the year in his footnote 61 should be 1910. Cf. also [Kreyszig 1978, 525].

44. Professor Béla Szökefalvi-Nagy informed us that it was most likely Gyula Vályi (1855–1913) at Kolozvár who influenced Riesz in this direction.

45. *Comptes Rendus* (Paris) **144** (1907), 734–736; *Göttinger Nachrichten* (1907), 116–122; Riesz 1960, 378–381, 389–395. Presented by Hilbert at Göttingen on March 9, whereas Fischer's seminar talk was on March 5; for Fischer's work, see *Comptes Rendus* (Paris) **144** (1907), 1022–1024.

46. For an extensive study of these relations, see J. D. Gray, *Archive for Rational Mechanics and Analysis*, in press.

47. In his 1894 paper Recherches sur les fractions continues, *Annales de la Faculté des Sciences de Toulouse* **8,** J1–J122. Riesz himself (p. 401) mentioned that his teacher, J. König, had used Stieltjes' results in class and published a (Hungarian) note on them already in 1897.

48. Due to a bad printing error in [Riesz 1960], the note is shown on pp. 396, 397, 405, 406, whereas pp. 398, 399 are not part of it.

49. Expressed on p. 215 of the *Festschrift Schwarz* [1914; C. Carathéodory et al., eds.], a volume of articles written to celebrate the 50th anniversary of a Ph.D. to Schwarz (1843–1921).

50. Linguistic barriers and national rivalries, in an era of intense nationalism, may also have had some effect.

51. See G. D. Birkhoff, *Bulletin of the American Mathematical Society* **17** (1911), 414–428. Birkhoff, then 27, had studied with both Moore and Bôcher. He also published a definitive appreciation of Bôcher's work in *Ibid.* **25** (1919), 195–215.

52. *Transactions of the American Mathematical Society* **9** (1908), 219–231, 375–395. See Bôcher's paper in the *Proceedings of the Fifth International Congress of Mathematicians*, Cambridge, 1912 (Vol. I, pp. 179, 187, 193) for appreciations of Birkhoff's work.

53. Reflexivity of l^p is essential here; cf. [Dunford & Schwartz 1958, 515, Ex. 30].

54. Actually, *Acta Mathematica* printed the paper near the end of 1916, but the completed volume bears the year 1918. A Hungarian version of the paper appeared in 1917.

55. Riesz' theory is also discussed in K. Yosida, *Functional Analysis*, Chap. X (Berlin: Springer, 1971); A. C. Zaanen, *Linear Analysis*, Chaps. 11–17 (Amsterdam: North-Holland, 1964); and [Kreyszig 1978, Chap. 8].

56. Caution! $\lambda = \infty$ in Riesz' (and Hilbert's) notation.

57. *Annales Scientifiques de l'École Normale Supérieure* **42** (1925), 293–323. For an extensive list of papers on the subject, and a thoughtful analysis of their contents, see A. E. Taylor, *Archive for History of Exact Sciences* **12** (1974), 355–383.

58. *Annals of Mathematics*, **19** (2) (1917–1918), 279–294; **20** (1918–1919), 281–288.

59. "Die Genesis des Raumbegriffs" [Riesz 1960, 110–154, in particular p. 119], presented to the Hungarian Academy of Science in 1906 and published in 1907. See also *Atti del IV Congresso Internazionale dei Matematici* (Rome, 1908), Vol. 2, pp. 18–24. For a historical discussion, see W. J. Thron, *Topological Structures* (New York: Holt, Rinehart & Winston, 1966).

60. Caution! The use of these terms is not uniform, even in the modern literature.

(T_1) For every pair x, y of distinct points there are neighborhoods U_x and U_y, respectively, such that $x \notin U_y$ and $y \notin U_x$ ("Fréchet's separation axiom").

61. *Mathematische Annalen* **92** (1924), 285–303; **94** (1925), 309–315. The main results had been presented to the Moscow Mathematical Society in 1922.

62. For more details, see C. Kuratowski, *A Half Century of Polish Mathematics* (Oxford: Pergamon, 1980).

63. Banach gave an outline of applications later, in 1936 at the Oslo Congress [*Comptes Rendus du Congrès International des Mathématiciens*, Vol. 1, pp. 261–268.

64. See also the comments by E. Hille in [Wiener 1976– , 3, 684] and by A. E. Taylor in *American Mathematical Monthly* **78**, (1971), 331–342.

65. M. Fréchet, Les espaces abstraits topologiquement affines, *Acta Mathematica* **47**, 25–52, preceded by a note of 1925 in *Comptes Rendus* (Paris) **180**, 419–421.

66. Caution! "Local convexity" (the existence of a basis of convex neighborhoods of 0) is not part of this definition, but was added only later, by Mazur and Bourbaki, to guarantee the existence of nontrivial bounded linear functionals.

67. Actually, it was S. Saks who first promoted the Baire category method, as a referee of the Banach–Steinhaus paper, by suggesting the original lengthy constructive proof be replaced with the now familiar category argument.

68. Quoted from G. D. Birkhoff, *Transactions of the American Mathematical Society* **14** (1913), 14–22 [Birkhoff 1950, 673–681], where the theorem was first proved. Poincaré had stated the conjecture in the *Rendiconti del Circolo Matematico di Palermo* **33** (1912), 375–407.

69. The English edition [Volterra 1930] was still to be published; it was reprinted, New York: Dover, 1959. It was only in 1936, in the second edition of Volterra-Pérès, that space is devoted to abstract ideas.

70. *Annals of Mathematics* **22** (1920), 182–211. For Hildebrandt's article, see *Bulletin of the American Mathematical Society* **24** (1917), 113–144, 177–202. See also G. Doetsch, *Mathematische Zeitschrift* **22** (1924), 284–306, and *Jahresbericht der Deutschen Mathematiker-Vereinigung* **36**, (1927), 1–30.

71. See A. S. Wightman's authoritative article in [Browder 1976, 147–240], from which we have drawn freely.

72. Von Neumann called them *Hermitian*; M. H. Stone's term "symmetric" is more common.

73. English translation by R. T. Beyer (Princeton, N.J.: Princeton Univ. Press, 1955).

74. Refer for details to A. Wintner, *Spektraltheorie der unendlichen Matrizen* (Leipzig: Hirzel, 1929).

75. *Zeitschrift für Physik* **43** (1927), 624; **47** (1929), 203 (also **49**, 73). See Robertson, cited in note 76, p. 404, (1).

76. Leipzig, 1928, 1930; translated by H. P. Robertson as *The Theory of Groups and Quantum Mechanics* (1931), reprinted, New York: Dover, 1950.

77. For these applications, see P. Hartman, *Ordinary Differential Equations*, Chap. 12 (New York: Wiley, 1964); D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order* (New York: Springer, 1977).

78. Koopman's article with G. D. Birkhoff in *Proceedings of the National Academy of Sciences* (USA) **18** (1932), 279–282, gives an excellent historical review of ergodic theory prior to 1932.

79. See E. Hopf's classic monograph *Ergodentheorie* (Berlin: Springer, 1937).

80. A Polish edition of the book appeared in 1931.

81. Personal communication from Stanislas Ulam, who was at the University of Lwów during the years 1925–1935.

82. Curiously, neither Banach nor Stone nor von Neumann seems to have used the term "functional analysis."

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