Contents

1 Preliminaria

Here we recall some facts needed from linear algebra:

During this course $\mathbb{K}$ will denote the scalar field -equal either to $\mathbb{R}$, or to $\mathbb{C}$. The vector spaces over $\mathbb{K}$ will be denoted X, Y, Z, M, H -some other capital letters may be used. In some cases arrows over some letters, like $\vec{u}, \vec{w}, \vec{x}, ...$ will be applied to mark the difference between vectors and scalars (usually denoted either by Greek lowercase letters: α, β, λ, or s, t -for real scalars). Later on this distinction will be clear and to simplify the notation, the arrows will be suppressed. The basics of linear algebra are assumed to be known, including the notions of linear independence of vectors, bases, the dimension, linear mappings and their relation to matrices, the Euclidean space $\mathbb{R}^n$, or $\mathbb{C}^n$ with its canonical 0-1 basis:

$$\epsilon_1 = (1, 0, \dots, 0),\ \epsilon_2 = (0, 1, 0 \dots, 0), \dots, \epsilon_n = (0, \dots, 0, 1)$$

and coordinate notations: $\vec{x} = (x_1, \dots, x_n)$.

If $n = 3$ (or 2), instead of $\epsilon_1, \epsilon_2, \epsilon_3$ one usually writes: $\vec{i}, \vec{j}, \vec{k}$ and $\vec{w} = (x, y, z)$.

The symbols: If $F(a)$ is some logical formula depending on the variable a, then
$\exists_{a \in A}! F(a)$ will denote "*there exists only one element $a \in A$ for which $F(a)$ holds*",
$:=$ -will denote "*equals, by definition*". Conjunction will be denoted p `and` q, or simply p, q rather than $p \wedge q$. The word "**iff**" will stand for "*if and only if*".

Given a mapping $\phi : D \to Y$, where D is the domain of ϕ, denoted $\mathcal{D}(\phi)$, we say that $F : X \to Y$ **extends** ϕ, (or -that ϕ is a restriction of F to D, notation: $F|_D = \phi$, or $\phi \subset F$, if

$$\mathcal{D}(\phi) \subset \mathcal{D}(F) \ \texttt{and} \ \forall_{x \in D} F(x) = \phi(x).$$

If moreover F is linear, we speak of a **linear extension**. Similarly, if it is continuous, we call F a **continuous extension** of ϕ.

Useful Fact 1: A subset $G \subset X$ is linearly independent **iff** any mapping $\phi : G \to Y$ has a linear extension (to some linear subspace containing its domain, D, or even to the entire space X.) In order to get one implication it suffices to require the existence of linear extensions only in the scalar-valued case: $Y = \mathbb{K}$.

We say, that G spans X, writing $X = \text{span}(G)$, if

$$\forall_{x \in X} \exists_{m \in \mathbb{N}} \exists_{\alpha_1, \dots, \alpha_m \in \mathbb{K}} \exists_{v_1, \dots, v_m \in G} \quad x = \sum_{j=1}^{m} \alpha_j v_j. \tag{1}$$

Useful Fact 2: A subset $G \subset X$ spans X, if linear extensions of any map $\phi : G \to Y$ are unique -provided they exist. This means that if F_1 and F_2 are linear mappings on X extending the same ϕ, one must have $F_1 = F_2$. Clearly, not all mappings ϕ have any linear extension- take $G = X$, or even $G = \{\epsilon_1, 2\epsilon_1\},\ X = \mathbb{R}^2$. (Again, it suffices to verify this with $Y = \mathbb{K}$).

For linear bases G in X (sets simultaneously lin. independent and spanning X) any mapping from G has exactly one linear extension. All bases of X have the same cardinality, called **the dimension** of X, denoted $\dim(X)$.
If the latter is finite, say $\dim(X) = m$, we may write $G = \{v_1, \dots, v_m\}$ and then the quantifiers:

$\exists_m, \exists_{v_1,\ldots v_m}$ are redundant in the formula 1, while the quantifier $\exists_{(\alpha_1,\ldots,\alpha_m)}$ can be written as $\exists!$. Then we call this m-tuple $(\alpha_1, \ldots, \alpha_m) \in \mathbb{K}^m$ of scalars - the coordinates of x in the basis $(v_1, \ldots, v_m)$. I will write now

$$v_j^*(x) := \alpha_j, \ \texttt{so that} \ x = \sum_{j=1}^{m} v_j^*(x) v_j. \tag{2}$$

The same notation can be used to define **the dual system** $\{e_j^* : j \in J\}$ to an infinite basis $\{e_j : j \in J\}$. Here some problems arise (see later -in Tutorials section).

When writing the value of a linear operator T on a given vector x one usually omits the parentheses – writing Tx rather than $T(x)$, when the range is clear. For example, in some formulae the vectors x themselves will be functions, eg. $x(s)$ for $s \in [0,1]$ and we write $(Tx)(s) = \int_0^s f(t)\,dt$ in the case of the so called Volterra operator. Here $(Tx)(s)$ looks better than $(T(x))(s)$, or $T(x)(s)$. Sometimes we need parentheses, otherwise it would be unclear, whether $Tx + y$ means $T(x) + y$, or $T(x + y)$.

NOTATION: $\mathcal{L}in(X, Y) := \{T : T : X \to Y \ \texttt{is linear} \ \}$ will be the space of all **linear operators** acting from X to Y. Members of $\mathcal{L}in(X, \mathbb{K})$ are called **linear functionals** on X. If we have normed spaces with norms $\| \|_X, \| \|_Y$, we say that $T \in \mathcal{L}in(X, Y)$ is bounded, if it is bounded on the unit ball denoted by

$$B_X(0, 1) := \{x \in X : \|x\|_X < 1\}.$$

Let us define the operator norm $\|T\|$ by

$$\|T\| := \sup\{\|Tv\|_Y : v \in B_X(0, 1)\}.$$

Denote the space of all **bounded linear operators** from X to Y by

$$\mathcal{B}(X, Y).$$

If $Y = X$, we write $\mathcal{B}(X)$ in place of $\mathcal{B}(X, Y)$ and if $Y = \mathbb{K}$, we write X' for the space $\mathcal{B}(X, \mathbb{K})$ of all bounded linear functionals on X, called **the dual space** for X. Some textbooks use the notation $\mathcal{L}(X, Y)$ for $\mathcal{B}(X, Y)$, writing X^* rather than X' is also frequent. An example of linear functionals is v_j^* of the dual system (2).

The vector space structure is defined (both on $\mathcal{B}(X, Y)$ and on $\mathcal{L}in(X, Y)$) by pointwise linear operations: given e.g. $T, S \in \mathcal{B}(X, Y)$ we write $T + S$ for the operator sending a vector $x \in X$ into $(T + S)(x) := T(x) + S(x) \in Y$. Similarly, $(\alpha T)(x) := \alpha T(x)$ defines the multiplication of an operator T by the scalar $\alpha \in \mathbb{K}$. The constant function 0, called the zero operator, is clearly bounded and linear. Thes is the zero element of $\mathcal{B}(X, Y)$. Boundedness of the sum of bounded operators results from the inequalities:

$$\|(T + S)x\| = \|Tx + Sx\| \le \|Tx\| + \|Sx\| \le \|T\| + \|S\| \ \texttt{valid for any} \ x \in B_X(0, 1),$$

which also shows that

$$\|S + T\| \le \|S\| + \|T\|.$$

Similarly, one shows that for $\alpha \in \mathbb{K}$ one has $\|\alpha T\| = |\alpha|\|T\|$. Apart from bounded, we often have to consider linear operators, that are unbounded and defined on domains $D(T)$ different from the entire space. Usually, the domains are dense subsets (in the norm topology of the space X). We call such mappings *densely defined operators* in X.

Let D_1, D_2 be now subspaces (in many applications -dense) of some two normed spaces X_1, X_2 -resp. and consider two operators $T_1 : D_1 \to X_2, T_2 : D_2 \to X_3$. Their composition, $T_2 \circ T_1$, denoted T_2T_1 –is defined on the domain

$$D(T_2T_1) := \{x \in D(T_1) : \ T_1(x) \in D(T_2)\} \quad \texttt{by} \quad T_2T_1x := T_2(T_1x). \tag{3}$$

We write T^2 for $T \circ T$ and -proceeding by induction- $T^{n+1} = T \circ T^n$. Let I_X denote the identity operator on X: $I_Xv = v$ $(\forall v \in X)$. Note that $\mathcal{L}in(X, X)$ is an algebra if the multiplication is defined to be the composition. If $\dim(X) > 1$ this algebra is noncommutative, but it has the unit, namely I_X. It is important to note that $\mathcal{B}(X, X)$ -i.e. $\mathcal{B}(X)$ is also an algebra. Moreover $\|T_2T_1\| \le \|T_2\|\|T_1\|$. This follows easily from the estimate:

$$\|Tw\|_Y \le \|T\|\|w\|_X \qquad \texttt{for any} \ w \in X, \ T \in \mathcal{B}(X, Y) \tag{4}$$

The continuity of linear operators and its invertibility are two central issues considered. Let us recall some results from functional analysis: (TFAE = "The Following Are Equivalent)

Theorem 1.1 *For a linear operator $T : X \to Y$ between two normed spaces TFAE:*

(a) T is continuous on X (even -uniformly continuous),

(b) T is continuous at some point $x_0 \in X$,

(c) T is bounded in some nonempty open set,

(d) T is bounded in the unit ball of X, i.e. $\|T\| < +\infty$,

(d) For some finite constant $M \geq 0$ one has $\|Tx\|_Y \leq M\|x\|_X$ for any $x \in X$.

The norm $\|T\|$ is the least $M \geq 0$ satisfying the estimate in (d).

Theorem 1.2 *Any linear mapping on a finite-dimensional normed vector space is continuous. Any finite-dimensional subspace of a normed space is closed.*

Theorem 1.3 *Any uniformly continuous mapping f defined on a dense subset D of a metric spaceX, whose values are in a complete normed space Y has a unique continuous extension to a continuous mapping $F : X \to Y$.*

Theorem 1.4 *If Y is complete, then the space $\mathcal{B}(X, Y)$ is also complete. (Conversely, if* $\dim(X) > 0$, *then from the completeness of $B(X, Y)$ it follows that Y is complete.)*

Theorem 1.5 *Any continuous linear functional ϕ on a subspace M of a normed space X has a continuous linear extension F having the same norm: $\|\phi\| = \|F\|$.*

Theorem 1.6 *For any $x \in X$ there exists (at least one) bounded linear functional $\phi \in X'$ having norm one: $\|\phi\| = 1$ such that $\phi(x) = \|x\|$. Hence the* ***dual formula for the norm*** *holds:*

$$\|x\| = \sup\{|f(x)| : f \in X', \|f\| \leq 1\}.$$

Theorem 1.7 *If X is complete and $\|T - I_X\| < 1$, then T is bijective, has a bounded inverse. Moreover,*

$$T^{-1} = I_X + \sum_{n=1}^{\infty} (I - T)^n.$$

Corollary 1.8 *The set of all invertible elements of the algebra $\mathcal{B}(X)$ is open and the operation of taking the inverse operator is continuous.*

Theorem 1.9 *(fundamental Banach results): Here X, Y are Banach spaces.*
Open Mapping Theorem *Any continuous linear surjection $T \in \mathcal{B}(X, Y)$ maps open subsets of X onto open subsets of Y.*
Inverse Mapping Theorem *The inverse of a continuous bijection $T \in \mathcal{B}(X, Y)$ is also continuous*
Closed Graph Theorem *If $T \in \mathcal{L}in(X, Y)$ has closed graph (i.e. the set $\Gamma_T := \{(x, y) \in X \times y : y = Tx\}$ is closed in the product topology), then T must be continuous.*
Banach – Steinhaus Theorem *If a sequence of bounded linear operators satisfies the pointwise - boundedness condition: $\forall_{x \in X} \sup_n \|T_n x\| < \infty$, then it is uniformly bounded on the unit ball:* $\sup_n \|T_n\| < \infty$.

We say that two norms, say $\| \,\|$ and $\| \,\|_*$ on the same linear space X are **equivalent norms**, if there exist positive constants $m, M > 0$ such that

$$\forall_{x \in X} \; m\|x\| \leq \|x\|_* \leq M\|x\|. \tag{5}$$

We say that a linear mapping $T : X \to Y$ is bounded below on X, if for some $m > 0$ we have estimates

$$\forall_{x \in X} \; \|Tx\| \geq m\|x\|. \tag{6}$$

Theorem 1.10 *Any two equivalent norms define the same topology. On a finitely-dimensional space all norms are equivalent.*

MORE NOTATION: For a linear mapping its kernel, known also as the nullspace is denoted either by $\mathcal{N}(T)$, or by $\ker(T)$ and is, by definition, the set

$$\mathcal{N}(T) := \{x \in X : Tx = 0\}.$$

The range space of $T \in \mathcal{L}in(X,Y)$ is denoted by $\mathcal{R}(T)$ (or in some books -by $Im(T)$. Here

$$\mathcal{R}(T) := \{y \in Y : \exists_{x\in X}\, y = Tx\}.$$

Both sets are linear subspaces. From linear algebra we know that

$$T \texttt{ is injective iff } \mathcal{N}(T) = \{0\}. \tag{7}$$

Surjectivity means that $\mathcal{R}(T) = Y$. In the finite-dimensional case we have the relation

$$\dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T) = \dim(X),$$

which for $X = Y$ gives the equivalence:

Lemma 1.11 $T \in \mathcal{L}in(X,X)$ *is invertible iff* $\mathcal{N}(T) = \{0\}$. *(This no longer applies in the infinite dimensional case!)*

1.1 Tutorials 1

Given a basis $(e_j)_{j\in J}$ the dual system of functionals $(e_j^*)_{j\in J}$ is defined by (2), where the e_j stand in place of v_j, the summation ranges through some finite subset $\{j_1,\ldots,j_m\}$ of the set J of indices rather than through $\{1,\ldots,m\}$. From linear algebra we know that these functionals e_j^* are linear. They just describe the coordinates of a vector x with respect to the given basis.

1. In the Euclidean space $\mathbb{K}^n$ the norm is given by $\|\vec{x}\| = \sqrt{|x_1|^2 + \ldots + |x_n|^2}$. Here $x_j = \epsilon_j^*(\vec{x})$ (according to the notation from equation (1)) are the coordinates of $\vec{x}$ in the canonical 0-1 basis $(\epsilon_1,\ldots,\epsilon_n)$. Any linear operator $T : \mathbb{K}^n \to Y$ can be represented as $T = \sum_{j=1}^n \epsilon_j^* T\epsilon_j$. Deduce the continuity of such T. Express the matrix entries a_{jk} in terms of the basis vectors, T and the dual basis functionals only.

2. Show that if $(e_1,\ldots,e_n)$ form a basis of X, then the dual system: $(e_1^*,\ldots,e_n^*)$ is a basis of X^*, hence $\dim(X) = \dim((X,\mathbb{K}))$ provided that $\dim(X) < \infty$.

3. In the infinitely dimensional case let $G = (e_j : j \in J)$ be a basis of X. Show that the dual system is linearly independent, but it fails to span the algebraic dual space $(X,\mathbb{K})$. In this case $\dim((X,\mathbb{K})) = 2^{\dim(X)}$.

4. Let μ be the counting measure defined on the σ-algebra of all subsets $A \subset \mathbb{N}$ of the set $\mathbb{N}$ of natural numbers. In other words, any one-point set $\{n\}$ has measure 1, so that $\mu(A) = \#A$, the number of elements of A. Denote a sequence $\mathbf{a} = (\alpha_n)_{n\in\mathbb{N}}$ as a formal sum $\mathbf{a} = \sum_{n=1}^\infty \alpha_n 1_{\{n\}}$, where $1_{\{n\}}$ is the characteristic function of the 1-point set $\{n\}$. If $\alpha_n = 0$ for n sufficiently large, we call such $\mathbf{a}$ a **finite sequence** and at least for finite sequences our sum represents a function on $\mathbb{N}$ (which is nothing else, but our sequence). Clearly, here all functions are measurable and in the case of finite sequences the integral of our function, that can be written as $\int \mathbf{a}\, d\mu$ or $\int \mathbf{a}(n)\, d\mu(n)$ is just the sum $\sum_{n=1}^\infty \alpha_n$.

   Verify that in this case $L^p(\mu)$ with $1 \le p < \infty$ can be identified isometrically with the sequence space ℓ^p, whose norm is

   $$\|\mathbf{a}\|_p := \big(\sum_{n=1}^\infty |\alpha_n|^p\big)^{\frac{1}{p}}.$$

5. Let H be the Hilbert space ℓ^2 of infinite, square summable sequences of scalars $\mathbf{a} = (\alpha_n)_{n\in\mathbb{N}}$ (i.e. such that $\|\mathbf{a}\|_2^2 := \sum_{n=1}^\infty |\alpha_n|^2 < \infty$). Let ϵ_j be the element of H which as a sequence has all but one zero terms, the only nonzero entry equal 1 appearing at the j-th position. This system generalizes the canonical 0-1 basis of $\mathbb{K}^n$, it is linearly independent but show that in this case its linear span (has the dimension equal $\aleph_0$) is strictly smaller than ℓ^2. Adjoin to it the vector $e_\bullet$ represented by the infinite sequence whose n-th member is $\frac{1}{n}$. So extended system is still linearly independent and it is contained in some algebraic basis of H. Show that the linear span of $\{e_j : j \in \mathbb{N}\}$ is dense in H, then analysing $e_\bullet^*(e_j)$ deduce that the coordinate functional $e_\bullet^*$ is discontinuous, while the e_j^* are norm-continuous on H.

6. Prove Theorem 1.10 in the case of the Euclidean space $X = \mathbb{R}^n$. Then try to transfer the result to any n-dimensional space.

7. Prove Theorem 1.1

8. If $L \in \mathcal{B}(X, Y)$ is a bounded linear operator and $x = \sum_{n=1}^{\infty} x_n$ is a sum of a convergent series in the normed space X, show the convergence of $x = \sum_{n=1}^{\infty} y_n$, where $y_n = Tx_n$.

9. Let $L_S : \mathcal{B}(X) \to \mathcal{B}(X)$ be the operator of left multiplication by a given operator $S \in \mathcal{B}(X)$. Namely, $L_S T := ST$ Show its continuity and compute its norm. This together with the previous point will allow you to interchange the left multiplication with convergent (in operator norm) series. Denote the right hand side of the equality in Theorem 1.7 as $\sum_{n=0}^{\infty}(I-T)^n$. Apply L_S to this sum, where $S = (I-T)$ and compute the result. Repeating the argument for the right -multiplication $R_S T := TS$ - conclude the proof of Theorem 1.7

2 Preliminaria on Hilbert space

In this course we mainly consider vector spaces X, H, V over the complex scalars field $\mathbb{C}$. Let us recall some notions related to the inner product.

DEFINITIONs: (1) A **sesquilinear form** on V is a mapping $q : V \times V \to \mathbb{C}$ assigning a scalar $q(u, v)$ to each pair of vectors $u, v \in V$, which is linear in the first variable and anti-linear i the second one, i.e. for any $\alpha \in \mathbb{C}, u, v, u_1, v_1, u_2, v_2 \in V$ we have

$$q(\alpha u, v) = \alpha q(u, v), \quad q(u, \alpha v) = \bar{\alpha} q(u, v),$$

$$q(u_1 + u_2, v) = q(u_1, v) + q(u_2, v), \quad q(u, v_1 + v_2) = q(u, v_1) + q(u, v_2).$$

(2) A form $q : V \times V \to \mathbb{C}$ is non-negative (or **positive semi-definite**), if $q(v, v) \geq 0$ for any $v \in V$. It is said to be positive, or **positive -definite**, if $q(v, v) > 0$ for any non-zero $v \in V$.

(3) A **hermitian form** is a sesquilinear form satisfying additionally the following "skew symmetry" postulate:

$$q(v, w) = \overline{q(w, v)} \quad (\forall_{w,v \in V}).$$

Finally, the **scalar product** on V is a hermitian, positive definite form, denoted usually

$$\langle u, v \rangle$$

rather than $q(u, v)$. The linear algebra textbooks use often the "dot notation" either $u \cdot v$, or $u \circ v$, unacceptable in the case where u, v are functions, which often is the case. The **orthogonality** relation $u \perp v$ means that $\langle u, v \rangle = 0$

The quadratic form $Q : V \to \mathbb{C}$ associated to a sesquilinear form $q : V \times V \to \mathbb{C}$ is defined for $w \in V$ by

$$Q(w) = q(w, w).$$

In the scalar product space we write $\|w\|^2$ for $Q(w)$, as it turns out that

$$\|w\| := \sqrt{\langle w, w \rangle}$$

defines then a norm on V. If V with respect to this norm is complete, then it is called a **Hilbert space**. (Etymology: from Latin *sēsqui* = one and a half)

Theorem 2.1 *Basic properties of sesquilinear forms q and their associated quadratic forms Q:*

(a) (Parallelogram Law) $Q(v + w) + Q(v - w) = 2Q(v) + 2Q(w)$.

(b) (Polarisation Identity) $q(f, g) = \frac{1}{4}(Q(f + g) - Q(f - g) + iQ(f + ig) - iQ(f - ig))$.

(c) (Phytagorean Theorem): If $q(u, w) = 0$, then $Q(u + w) = Q(u) + Q(w)$

(d) (Schwarz Inequality) If q is nonnegative-definite ($\Rightarrow$`hermitian`), then $|q(u, w)|^2 \leq Q(u)Q(w)$.

(e) q is hermitian if and only if Q assumes only real values `(cf. previous point (d))`

If the inner product notation is used, the Schwarz Inequality takes the form:

$$|\langle u, v \rangle| \leq \|u\| \|v\|. \tag{8}$$

The proofs of (a),(b), (c) reduce to direct calculations, (e) follows from (b) by expressing the real part of $q(u,v)$ as : $\texttt{Re}\ q(u,w) = \frac{1}{4}(Q(u+v) - Q(u-v))$ if $Q(H) \subset \mathbb{R}$. Ihe imaginary part of $q(u,v)$ -as a linear functional in the variable u is equal to $-\texttt{Re}\ q(iu,w)$. Using these formulae one computes the adjoint of $q(u,v)$ and compares it to $q(v,u)$ by elementary linear algebra. The converse implication in (e) is obvious. Now using both non-negativity and hermitian property, for any real t we get $0 \le p(t) := Q(u+tw) = Q(u) + 2t\texttt{Re}\ q(u,w) + t^2 Q(w)$, which is a polynomial of degree 2 in t. It cannot have two distinct roots, so -non-positive must be its discriminant ("Delta"): $0 \ge (2\texttt{Re}\ q(u,w))^2 - 4Q(u)Q(v)$. Hence $|\texttt{Re}\ q(u,w)| \le \sqrt{Q(u)Q(w)}$. For any fixed w the right-hand side is a seminorm of v and the following easy lemma (replace u by $e^{-i\phi}u$, if $q(u,v) = |q(u,v)|e^{i\phi}$) concludes the proof

Lemma 2.2 *If $F : H \to \mathbb{C}$ is a $\mathbb{C}$-linear functional and $\rho : X \to [0,+\infty)$ is a seminorm on H, then*

$$\big(\forall_{x\in H}\ \texttt{Re}\ F(x) \le \rho(x)\big) \Leftrightarrow \big(\forall_{x\in H}\ |F(x)| \le \rho(x)\big).$$

IMPORTANT NOTE: If one considers the real scalar field $\mathbb{R}$, the sesquilinear forms become just the bilinear ones, but the **Polarisation Identity fails in the real case, unless we assume the symmetry**. For symmetric $\mathbb{R}$-valued bilinear q, i.e. satisfying $q(u,v) = q(v,u)\forall_{u,v}$ we obtain $q(f,g) = \frac{1}{4}(Q(f+g) - Q(f-g))$ -just by subtracting side-by-side the equalities expressing $Q(f \pm g)$. Another algebraic result (Jordan – von-Neumann Theorem) says that any norm obeying the Parallelogram Law is defined by some inner product. This holds both for $\mathbb{B} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

The sesquilinear form q_T and the corresponding quadratic form Q_T defined by a linear operator $T : H \to H$ are given by

$$q_T(x,y) = \langle Tx, y\rangle, \quad Q_T(x) = \langle Tx, x\rangle, \qquad x,y \in H. \tag{9}$$

The rotation 2×2 matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generates the (isometric) linear mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates any vector by 90 degrees, hence $Q_A(x) = 0\,\forall x$. It is therefore important to stress that in the complex inner product spaces the quadratic form does determine the operator: If $Q_T(x) = Q_S(x)$ for all $x \in H$, then $T = S$ (even without assuming any symmetry). This is so because the polarisation formula holds in this case ($\mathbb{K} = \mathbb{C}$). Hence from Q_T we recover q_T. The way of getting the vector Tx from the values of $q_T(x,y)$, where y runs through H, comes from the Fourier series theory. If $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis, then

$$Tx = \sum_{n=1}^{\infty} \langle Tx, e_n\rangle e_n = \sum_{n=1}^{\infty} q_T(x, e_n) e_n$$

.

The orthogonal projection of a vector $x \in H$ **onto a convex closed set** M in a Hilbert space H, denoted P_M is the unique point $y \in M$ that minimises the distance from x, meaning that for any $z \in M$ one has $\|x-y\| \le \|x-z\|$. In other words, if $\delta = \texttt{dist}(x,M) := \inf\{\|x-z\| : z \in M\}$, then $P_M x$ is the only vector y such that $y \in M$ and $\|x-y\| = \delta$. In any inner product space V one finds a sequence of the points $z_n \in M$ with $\delta = \lim \|x - z_n\|$ Using (a) above and the convexity (so that still $\frac{1}{2}(z_n + z_k) \in M$, which implies $\|2x - (z_n + z_k)\|^2 \ge 4\delta^2$) we show the Cauchy's condition for the sequence (z_n). Unless V is complete (i.e. -a Hilbert space), or at least M is complete as a subspace, nothing else can be done. But using the completeness -one obtains the limit of (z_n), say $y = \lim z_n$. Since M is closed, $y \in M$. By the continuity of the norm, $\|x-y\| = \lim \|x - z_n\|$, proving the existence. The uniqueness results from the inequalities used earlier in the proof.

Theorem 2.3 *If $P = P_M$ is the orthoprojection onto a closed linear subspace $M \ne \{0\}$ of a Hilbert space H, then*

1. *For $x \in H, y \in M$ we have $y = P_M x \Leftrightarrow x - y \perp M$,*
2. *The mapping $P_M : H \to H$ is bounded and linear, of norm $\|P_M\| = 1$*
3. *If a linear mapping $P : H \to H, P \ne 0$ is bounded, linear, then there exists a closed linear subspace $M \ne \{0\}$ such that $P = P_M$ if and only if $PP = P$ and P satisfies one of the additional conditions: $\|P\| \le 1$ or $P^* = P$, the latest meaning $\langle Pv, w\rangle = \langle v, Pw\rangle\ \forall_{v,w\in H}$.*
4. *The orthogonal decomposition holds : Any $x \in H$ can be written uniquely in the form $x = y + r$, where $y \in M, r \perp M$. Here $r \perp M$ means $r \perp z \quad \forall_{z\in M}$.*

5. *for $P = P_M$ we have $M = \mathcal{N}(I-P) = \mathcal{R}(P)$ and $I-P$ is the orthoprojection onto $M^\perp$ -the orthocomplement of M in H, denoted also $H \ominus M$.*

Note that $\mathcal{N}(I-P)$, the nullspace (=kernel) of the identity minus P is exactly the set of all fixpoints of P, i.e. such points $v \in H$ that $Pv = v$. In the remaining case $M = \{0\}$, we clearly have $P_M = 0$.

There are also "skew projections" -corresponding to a direct sum decomposition. If

$$H = M_1 + M_2,\ M_1 \cap M_2 = \{0\},$$

we say that H is a direct sum of the subspaces M_1, M_2. From linear algebra we know that this corresponds to a unique decomposition: $x = x_1 + x_2$ with $x_j \in M_j, j = 1, 2$. The projection of x onto M_1 in the direction of M_2, denoted $P_{M_1,M_2}x$ is simply the summand x_1. One can prove that unless $M_1 \perp M_2$, we have the norm of this projection > 1. Also this operator's adjoint is different from $P_{M_1,M_2}x$ in this case. If only M_1, M_2 are both closed and H is complete, the continuity of the corresponding projection can be deduced from Banach's Inverse Mapping Theorem applied to the addition mapping: $S : M_1 \times M_2 \ni (u, v) \to u + v \in H$.

EXAMPLES OF ORTHOPROJECTIONS

1. Any diagonal matrix whose diagonal entries are either 0 or 1

2. If an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of H is given, for any $k \in \mathbb{N}$ the k-th partial sum of the Fourier series,

$$S_k(x) := \sum_{n=1}^{k} \langle x, e_n\rangle e_n$$

defines the operator $S_k : H \to H$. It is easy to verify the condition from point (1.) of theorem 2.3. Hence S_k are the orthoprojections onto the linear span of $\{e_1, \dots, e_k\}$. Hence $\|S_k(x)\|^2 \le \|x\|^2$, which due to the Pythagorean Theorem gives $\sum_{n=1}^{k} |\langle x, e_n\rangle|^2 \le \|x\|^2$ and passing with $k \to \infty$ we get Bessel's Inequality:

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \le \|x\|^2 \tag{10}$$

Last Thursday I have discussed 3 theorems describing arbitrary bounded linear functionals φ on specific Banach spaces (the so called F.Riesz' Representation Theorems): On $L^p(\mu)$ one finds $\rho \in L^q(\mu)$ with $\frac{1}{p} + \frac{1}{q} = 1$ so that

$$\varphi(f) = \int f(\omega)\rho(\omega)d\mu,$$

while on $C(X)$ with X -a compact topological space, there exist a Borel-measurable $\rho : X \to \{z \in \mathbb{C} : |z| = 1\}$ and a Borel, regular measure ν on X so that

$$\varphi(f) = \int f(\omega)\rho(\omega)d\nu,$$

In Hilbert spaces any bounded linear functional comes from a vector $v \in H$ via the inner product:

$$\varphi(x) = \langle x, v\rangle.$$

FOR THE NEXT TUTORIALS please solve the following (in addition to the previously given problrms): Given nonzero vectors $u, v \in H$ define the linear operator $L : H \ni x \to \langle x, v\rangle u \in H$, denoted by $u \otimes v$, or by uv^*. Hence $(u \otimes v)(x) := \langle x, v\rangle u$. This "tensor product -style notation is a bit misleading, so later I will replace it by uv^*- consistent with column-vector u multiplied (as a matrix) with its Hermitian conjugate (a row vector v^* having the complex conjugates of the coefficients of v).

This is a rank-one operator (rank (L) is the dimension of the range space L(H). L is the only linear mapping that sends the vector v to $\|v\|^2u$ and for which the nullspace $\mathcal{N}(L)$ $(= \ker(L))$ equals the orthocomplement of $\{v\}$, i.e to the set $\{v\}^\perp := \{y \in H : y \perp v\}$.

(1) Show, that the adjoint to this L is obtained by interchanging the vectors: $(v \otimes u)^* = u \otimes v$

(2) In particular, $v \otimes v$ is self-adjoint. Show that this operator is of the form $\|v\|^2P$, where P is the orthoprojection onto the 1-dimensional subspace spanned by v

(3) Show that any finite-rank operator $T \in \mathcal{B}(H)$ can be written as a finite sum of the form $\sum_{n=1}^{k} v_n \otimes w_n$ with some vectors $v_j, w_j, j \le k$. (Hint: take any basis of the range space $\mathcal{R}(T)$.=

(4) Show that $\|u \otimes v\| = \|u\|\|v\|$

(5) Find a condition (in terms of the matrix' entries) for a nonzero 2×2 matrix to be a matrix for some rank-one operator. Generalize to $d \times d$ matrices.

3 More basic facts on Hilbert space operators

DEFINITIONs: (1)(Eigenvectors and eigenvalues) : If $T : H \to H$ is a linear operator, then a scalar $\lambda \in \mathbb{K}$ is called an *eigenvalue* of T, if for some **nonzero** $w \in H$ (called eigenvector of T) we have

$$Tw = \lambda w. \quad (\texttt{equivalently, if} \quad \mathcal{N}(T - \lambda I) \neq \{0\}).$$

The set of all eigenvalues is called the *point spectrum of* T and is denoted by

$$\sigma_p(T).$$

The set of all eigenvectors corresponding to λ, equal to $\mathcal{N}(T - \lambda I)$, is a linear subspace -called the *eigenspace* of T corresponding to the eigenvalue λ. (Here we treat 0 also as an eigenvector.)

The simplest linear operators on the sequence Hilbert space ℓ^2 are the diagonal operators

$$T = \mathrm{diag}(a_n)$$

corresponding to a given sequence (a_n) of complex numbers. Its domain,

$$\mathcal{D}(\mathrm{diag}(a_n)) := \{(x_n) \in \ell^2 : (a_n x_n) \in \ell^2\}$$

is the maximal "natural domain", where the following definition makes sense: $T(x_n) = (a_n x_n)$. Recall that the sequence $(a_n x_n)$ belongs to ℓ^2 iff $\sum_n |a_n x_n|^2 < \infty$. The domain is the entire ℓ^2 and the operator is bounded iff the sequence (a_n) is bounded, i.e. it belongs to ℓ^∞ and then the operator norm, $\|T\|$ is equal to the ℓ^∞ norm of (a_n), which is $\|(a_n)\|_\infty = \sup\{ |a_n| : n \in \mathbb{N}\}$. We have checked it substituting the basic 0-1 vectors ε_j forming an orthonormal basis in ℓ^2. A converse result holds true:

Theorem 3.1 *If some orthonormal basis (f_n) of a Hilbert space H consists of eigenvectors of a bounded linear operator T, so that $Tf_n = a_n f_n$ for some scalars a_n, then the infinite matrix representing T in this basis (whose entries are equal to $\langle Tf_j, f_k\rangle$) is diagonal, with the main diagonal's n-th entry equal a_n.*

Which operators can be diagonalised in the above manner? -is a nontrivial question. A partial answer will be given by the spectral theorem. The existence of such an orthonormal basis of eigenvectors for T will be proved for compact normal operators defined below.

Definition 3.2 *A linear operator $T : H_1 \to H_2$ is called*

- an isometry, *if $\|Tx\| = \|x\| \forall x \in H_1$*
- a partial isometry, *if $\|Tx\| = \|x\| \quad \forall x \in H_1 \ominus \mathcal{N}(T)$*
- a unitary operator, *if T is a surjective isometry*
- a normal operator, *if $\mathcal{D}(T^*) = \mathcal{D}(T)$ and $\|T^*x\| = \|Tx\| \; \forall x \in \mathcal{D}(T)$*
- a selfadjoint operator, *if $\mathcal{D}(T^*) = \mathcal{D}(T)$ and $T^*x = Tx \; \forall x \in \mathcal{D}(T)$*

Finally, an operator $T : H_1 \to H_2$ is compact, *if the image of any bounded sequence of vectors $x_n \in H_1$ contains a convergent subsequence. The set of all compact linear operators from H_1 to H_2 will be denoted $\mathcal{B}_0(H_1, H_2)$ and by $\mathcal{B}_0(H_1)$, if $H_1 = H_2$. In other words, if $\sup_n \|x_n\| < \infty$, then for some integers $1 \le n_1 < n_2 < \dots$ there should exist a limit $\lim_{k\to\infty} Tx_{n_k}$ in the norm topology of H_2.*

We have the following equivalent formulations for the above properties:

Theorem 3.3 *If $\mathcal{D}(T) = H_1$ for a bounded linear operator $T : H_1 \to H_2$, then*

(a) T is an isometry iff $T^*Tx = x \; \forall x \in H_1$, i.e. iff $T^*T = I_{H_1}$

(b) T is unitary iff ($T^*T = I_{H_1}$ and $TT^* = I_{H_2}$), i.e. iff $T^* = T^{-1}$.

(c) T is normal iff $T^*T = TT^*$

(d) T is selfadjoint, iff $\mathcal{D}(T^*) = \mathcal{D}(T)$ and the form $Q_T(x) = \langle Tx, x\rangle$ is real-valued.

Recall, that by c_0 we denote the space of all scalar sequences converging to zero, called sometimes "null sequences".

Theorem 3.4 *Let* $T = \operatorname{diag}(a_n) : \ell^2 \to \ell^2$ *be a diagonal operator corresponding to a sequence* $(a_n) \in \ell^\infty$. *Then* T *is normal and its adjoint is also a diagonal operator,* $T^* = \operatorname{diag}(\bar{a}_n)$*, defined by the complex conjugates of its diagonal sequence. Moreover,* T *is an isometry iff* $|a_n| = 1 \, \forall n \in \mathbb{N}$. T *is compact iff* $(a_n) \in c_0$. *The composition* TS *of two diagonal operators is a diagonal operator with diagonal* $a_n b_n$*, if* $T = \operatorname{diag}(a_n), S = \operatorname{diag}(b_n)$. *In particular,* $TS = ST$.

The compactness of a subset $E \subset H$ in a normed space is equivalent to sequential compactness (the existence of convergent subsequences y_{n_k} for any sequence y_n in E, so that $\lim_k y_{n_k} \in E$). If we only assume the existence of $\lim_k y_{n_k} \in H$, the condition means the compactness of the closure of E, called the relative compactness of E. If $\bar{B}$ denotes the closed unit ball in H_1, then the compactness of a linear operator $T : H_1 \to H_2$ means the relative compactness of the image $T(\bar{B})$ of the closed unit ball. As in the case of the equivalent conditions for continuity of T, the compactness of T is equivalent to the relative compactness of the image of any ball of positive radius. Since relatively compact subsets are bounded, any compact operator must be bounded. The definition, as stated, applies to Banach spaces as well. In the case of Hilbert spaces, or reflexive Banach spaces, the image $T(\bar{B})$ of any closed ball is also closed. This is a consequence of the compactness of $\bar{B}$ in the weak topology (Banach -Alaoglu's Theorem) and the continuity of bounded linear maps also with respect to weak topologies. We are not going to use this fact, however in this course. More important is the following remark: The unit closed ball $\bar{B}$ in H cannot be compact unless $\dim(H) < \infty$.

Another important characterisation of (relative) compactness, similar to the classical Bolzano-Weierstrass theorem: ("Bounded subsets of the Euclidean space $\mathbb{R}^n$ are relatively compact") is due to Hausdorff. We know, that in the infinite dimensional case boundedness will not suffice (why?). We need the stronger property than just boundedness:

Definition 3.5 *A subset* E *in a metric space* (X, d) *is* totally bounded *if it has finite coverings by finite families of sets with arbitrarily small diameter. In other words,*

$$\forall_{\epsilon>0} \exists_{k \in \mathbb{N}} \exists_{x_1, \ldots, x_k \in E} \forall_{x \in E} \exists_{j \leq k} d(x, x_j) < \epsilon.$$

The set $\{x_1, \ldots x_n\}$ is then called an ϵ-network for E and the balls with radii ϵ, centered at x_j cover E. (Their diameters satisfy, of course, $\operatorname{diam} B(x_j, \epsilon) \leq 2\epsilon$). If we just require that the x_j are points from the space X, we get an equivalent definition (why?). We may call a sequence $(z_n) \subset X$ a *uniformly separated sequence*, if for some $\delta > 0$ and for any $k, j \in \mathbb{N}$ one has $d(z_j, z_k) \geq \delta$. Clearly, any of its subsequences is also uniformly separated and it cannot satisfy Cauchy's condition. Therefore to prove that a set Z is not relatively compact it suffices to find a uniformly separated sequence of its points. Check that orthonormal sequences are uniformly separated and that completely bounded sets are bounded. Now the positive result:

Theorem 3.6 *(*Hausdorff*) A set* E *in a complete metric space is relatively compact if and only if it is completely bounded. In particular,* E *is compact iff* E *is closed and completely bounded.*

Using this characterisation it is relatively easy to describe the set $B_0(H_1, H_2)$ of compact operators. Recall that a *sequence* (x_n) *converges weakly* to x_0 in a Banach space X, if for any continuous linear functional $\phi \in X'$ the scalar sequence $\phi(x_n)$ converges to $\phi(x_0)$. In Hilbert spaces this is equivalent to $\lim \langle x_n - x_0, z \rangle = 0 \forall_{z \in X}$.

Theorem 3.7 *Let* $T \in \mathcal{B}(H_1, H_2)$ *be a bounded linear operator in a Hilbert space. Then TFAE:*

1. *T is compact (in symbols,* $T \in \mathcal{B}_0(H_1, H_2)$*).*
2. *If* $(x_n) \subset H_1$ *converges weakly to zero, then* $\|Tx_n\| \to 0$.
3. *There exists a sequence of finite rank operators* $T_n : H_1 \to H_2$ *such that* $\|T_n - T\| \to 0$.

The set $\mathcal{B}_0(H)$ of compact linear operators $S : H \to H$ is a two-sided closed ideal in $\mathcal{B}(H)$. In other words, the limit of norm-convergent sequence (S_n) of compact operators is compact and so is any linear combination of two compact operators. If S is compact and T is bounded, then both ST and TS are compact.

4 Compact operators

4.1 Weak convergence

The weak topology is a bit exotic, since it is non-metrizable in any $\infty-$ dimensional Banach space. Denote the weak convergence by $\rightharpoonup$. Recall, that a sequence of vectors x_j in a Banach space X *converges weakly to* $x_0 \in X$, a fact denoted by $x_j \rightharpoonup x_0$, if for any fixed continuous linear functional $\varphi \in X'$ the sequence of (real or complex) numbers $\varphi(x_j)$ converges to $\varphi(x_0)$. In Hilbert spaces we have $\varphi(x)$ uniquely represented as $\langle x, z\rangle$ for some $z \in H$, so the defining condition becomes simply

$$x_j \rightharpoonup x_0 \Leftrightarrow \forall_{z\in H} \lim_j \langle x_j, z\rangle = \langle x, z\rangle.$$

In any case this convergence comes from *the weak topology* defined as the weakest topology making all $\varphi \in X'$ continuous. This is the topology defined by a family of seminorms $\{p_\phi(\cdot) : \phi \in X'\}$, where $p_\phi(x) = |\phi(x)|$ for $x \in X$. Basis of weak neighbourhoods of zero is formed by the family of sets

$$W(\phi_1, \dots \phi_k) = \{x \in X : |\phi_1(x)| < 1, \dots |\phi_k(x)| < 1\}, \quad \textit{where } k \in \mathbb{N}, \phi_1, \dots \phi_k \in X'.$$

Hence as an indexing set we may take the family of all finite subsets of X'. Note that for families of seminorms p_j one should consider finite intersections of balls $\{x \in X : p_j(x) < r\}$ of radii $r > 0$. This is because for one neighbourhood of zero the minimum (taken over $j \in \{1, \dots, k\}$) of a finite set of radii r_j can be put instead of the radii $r_1, \dots, r_k$. In the case of the weak topology it simplifies further -we can take $r = 1$ throughout, since ϕ can be replaced by the functional $r^{-1}\phi$, since $|\frac{1}{r}\phi(x)| < 1 \Leftrightarrow |\phi(x)| < r$.

In any Hilbert space with orthonormal sequence (e_n) it follows from Bessel's inequality that $e_n \rightharpoonup 0$ as $n \to \infty$. The famous example due to von Neumann shows this: If $E = \{e_n + ne_m,\ n, m \in \mathbb{N}\}$, then the iterated weak limit is zero:

$$\lim_n(\lim_m e_n + ne_m) = \lim_n(e_n + n0) = 0,$$

hence zero belongs to the weak closure of E. But -unlike in the metric spaces - we have the following fact: there is **no sequence of vectors** $x_k \in E$ **weakly convergent to zero**. Indeed, by the Uniform Boundedness Principle (or directly from the Banach-Steinhaus Theorem), as a weakly convergent sequence, such a sequence should be bounded. As $x_k = e_{n_k} + n_k e_{m_k}$ for some $n_k, m_k \in \mathbb{N}$ and $\|x_k\|^2 = 1 + n_k^2$, the sequence n_k should be bounded, hence it should contain a constant subsequence n_{k_j} -equal to some n_0. But then either m_k is eventually constant (say, equal m_0 for sufficiently large k, leading to $x_{k_j} \rightharpoonup e_{n_0} + n_0 e_{m_0}$), or we should have $m_{k_j} \to \infty$, yielding $x_{k_j} \rightharpoonup e_{n_0} + 0$, a contradiction.

For general normed spaces X – the non-metrizability follows from the fact that any countable intersection of a sequence of weak neighbourhoods of zero is of the form $\bigcap_{n=1}^\infty W(\phi_n)$ and contains more than one point (show that codim $\bigcap_{n=1}^\infty \ker(\phi_n) \leq \aleph_0$ – exercise). In metric spaces the balls centered at 0, of radii $\frac{1}{n}$ have one-point intersection.

Any ball in X with respect to the norm (and any bounded set) has empty interior in the weak topology.

Despite of these drawbacks, the weak convergence (of ordinary sequences) is in a sense quite natural:

In the sequence spaces $\ell^p, 1 < p < \infty$ a sequence $\mathbf{x}_n \in \ell^p$ converges weakly to $\mathbf{x}_0$ iff it is bounded ($\sup_n \|\mathbf{x}_n\|_p < \infty$) and each coordinate converges: $(\mathbf{x}_n)_k = e_k^*(\mathbf{x}_n) \to (\mathbf{x}_0)_k$ as $n \to \infty (\forall_k \in \mathbb{N})$. This is a particular case of the following general principle (with $D = \{e_k^* : k \in \mathbb{N}\}$):

Theorem 4.1 *If the linear span of a subset $D \subset X'$ is dense in X' (with its norm), then a sequence (x_n) converges weakly to x_0 iff it is bounded and $\phi(x_n) \to \phi(x_0)$ for any $\phi \in D$. If moreover such a "generating set" $D = \{\phi_j\}_{j\in\mathbb{N}}$ is countable, then the metric $\rho(x, y) := \sum_{n=1}^\infty 2^{-j}|\phi_j(x) - \phi_j(y)|$ defines the weak topology on bounded subsets of X.*

A surprising result at $p = 1$ is due to I. Schur: *A sequence in ℓ^1 converges weakly if and only if it norm- converges to the same limit.* (Note that the two topologies are different!)

In Hilbert spaces any bounded sequence has weakly convergent subsequence and any closed ball is weakly compact. In separable Hilbert spaces the weak topology, when restricted to bounded subsets is metrizable! The **weak compactness of the closed unit ball** (and of any closed ball) takes place iff the space is **reflexive** (i.e. the canonical embedding in its second dual space $j : X \to X''$, $(j(\phi))(x) := \phi(x), \phi \in X', x \in X$ is surjective -a fact denoted often as $X = X''$.

For convex sets their weak closures are equal to their norm-closures, due to Hahn-Banach theorem (precisely, from its corollary on separation of points from closed convex sets).

4.2 Convergence of sequences of operators

We say that a sequence (or a generalized sequence, i.e. a net) of operators $T_j \in \mathcal{B}(X, Y)$ acting between two Banach spaces converges to an operator $T \in \mathcal{B}(X, Y)$

- uniformly, or in norm (notation $T_j \to T$), if the operator norms $\|T_j - T\|$ converges to zero
- strongly (notation $T_j \to T(SOT)$), if for any $x \in X$ we have $\|T_j x - Tx\| \to 0$
- weakly (notation $T_j \to T$ (WOT)) if for any $x \in X$ we have $T_j x - Tx \rightharpoonup 0$

Clearly, the uniform convergence implies the strong convergence. The strong convergence implies the weak convergence. None of these implications is reversible.

The remaining part of this page can be omitted at the first reading, we are going to concentrate on ordinary (not generalised) sequences. The (SOT) convergence corresponds to the *strong operator topology* given by the family of seminorms $\{p_x : x \in X$, where $p_x(T) := \|Tx\|$. Similarly, the weak operator topology is defined by $p_{x\varphi}(T) := |\varphi(Tx)|$, a family of seminorms indexed by $(x, \varphi) \in X \times Y'$. Let us concentrate for a while on the latter two convergences in the Hilbert spaces case: Neither of these two convergences is metrizable (but their restrictions to bounded sets of operators are). Clearly, in the Hilbert space case for $T_j, T \in \mathcal{B}(H)$

- the weak convergence: $T_j \to T$ (WOT) means that $\langle T_j x, y\rangle \to \langle Tx, y\rangle\ \forall_{x,y\in H}$.

It turns out, that **the multiplication in $\mathcal{B}(H)$ is discontinuous -but is sequentially continuous!** Here the multiplication TS is just the composition: $(TS)(x) = T(Sx)$. The relevant example can be found in a separate file entitled "example1.pdf". The difference between nets and sequences is that for sequences the index j runs through the set $\mathbb{N}$ of natural numbers and only finitely many terms of a convergent sequence can stay outside a given neighbourhood of the limit. By Uniform Boundedness Principle, WOT- convergent sequences are bounded. The same cannot be asserted for convergent nets, where we have the index set J possibly uncountable, directed by some transitive relation $j \preceq i$ such that $\forall_{j,k\in J}\exists_{i\in J} j \preceq i$ and $k \preceq i$. Here the convergence of T_j to T means that for any neighbourhood W of T there exists $j_0 \in J$ such that $j \in J, j_0 \preceq j \Rightarrow T_j \in W$. Any accumulation point (or a point from the closure $\bar{E}$) of a set E is always a limit of some generalized sequence of elements of E. The example from the previous subsection on weak convergence with $E = \{e_n + ne_m,\ n, m \in \mathbb{N}\}$ shows, that the ordinary sequences can be inadequate for the latter purpose -even for countable sets E.

EXERCISE 1. Let us direct the set of integers $\mathbb{Z}$ by the ordinary relation $j \leq k$. Is the net $(2^{-j})_{j\in\mathbb{Z}}$ bounded? Does it converge?

EXERCISE 2. If we direct $\mathbb{N} \times \mathbb{N}$ by $(j, k) \preceq (n, m)$ meaning that $\max(j, k) \leq \min(n, m)$, is the convergence $\|x_n - x_m\|$ corresponding to this direction of $\mathbb{N} \times \mathbb{N}$ equivalent to the Cauchy condition (in a normed space X)?

EXERCISE 3. The diameter $\delta(\mathcal{T})$ of a partition $\mathcal{T} = (t_0 = a < t_1 < \ldots < t_n = b$ of an interval $[a, b]$ is defined as $\delta(\mathcal{T}) := \max\{t_j - t_{j-1} : j = 1, \ldots n\}$. In the set of pairs $(\mathcal{T}, \Lambda)$, where $\Lambda = \{\lambda_1, \ldots \lambda_k\}$ is a collection of intermediate points for $\mathcal{T}$, so that $\lambda_j \in [t_{j-1}, t_j]$ define a direction $(\mathcal{T}_1, \Lambda_1) \preceq (\mathcal{T}_2, \Lambda_2)$, if $\delta(\mathcal{T}_1) \leq \delta(\mathcal{T}_2)$.

Now let $f : [a, b] \to \mathbb{R}$ be a given function. Define

$$S(f, \mathcal{T}, \Lambda) = \sum_{j=1}^{n} f(\lambda_j)(t_j - t_{j-1}).$$

If we treat $S(f, \mathcal{T}, \Lambda) \in \mathbb{R}$ as a generalised sequence indexed by such pairs $(\mathcal{T}, \Lambda)$, what is the meaning for its convergence to some limit $S \in \mathbb{R}$? (answer using a notion from basic calculus)

EXERCISE 4. If we have a neighbourhood basis $(W_j)_{j\in J}$ of a point x_0 in some topological space, so that for any neighbourhood U of x_0 there exists $j \in J$ such that $W_j \subset U$, let us pick arbitrary points $x_j \in W_j$ and direct J by $j \preceq k$ meaning $W_j \supset W_k$ (the reverse inclusion!). Show that the net x_j converges to x_0. (Using this one can show that the continuity at a point x_0 of a mapping F between two topological spaces takes place iff for any generalised sequence x_j converging to x_0 the values $F(x_j)$ converge to x_0.)

EXERCISE 5. Let Ω be the set of all continuous functions on $[0, 1]$, taking values from $[0, 1]$. Consider the topology of pointwise convergence defined on $C[0, 1]$ by the seminorms $p_t(x) := |x(t)|$, where $t \in [0, 1], x \in C[0, 1]$. Show that the functional $F : \Omega \ni x \to F(x) : \int_0^1 x(t)\, dt$ **is sequentially continuous**, i.e. it satisfies the Heine condition: for all ordinary sequences $x_n \in \Omega$ convergent pointwise to x_0 one has $F(x_n) \to F(x_0)$. On the other hand, any neighbourgood of zero in Ω contains a basic neighbourghood of the form $W_{t_1,\ldots,t_k} := \{x \in \Omega : |x(t_1)| < \epsilon, \ldots |x(t_k)| < \epsilon\}$ corresponding to some $\epsilon > 0$ and a finite set of points $t_1, \ldots t_k \in [0, 1]$. We can pick a function $z \in W_{t_1,\ldots,t_k}$ (e.g. vanishing at these finite set of points -then any $\epsilon > 0$ will be "good") such that $\int_0^1 z(t)\, dt > \frac{1}{2}$ (draw a picture of a piecewise-linear function having these properties). This shows **the discontinuity** of F on Ω.

WE HAVE PROVED THIS: If a sequence of operators $P_n \to P$ (SOT) and K is a compact operator, then $P_n K \to PK$ uniformly. If a Schauder basis exists in X, then $\exists P_n$ of finite rank, (SOT)-convergent to I (identity). Compact operators are then uniform limits of finite rank operators. Continuous finite rank operators and uniform limits of compact operators are compact.