Additional notes to "Operator Theory" course by Krzysztof Rudol, at WMS AGH

Contents

1. Preliminaria

Here we recall some facts needed from linear algebra:

During this course $\mathbb{K}$ will denote the scalar field -equal either to $\mathbb{R}$, or to $\mathbb{C}$. The vector spaces over $\mathbb{K}$ will be denoted X, Y, Z, M, H -some other capital letters may be used. In some cases arrows over some letters, like $\vec{u}, \vec{w}, \vec{x}, ...$ will be applied to mark the difference between vectors and scalars (usually denoted either by Greek lowercase letters: α, β, λ, or s, t -for real scalars). Later on this distinction will be clear and to simplify the notation, the arrows will be suppressed. The basics of linear algebra are assumed to be known, including the notions of linear independence of vectors, bases, the dimension, linear mappings and their relation to matrices, the Euclidean space $\mathbb{R}^n$, or $\mathbb{C}^n$ with its canonical 0-1 basis:

$$\epsilon_1 = (1, 0, \dots, 0),\ \epsilon_2 = (0, 1, 0 \dots, 0), \dots, \epsilon_n = (0, \dots, 0, 1)$$

and coordinate notations: $\vec{x} = (x_1, \dots, x_n)$.

If $n = 3$ (or 2), instead of $\epsilon_1, \epsilon_2, \epsilon_3$ one usually writes: $\vec{i}, \vec{j}, \vec{k}$ and $\vec{w} = (x, y, z)$.

The symbols: If $F(a)$ is some logical formula depending on the variable a, then $\exists_{a\in A}!F(a)$ will denote "*there exists only one element $a \in A$ for which $F(a)$ holds*", $:=$ -will denote "*equals, by definition*". Conjunction will be denoted p `and` q, or simply p, q rather than $p \wedge q$. The word "**iff**" will stand for "*if and only if*".

Given a mapping $\phi : D \to Y$, where D is the domain of ϕ, denoted $\mathcal{D}(\phi)$, we say that $F : X \to Y$ **extends** ϕ, (or -that ϕ is a restriction of F to D, notation: $F|_D = \phi$, or $\phi \subset F$, if

$$\mathcal{D}(\phi) \subset \mathcal{D}(F) \ \texttt{and}\ \forall_{x\in D}\, F(x) = \phi(x).$$

If moreover F is linear, we speak of a **linear extension**. Similarly, if it is continuous, we call F a **continuous extension** of ϕ.

Useful Fact 1: A subset $G \subset X$ is linearly independent **iff** any mapping $\phi : G \to Y$ has a linear extension (to some linear subspace containing its domain, D, or even to the entire space X.) In order to get one implication it suffices to require the existence of linear extensions only in the scalar-valued case: $Y = \mathbb{K}$.

We say, that G spans X, writing $X = \operatorname{span}(G)$, if

$$\forall_{x\in X}\exists_{m\in\mathbb{N}}\exists_{\alpha_1,\dots,\alpha_m\in\mathbb{K}}\exists_{v_1,\dots,v_m\in G} \quad x = \sum_{j=1}^{m}\alpha_j v_j. \tag{1}$$

Useful Fact 2: A subset $G \subset X$ spans X, if linear extensions of any map $\phi : G \to Y$ are unique -provided they exist. This means that if F_1 and F_2 are linear mappings on X extending the same ϕ, one must have $F_1 = F_2$. Clearly, not all mappings ϕ have any linear extension- take $G = X$, or even $G = \{\epsilon_1, 2\epsilon_1\},\ X = \mathbb{R}^2$. (Again, it suffices to verify this with $Y = \mathbb{K}$).

For linear bases G in X (sets simultaneously lin. independent and spanning X) any mapping from G has exactly one linear extension. All bases of X have the same cardinality, called **the**

dimension of X, denoted $\dim(X)$. If the latter is finite, say $\dim(X) = m$, we may write $G = \{v_1, \dots, v_m\}$ and then the quantifiers: $\exists_m, \exists_{v_1,\dots v_m}$ are redundant in the formula 1, while the quantifier $\exists_{(\alpha_1,\dots,\alpha_m)}$ can be written as $\exists!_{(\alpha_1,\dots,\alpha_m)}$. Then we call this m-tuple $(\alpha_1, \dots, \alpha_m) \in \mathbb{K}^m$ of scalars - the coordinates of x in the basis $(v_1, \dots, v_m)$. I will write now

$$v_j^*(x) := \alpha_j, \ \texttt{so that} \ x = \sum_{j=1}^{m} v_j^*(x) v_j. \tag{2}$$

The same notation can be used to define **the dual system** $\{e_j^* : j \in J\}$ to an infinite basis $\{e_j : j \in J\}$. Here some problems arise (see later -in Tutorials section).

When writing the value of a linear operator T on a given vector x one usually omits the parentheses – writing Tx rather than $T(x)$, when the range is clear. For example, in some formulae the vectors x themselves will be functions, eg. $x(s)$ for $s \in [0,1]$ and we write $(Tx)(s) = \int_0^s f(t)\,dt$ in the case of the so called Volterra operator. Here $(Tx)(s)$ looks better than $(T(x))(s)$, or $T(x)(s)$. Sometimes we need parentheses, it would be unclear, whether $Tx+y$ means $T(x)+y$, or $T(x+y)$.

NOTATION: $\mathcal{L}in(X,Y) := \{T : T \ \texttt{is linear,} \ T : X \to Y\}$ will be the space of all **linear operators** acting from X to Y. Members of $\mathcal{L}in(X, \mathbb{K})$ are called **linear functionals** on X. If we have normed spaces with norms $\| \, \|_X, \| \, \|_Y$, we say that $T \in \mathcal{L}in(X,Y)$ is bounded, if it is bounded on the unit ball denoted by

$$B_X(0,1) := \{x \in X : \|x\|_X < 1\}.$$

Let us define the operator norm $\|T\|$ by

$$\|T\| := \sup\{\|Tv\|_Y : v \in B_X(0,1)\}.$$

Denote the space of all **bounded linear operators** from X to Y by

$$\mathcal{B}(X,Y).$$

If $Y = X$, we write $\mathcal{B}(X)$ in place of $\mathcal{B}(X,Y)$ and if $Y = \mathbb{K}$, we write X' for the space $\mathcal{B}(X,\mathbb{K})$ of all bounded linear functionals on X, called **the dual space** for X. Some textbooks use the notation $\mathcal{L}(X,Y)$ for $\mathcal{B}(X,Y)$, writing X^* rather than X' is also frequent. An example of linear functionals is v_j^* of the dual system (2).

The vector space structure is defined (both on $\mathcal{B}(X,Y)$ and on $\mathcal{L}in(X,Y)$) by pointwise linear operations: given e.g. $T, S \in \mathcal{B}(X,Y)$ we write $T+S$ for the operator sending a vector $x \in X$ into $(T+S)(x) := T(x) + S(x) \in Y$. Similarly, $(\alpha T)(x) := \alpha T(x)$ defines the multiplication of an operator T by the scalar $\alpha \in \mathbb{K}$. The constant function 0, called the zero operator, is clearly bounded and linear. This is the zero element of $\mathcal{B}(X,Y)$. Boundedness of the sum of bounded operators results from the inequalities:

$$\|(T+S)x\| = \|Tx+Sx\| \le \|Tx\| + \|Sx\| \le \|T\| + \|S\| \ \texttt{valid for any} \ x \in B_X(0,1),$$

which also shows that

$$\|S+T\| \le \|S\| + \|T\|.$$

Similarly, one shows that for $\alpha \in \mathbb{K}$ one has $\|\alpha T\| = |\alpha|\|T\|$. Apart from bounded, we often have to consider linear operators, that are unbounded and defined on domains $D(T)$ different from the entire space. Usually, the domains are dense subsets (in the norm topology of the space X). We call such mappings *densely defined operators* in X.

Let D_1, D_2 be now subspaces (in many applications -dense) of some two normed spaces X_1, X_2 -respectively and consider two operators $T_1 : D_1 \to X_2, T_2 : D_2 \to X_3$. Their composition, $T_2 \circ T_1$, denoted T_2T_1 –is defined on the domain

$$D(T_2T_1) := \{x \in D(T_1) : \ T_1(x) \in D(T_2)\} \quad \texttt{by} \quad T_2T_1x := T_2(T_1x). \tag{3}$$

We write T^2 for $T \circ T$ and -proceeding by induction- $T^{n+1} = T \circ T^n$. Let I_X denote the identity operator on X: $I_Xv = v$ $(\forall v \in X)$. Note that $\mathcal{L}in(X,X)$ is an algebra where the multiplication is defined to be the composition. If $\dim(X) > 1$, this algebra is noncommutative, but it has the unit, namely I_X. It is important to note that $\mathcal{B}(X,X)$ -denoted as $\mathcal{B}(X)$ is also an algebra. Moreover $\|T_2T_1\| \le \|T_2\|\|T_1\|$. This follows easily from the estimate:

$$\|Tw\|_Y \le \|T\|\|w\|_X \qquad \texttt{for any} \ w \in X, \ T \in \mathcal{B}(X,Y) \tag{4}$$

The continuity of linear operators and its invertibility are two central issues considered. Let us recall some results from functional analysis: (TFAE = "The Following Are Equivalent)

Theorem 1.1. *For a linear operator $T : X \to Y$ between two normed spaces TFAE:*

(a) *T is continuous on X (even -uniformly continuous),*
(b) *T is continuous at some point $x_0 \in X$,*
(c) *T is bounded in some nonempty open set,*

(d) *T is bounded in the unit ball of X, i.e.* $\|T\| < +\infty$,

(e) *For some finite constant $M \geq 0$ one has $\|Tx\|_Y \leq M\|x\|_X$ for any $x \in X$.*

The norm $\|T\|$ is the least $M \geq 0$ satisfying the estimate in (e). Also $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$.

Theorem 1.2. *Any linear mapping on a finite-dimensional normed vector space is continuous. Any finite-dimensional subspace of a normed space is closed.*

Theorem 1.3. *Any uniformly continuous mapping f defined on a dense subset D of a metric spaceX, whose values are in a complete normed space Y has a unique continuous extension to a continuous mapping $F : X \to Y$.*

Theorem 1.4. *If Y is complete, then the space $\mathcal{B}(X,Y)$ is also complete. (Conversely, if* $\dim(X) > 0$, *then from the completeness of $B(X,Y)$ it follows that Y is complete.)*

Theorem 1.5. *Any continuous linear functional ϕ on a subspace M of a normed space X has a continuous linear extension F having the same norm: $\|\phi\| = \|F\|$.*

Theorem 1.6. *For any $x \in X$ there exists (at least one) bounded linear functional $\phi \in X'$ having norm one: $\|\phi\| = 1$ such that $\phi(x) = \|x\|$. Hence the **dual formula for the norm** holds:*

$$\|x\| = \sup\{|f(x)| : f \in X', \|f\| \leq 1\}.$$

Theorem 1.7. *If X is complete and $\|T - I_X\| < 1$, then T is bijective, has a bounded inverse. Moreover,*

$$T^{-1} = I_X + \sum_{n=1}^{\infty}(I - T)^n.$$

Corollary 1.8. *The set of all invertible elements of the algebra $\mathcal{B}(X)$ is open and the operation of taking the inverse operator is continuous.*

Theorem 1.9. *(fundamental Banach results): Here X, Y are Banach spaces.*
OPEN MAPPING THEOREM *Any continuous linear surjection $T \in \mathcal{B}(X,Y)$ maps open subsets of X onto open subsets of Y.*
INVERSE MAPPING THEOREM *The inverse of a continuous bijection $T \in \mathcal{B}(X,Y)$ is also continuous*
CLOSED GRAPH THEOREM *If $T \in \mathcal{L}in(X,Y)$ has closed graph (i.e. the set $\Gamma_T := \{(x,y) \in X \times y : y = Tx\}$ is closed in the product topology), then T must be continuous.*
BANACH – STEINHAUS THEOREM *If a sequence of bounded linear operators satisfies the pointwise - boundedness condition: $\forall_{x\in X} \sup_n \|T_n x\| < \infty$, then it is uniformly bounded on the unit ball:* $\sup_n \|T_n\| < \infty$.

We say that two norms, say $\|\ \|$ and $\|\ \|_*$ on the same linear space X are **equivalent norms**, if there exist positive constants $m, M > 0$ such that

$$\forall_{x\in X}\ m\|x\| \leq \|x\|_* \leq M\|x\|. \tag{5}$$

We say that a linear mapping $T : X \to Y$ is bounded below on X, if for some $m > 0$ we have estimates

$$\forall_{x\in X}\ \|Tx\| \geq m\|x\|. \tag{6}$$

Theorem 1.10. *Any two equivalent norms define the same topology. On a finitely-dimensional space all norms are equivalent.*

MORE NOTATION: For a linear mapping its kernel, known also as the nullspace is denoted either by $\mathcal{N}(T)$, or by $\ker(T)$ and is, by definition, the set

$$\mathcal{N}(T) := \{x \in X : Tx = 0\}.$$

The range space of $T \in \mathcal{L}in(X,Y)$ is denoted by $\mathcal{R}(T)$ (or in some books -by $Im(T)$. Here

$$\mathcal{R}(T) := \{y \in Y : \exists_{x\in X}\, y = Tx\}.$$

Both sets are linear subspaces. From linear algebra we know that

$$T \texttt{ is injective iff } \mathcal{N}(T) = \{0\}. \tag{7}$$

Surjectivity means that $\mathcal{R}(T) = Y$. In the finite-dimensional case we have the relation

$$\dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T) = \dim(X),$$

which for $X = Y$ gives the equivalence:

Lemma 1.11. *$T \in \mathcal{L}in(X,X)$ is invertible iff $\mathcal{N}(T) = \{0\}$. (This no longer applies in the infinite dimensional case!)*

1.1. **Tutorials 1.** Given a basis $(e_j)_{j\in J}$ the dual system of functionals $(e_j^*)_{j\in J}$ is defined by (2), where the e_j stand in place of v_j, the summation ranges through some finite subset $\{j_1,\dots,j_m\}$ of the set J of indices rather than through $\{1,\dots,m\}$. From linear algebra we know that these functionals e_j^* are linear. They just describe the coordinates of a vector x with respect to the given basis.

(1) In the Euclidean space $\mathbb{K}^n$ the norm is given by $\|\vec{x}\| = \sqrt{|x_1|^2+\cdots+|x_n|^2}$. Here $x_j = \epsilon_j^*(\vec{x})$ (according to the notation from equation (1)) are the coordinates of $\vec{x}$ in the canonical 0-1 basis $(\epsilon_1,\dots,\epsilon_n)$. Any linear operator $T:\mathbb{K}^n\to Y$ can be represented as $T = \sum_{j=1}^n \epsilon_j^* T\epsilon_j$. Deduce the continuity of such T. Express the matrix entries a_{jk} in terms of the basis vectors, T and the dual basis functionals only.

(2) Show that if $(e_1,\dots,e_n)$ form a basis of X, then the dual system: $(e_1^*,\dots,e_n^*)$ is a basis of X^*, hence $\dim(X) = \dim((X,\mathbb{K}))$ provided that $\dim(X)<\infty$.

(3) In the infinitely dimensional case let $G = (e_j: \ j\in J)$ be a basis of X. Show that the dual system is linearly independent, but it fails to span the algebraic dual space $(X,\mathbb{K})$. In this case $\dim((X,\mathbb{K})) = 2^{\dim(X)}$.

(4) Let μ be the counting measure defined on the σ-algebra of all subsets $A\subset\mathbb{N}$ of the set $\mathbb{N}$ of natural numbers. In other words, any one-point set $\{n\}$ has measure 1, so that $\mu(A) = \#A$, the number of elements of A. Denote a sequence $\mathbf{a} = (\alpha_n)_{n\in\mathbb{N}}$ as a formal sum $\mathbf{a} = \sum_{n=1}^\infty \alpha_n 1_{\{n\}}$, where $1_{\{n\}}$ is the characteristic function of the 1-point set $\{n\}$. If $\alpha_n = 0$ for n sufficiently large, we call such $\mathbf{a}$ a **finite sequence** and at least for finite sequences our sum represents a function on $\mathbb{N}$ (which is nothing else, but our sequence). Clearly, here all functions are measurable and in the case of finite sequences the integral of our function, that can be written as $\int \mathbf{a}\,d\mu$ or $\int \mathbf{a}(n)\,d\mu(n)$ is just the sum $\sum_{n=1}^\infty \alpha_n$.

Verify that in this case $L^p(\mu)$ with $1\le p<\infty$ can be identified isometrically with the sequence space ℓ^p, whose norm is

$$\|\mathbf{a}\|_p := \big(\sum_{n=1}^\infty |\alpha_n|^p\big)^{\frac{1}{p}}.$$

(5) Let H be the Hilbert space ℓ^2 of infinite, square summable sequences of scalars $\mathbf{a} = (\alpha_n)_{n\in\mathbb{N}}$ (i.e. such that $\|\mathbf{a}\|_2^2 := \sum_{n=1}^\infty |\alpha_n|^2<\infty$). Let ϵ_j be the element of H which as a sequence has all but one zero terms, the only nonzero entry equal 1 appearing at the j-th position. This system generalizes the canonical 0-1 basis of $\mathbb{K}^n$, it is linearly independent but show that in this case its linear span (has the dimension equal $\aleph_0$) is strictly smaller than ℓ^2. Adjoin to it the vector $e_\bullet$ represented by the infinite sequence whose n-th member is $\frac{1}{n}$. So extended system is still linearly independent and it is contained in some algebraic basis of H. Show that the linear span of $\{e_j: j\in\mathbb{N}\}$ is dense in H, then analysing $e_\bullet^*(e_j)$ deduce that the coordinate functional $e_\bullet^*$ is discontinuous, while the e_j^* are norm-continuous on H.

(6) Prove Theorem 1.10 in the case of the Euclidean space $X=\mathbb{R}^n$. Then try to transfer the result to any n-dimensional space.

(7) Prove Theorem 1.1

(8) If $L\in\mathcal{B}(X,Y)$ is a bounded linear operator and $x = \sum_{n=1}^\infty x_n$ is a sum of a convergent series in the normed space X, show the convergence of $x = \sum_{n=1}^\infty y_n$, where $y_n = Tx_n$.

(9) Let $L_S:\mathcal{B}(X)\to\mathcal{B}(X)$ be the operator of left multiplication by a given operator $S\in\mathcal{B}(X)$. Namely, $L_ST := ST$ Show its continuity and compute its norm. This together with the previous point will allow you to interchange the left multiplication with convergent (in operator norm) series. Denote the right hand side of the equality in Theorem 1.7 as $\sum_{n=0}^\infty (I-T)^n$. Apply L_S to this sum, where $S = (I-T)$ and compute the result. Repeating the argument for the right -multiplication $R_ST := TS$ - we conclude the proof of Theorem 1.7

2. Preliminaria on Hilbert space

In this course we mainly consider vector spaces X,H,V over the complex scalars field $\mathbb{C}$. Let us recall some notions related to the inner product.

DEFINITIONs: (1) A **sesquilinear form** on V is a mapping $q: V\times V\to\mathbb{C}$ assigning a scalar $q(u,v)$ to each pair of vectors $u,v\in V$, which is linear in the first variable and anti-linear i the second one, i.e. for any $\alpha\in\mathbb{C}, u,v,u_1,v_1,u_2,v_2\in V$ we have

$$q(\alpha u,v) = \alpha q(u,v),\quad q(u,\alpha v) = \bar{\alpha} q(u,v),$$

$$q(u_1+u_2,v) = q(u_1,v)+q(u_2,v),\quad q(u,v_1+v_2) = q(u,v_1)+q(u,v_2).$$

(2) A form $q : V \times V \to \mathbb{C}$ is non-negative (or **positive semi-definite**), if $q(v, v) \geq 0$ for any $v \in V$. It is said to be positive, or **positive -definite**, if $q(v, v) > 0$ for any non-zero $v \in V$.

(3) A **hermitian form** is a sesquilinear form satisfying additionally the following "skew symmetry" postulate:

$$q(v, w) = \overline{q(w, v)} \quad (\forall_{w,v \in V}).$$

Finally, the **scalar product** on V is a hermitian, positive definite form, denoted usually

$$\langle u, v \rangle$$

rather than $q(u, v)$. The linear algebra textbooks use often the "dot notation" either $u \cdot v$, or $u \circ v$, unacceptable in the case where u, v are functions, which often is the case. The **orthogonality** relation $u \perp v$ means that $\langle u, v \rangle = 0$

The quadratic form $Q : V \to \mathbb{C}$ associated to a sesquilinear form $q : V \times V \to \mathbb{C}$ is defined for $w \in V$ by

$$Q(w) = q(w, w).$$

In the scalar product space we write $\|w\|^2$ for $Q(w)$, as it turns out that

$$\|w\| := \sqrt{\langle w, w \rangle}$$

defines then a norm on V. If V with respect to this norm is complete, then it is called a **Hilbert space**. (Etymology: from Latin *sēsqui* = one and a half)

Theorem 2.1. *Basic properties of sesquilinear forms q and their associated quadratic forms Q:*

(a) *(Parallelogram Law) $Q(v + w) + Q(v - w) = 2Q(v) + 2Q(w)$.*
(b) *(Polarisation Identity) $q(f, g) = \frac{1}{4}(Q(f + g) - Q(f - g) + iQ(f + ig) - iQ(f - ig))$.*
(c) *(Phytagorean Theorem): If $q(u, w) = 0$, then $Q(u + w) = Q(u) + Q(w)$*
(d) *(Schwarz Inequality) If q is nonnegative-definite ($\Rightarrow$hermitian), then $|q(u, w)|^2 \leq Q(u)Q(w)$.*
(e) *q is hermitian if and only if Q assumes only real values (cf. previous point (d))*

If the inner product notation is used, the Schwarz Inequality takes the form:

$$|\langle u, v \rangle| \leq \|u\|\|v\|. \tag{8}$$

The proofs of (a),(b), (c) reduce to direct calculations, (e) follows from (b) by expressing the real part of $q(u, v)$ as : $\mathtt{Re}\ q(u, w) = \frac{1}{4}(Q(u + v) - Q(u - v))$ if $Q(H) \subset \mathbb{R}$. The imaginary part of $q(u, v)$ -as a linear functional in the variable u is equal to $-\mathtt{Re}\ q(iu, w)$. Using these formulae one computes the adjoint of $q(u, v)$ and compares it to $q(v, u)$ by elementary linear algebra. The converse implication in (e) is obvious. Now using both non-negativity and hermitian property, for any real t we get $0 \leq p(t) := Q(u + tw) = Q(u) + 2t\mathtt{Re}\ q(u, w) + t^2 Q(w)$, which is a polynomial of degree 2 in t. It cannot have two distinct roots, so -non-positive must be its discriminant ("Delta"): $0 \geq (2\mathtt{Re}\ q(u, w))^2 - 4Q(u)Q(v)$. Hence $|\mathtt{Re}\ q(u, w)| \leq \sqrt{Q(u)Q(w)}$. For any fixed w the right-hand side is a seminorm of v and the following easy lemma (replace u by $e^{-i\phi}u$, if $q(u, v) = |q(u, v)|e^{i\phi}$) concludes the proof

Lemma 2.2. *If $F : H \to \mathbb{C}$ is a $\mathbb{C}$-linear functional and $\rho : H \to [0, +\infty)$ is a seminorm, then*

$$\big(\forall_{x \in H}\ \mathit{Re}\ F(x) \leq \rho(x)\big) \Leftrightarrow \big(\forall_{x \in H}\ |F(x)| \leq \rho(x)\big).$$

IMPORTANT NOTE: If one considers the real scalar field $\mathbb{R}$, the sesquilinear forms become just the bilinear ones, but the **Polarisation Identity fails in the real case, unless we assume the symmetry**. For symmetric $\mathbb{R}$-valued bilinear q, i.e. satisfying $q(u, v) = q(v, u)\forall_{u,v}$ we obtain $q(f, g) = \frac{1}{4}(Q(f + g) - Q(f - g))$ -just by subtracting side-by-side the equalities expressing $Q(f \pm g)$. Another algebraic result (Jordan – von-Neumann Theorem) says that any norm obeying the Parallelogram Law is defined by some inner product. This holds both for $\mathbb{B} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

The sesquilinear form q_T and the corresponding quadratic form Q_T defined by a linear operator $T : H \to H$ are given by

$$q_T(x, y) = \langle Tx, y \rangle, \quad Q_T(x) = \langle Tx, x \rangle, \qquad x, y \in H. \tag{9}$$

The rotation 2×2 matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generates the (isometric) linear mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates any vector by 90 degrees, hence $Q_A(x) = 0\,\forall x$. It is therefore important to stress that in the complex inner product spaces the quadratic form does determine the operator: If $Q_T(x) = Q_S(x)$ for all $x \in H$, then $T = S$ (even without assuming any symmetry). This is so because the polarisation formula holds in this case ($\mathbb{K} = \mathbb{C}$). Hence from Q_T we recover q_T. The way of getting the vector Tx from the values of $q_T(x,y)$, where y runs through H, comes from the Fourier series theory. If $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis, then

$$Tx = \sum_{n=1}^{\infty} \langle Tx, e_n\rangle e_n = \sum_{n=1}^{\infty} q_T(x, e_n) e_n$$

.

The orthogonal projection of a vector $x \in H$ **onto a convex closed set** M in a Hilbert space H, denoted P_M is the unique point $y \in M$ that minimises the distance from x, meaning that for any $z \in M$ one has $\|x-y\| \le \|x-z\|$. In other words, if $\delta = \mathtt{dist}(x, M) := \inf\{\|x-z\| : z \in M\}$, then $P_M x$ is the only vector y such that $y \in M$ and $\|x-y\| = \delta$. In any inner product space V one finds a sequence of the points $z_n \in M$ with $\delta = \lim \|x - z_n\|$ Using (a) above and the convexity (so that still $\frac{1}{2}(z_n + z_k) \in M$, which implies $\|2x - (z_n + z_k)\|^2 \ge 4\delta^2$) we show the Cauchy's condition for the sequence (z_n). Unless V is complete (i.e. -a Hilbert space), or at least M is complete as a subspace, nothing else can be done. But using the completeness -one obtains the limit of (z_n), say $y = \lim z_n$. Since M is closed, $y \in M$. By the continuity of the norm, $\|x-y\| = \lim \|x - z_n\|$, proving the existence. The uniqueness results from the inequalities used earlier in the proof.

Theorem 2.3. *If $P = P_M$ is the orthoprojection onto a closed linear subspace $M \neq \{0\}$ of a Hilbert space H, then*

(1) *For $x \in H, y \in M$ we have $y = P_M x \Leftrightarrow x - y \perp M$,*
(2) *The mapping $P_M : H \to H$ is bounded and linear, of norm $\|P_M\| = 1$*
(3) *If a linear mapping $P : H \to H, P \neq 0$ is bounded, linear, then there exists a closed linear subspace $M \neq \{0\}$ such that $P = P_M$ if and only if $PP = P$ and P satisfies one of the additional conditions: $\|P\| \le 1$ or $P^* = P$, the latest meaning $\langle Pv, w\rangle = \langle v, Pw\rangle \ \forall_{v,w\in H}$.*
(4) *The orthogonal decomposition holds : Any $x \in H$ can be written uniquely in the form $x = y + r$, where $y \in M, r \perp M$. Here $r \perp M$ means $r \perp z \quad \forall_{z\in M}$.*
(5) *for $P = P_M$ we have $M = \mathcal{N}(I - P) = \mathcal{R}(P)$ and $I - P$ is the orthoprojection onto $M^\perp$ -the orthocomplement of M in H, denoted also $H \ominus M$.*

Note that $\mathcal{N}(I-P)$, the nullspace (=kernel) of the identity minus P is exactly the set of all fixpoints of P, i.e. such points $v \in H$ that $Pv = v$. In the remaining case $M = \{0\}$, we clearly have $P_M = 0$.

There are also "skew projections" -corresponding to a direct sum decomposition. If

$$H = M_1 + M_2,\ M_1 \cap M_2 = \{0\},$$

we say that H is a direct sum of the subspaces M_1, M_2. From linear algebra we know that this corresponds to a unique decomposition: $x = x_1 + x_2$ with $x_j \in M_j, j = 1, 2$. The projection of x onto M_1 in the direction of M_2, denoted $P_{M_1,M_2}x$ is simply the summand x_1. One can prove that unless $M_1 \perp M_2$, we have the norm of this projection > 1. Also this operator's adjoint is different from $P_{M_1,M_2}x$ in this case. If only M_1, M_2 are both closed and H is complete, the continuity of the corresponding projection can be deduced from Banach's Inverse Mapping Theorem applied to the addition mapping: $S : M_1 \times M_2 \ni (u, v) \to u + v \in H$.

EXAMPLES OF ORTHOPROJECTIONS

1. Any diagonal matrix whose diagonal entries are either 0 or 1 is a projection

2. If an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of H is given, for any $k \in \mathbb{N}$ the k-th partial sum of the Fourier series,

$$S_k(x) := \sum_{n=1}^{k} \langle x, e_n\rangle e_n$$

defines the operator $S_k : H \to H$. It is easy to verify the condition from point (1.) of theorem 2.3. Hence S_k are the orthoprojections onto the linear span of $\{e_1, \ldots, e_k\}$ and $\|S_k(x)\|^2 \le \|x\|^2$, which due to the Pythagorean Theorem gives $\sum_{n=1}^{k} |\langle x, e_n\rangle|^2 \le \|x\|^2$ and passing with $k \to \infty$ we get Bessel's Inequality:

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \le \|x\|^2 \tag{10}$$

Last Thursday I have discussed 3 theorems describing arbitrary bounded linear functionals φ on specific Banach spaces (the so called F.Riesz' Representation Theorems): On $L^p(\mu)$ one finds $\rho \in L^q(\mu)$ with $\frac{1}{p} + \frac{1}{q} = 1$ so that

$$\varphi(f) = \int f(\omega)\rho(\omega)d\mu,$$

while on $C(X)$ with X -a compact topological space, there exist Borel, regular, nonnegative measure ν on X and a a Borel-measurable $\rho : X \to \{z \in \mathbb{C} : \ |z| = 1\}$ such that

$$\varphi(f) = \int f(\omega)\rho(\omega)d\nu, \quad \mu(X) = \|\varphi\|.$$

In Hilbert spaces any bounded linear functional comes from a vector $v \in H$ via the inner product:

$$\varphi(x) = \langle x, v\rangle.$$

FOR THE NEXT TUTORIALS please solve the following.

> Given nonzero vectors $u, v \in H$ define the linear operator $L : H \ni x \to \langle x, v\rangle u \in H$, denoted by $u \otimes v$, or by uv^*. Hence $(u \otimes v)(x) := \langle x, v\rangle u$. This "tensor product - style" notation is a bit misleading, so later I will replace it by uv^*- consistent with column-vector u multiplied (as a matrix) with its Hermitian conjugate (a row vector v^* having the complex conjugates of the coefficients of v).

This is a rank-one operator [rank (L) is the dimension of the range space L(H)]. L is the only linear mapping that sends the vector v to $\|v\|^2 u$ and for which the nullspace $\mathcal{N}(L)$ $(= \ker(L))$ equals the orthocomplement of $\{v\}$, i.e to the set $\{v\}^{\perp} := \{y \in H : y \perp v\}$.

(1) Show, that the adjoint to this L is obtained by interchanging the vectors: $(v \otimes u)^* = u \otimes v$

(2) In particular, $v \otimes v$ is self-adjoint. Show that this operator is of the form $\|v\|^2 P$, where P is the orthoprojection onto the 1-dimensional subspace spanned by v

(3) Show that any finite-rank operator $T \in \mathcal{B}(H)$ can be written as a finite sum of the form $\sum_{n=1}^{k} v_n \otimes w_n$ with some vectors $v_j, w_j, j \leq k$. (Hint: take any basis of the range space $\mathcal{R}(T)$.=

(4) Show that $\|u \otimes v\| = \|u\|\|v\|$

(5) Find a condition (in terms of the matrix' entries) for a nonzero 2×2 matrix to be a matrix for some rank-one operator. Generalize to $d \times d$ matrices.

3. More basic facts on Hilbert space operators

DEFINITIONs: (1)(Eigenvectors and eigenvalues) : If $T : H \to H$ is a linear operator, then a scalar $\lambda \in \mathbb{K}$ is called an *eigenvalue* of T, if for some **nonzero** $w \in H$ (called eigenvector of T) we have

$$Tw = \lambda w. \quad (\texttt{equivalently, if} \quad \mathcal{N}(T - \lambda I) \neq \{0\}).$$

The set of all eigenvalues is called the *point spectrum of* T and is denoted by

$$\sigma_p(T).$$

The set of all eigenvectors corresponding to λ, equal to $\mathcal{N}(T - \lambda I)$, is a linear subspace -called the *eigenspace* of T corresponding to the eigenvalue λ. (Here we treat 0 also as an eigenvector.)

The simplest linear operators on the sequence Hilbert space ℓ^2 are the diagonal operators

$$T = \mathrm{diag}(a_n)$$

corresponding to a given sequence (a_n) of complex numbers. Its domain,

$$\mathcal{D}(\mathrm{diag}(a_n)) := \{(x_n) \in \ell^2 : \ (a_n x_n) \in \ell^2\}$$

is the maximal "natural domain", where the following definition makes sense: $T(x_n) = (a_n x_n)$. Recall that the sequence $(a_n x_n)$ belongs to ℓ^2 iff $\sum_n |a_n x_n|^2 < \infty$. The domain is the entire ℓ^2 and the operator is bounded iff the sequence (a_n) is bounded, i.e. it belongs to ℓ^∞ and then the operator norm, $\|T\|$ is equal to the ℓ^∞ norm of (a_n), which is $\|(a_n)\|_\infty = \sup\{\, |a_n| \, : \, n \in \mathbb{N}\}$. We have checked it substituting the basic 0-1 vectors ε_j forming an orthonormal basis in ℓ^2. A converse result holds true:

Theorem 3.1. *If some orthonormal basis (f_n) of a Hilbert space H consists of eigenvectors of a bounded linear operator T, so that $Tf_n = a_n f_n$ for some scalars a_n, then the infinite matrix representing T in this basis (whose entries are equal to $\langle Tf_j, f_k\rangle$) is diagonal, with the main diagonal's n-th entry equal a_n.*

Which operators can be diagonalised in the above manner? -is a nontrivial question. A partial answer will be given by the spectral theorem. The existence of such an orthonormal basis of eigenvectors for T will be proved for compact normal operators defined below.

Definition 3.2. *A linear operator $T : H_1 \to H_2$ is called*

- an isometry, *if* $\|Tx\| = \|x\| \forall x \in H_1$
- a partial isometry, *if* $\|Tx\| = \|x\| \quad \forall x \perp \mathcal{N}(T)$
- a unitary operator, *if T is a surjective isometry*
- a normal operator, *if* $\mathcal{D}(T^*) = \mathcal{D}(T)$ *and* $\|T^*x\| = \|Tx\| \; \forall x \in \mathcal{D}(T)$

- a selfadjoint operator, *if* $\mathcal{D}(T^*) = \mathcal{D}(T)$ *and* $T^*x = Tx \; \forall x \in \mathcal{D}(T)$

Finally, an operator $T : H_1 \to H_2$ *is* compact, *if the image of any bounded sequence of vectors* $x_n \in H_1$ *contains a convergent subsequence. The set of all compact linear operators from* H_1 *to* H_2 *will be denoted* $\mathcal{B}_0(H_1, H_2)$ *and by* $\mathcal{B}_0(H_1)$, *if* $H_1 = H_2$. *In other words, if* $\sup_n \|x_n\| < \infty$, *then for some integers* $1 \le n_1 < n_2 < \dots$ *there should exist a limit* $\lim_{k\to\infty} Tx_{n_k}$ *in the norm topology of* H_2.

We have the following equivalent formulations for the above properties:

Theorem 3.3. *If* $\mathcal{D}(T) = H_1$ *for a bounded linear operator* $T : H_1 \to H_2$, *then*

(a) T *is an isometry iff* $T^*Tx = x \; \forall x \in H_1$, *i.e. iff* $T^*T = I_{H_1}$
(b) T *is unitary iff (* $T^*T = I_{H_1}$ *and* $TT^* = I_{H_2}$ *), i.e. iff* $T^* = T^{-1}$.
(c) T *is normal iff* $T^*T = TT^*$
(d) T *is selfadjoint, iff* $\mathcal{D}(T^*) = \mathcal{D}(T)$ *and the form* $Q_T(x) = \langle Tx, x\rangle$ *is real-valued.*

Recall, that by c_0 we denote the space of all scalar sequences converging to zero, called sometimes "null sequences".

Theorem 3.4. *Let* $T = \operatorname{diag}(a_n) : \ell^2 \to \ell^2$ *be a diagonal operator corresponding to a sequence* $(a_n) \in \ell^\infty$. *Then* T *is normal and its adjoint is also a diagonal operator,* $T^* = \operatorname{diag}(\bar{a}_n)$, *defined by the complex conjugates of its diagonal sequence. Moreover,* T *is an isometry iff* $|a_n| = 1 \, \forall n \in \mathbb{N}$. T *is compact iff* $(a_n) \in c_0$. *The composition* TS *of two diagonal operators is a diagonal operator with diagonal* $a_n b_n$, *if* $T = \operatorname{diag}(a_n), S = \operatorname{diag}(b_n)$. *In particular,* $TS = ST$.

3.1. **Compactness.** The compactness of a subset $E \subset H$ in a normed space is equivalent to sequential compactness (the existence of convergent subsequences y_{n_k} for any sequence y_n in E, so that $\lim_k y_{n_k} \in E$). If we only assume the existence of $\lim_k y_{n_k} \in H$, the condition means the compactness of the closure of E, called the relative compactness of E. If $\bar{B}$ denotes the closed unit ball in H_1, then the **compactness of a linear operator** $T : H_1 \to H_2$ **means the relative compactness of the image** $T(\bar{B})$ **of the closed unit ball.** As in the case of the equivalent conditions for continuity of T, the compactness of T is equivalent to the relative compactness of the image of any ball of positive radius. Since relatively compact subsets are bounded, any compact operator must be bounded. The definition, as stated, applies to Banach spaces as well. In the case of Hilbert spaces, or reflexive Banach spaces, the image $T(\bar{B})$ of any closed ball is also closed. This is a consequence of the compactness of $\bar{B}$ in the weak topology (Banach -Alaoglu's Theorem) and the continuity of bounded linear maps also with respect to weak topologies. We are not going to use this fact, however in this course. More important is the following remark:

The unit closed ball $\bar{B}$ **in** H **cannot be compact unless** $\dim(H) < \infty$**.**

Another important characterisation of (relative) compactness, similar to the classical Bolzano-Weierstrass theorem: ("Bounded subsets of the Euclidean space $\mathbb{R}^n$ are relatively compact") is due to Hausdorff. We know, that in the infinite dimensional case boundedness will not suffice (why?). We need the stronger property than just boundedness:

Definition 3.5. *A subset* E *in a metric space* (X, d) *is* ***totally bounded*** *if it has finite coverings by finite families of sets with arbitrarily small diameter. In other words,*

$$\forall_{\epsilon>0} \exists_{k\in\mathbb{N}} \exists_{x_1,\dots,x_k\in E} \forall_{x\in E} \exists_{j\le k} \; d(x, x_j) < \epsilon.$$

The set $\{x_1, \dots x_n\}$ is then called an ϵ-network for E. The balls with radii ϵ, centered at x_j cover E. (Their diameters satisfy, of course, $\operatorname{diam} B(x_j, \epsilon) \le 2\epsilon$). If we just require that the x_j are points from the space X, we get an equivalent definition (why?). Check that *the closure of a totally bounded set is also totally bounded.*

We may call a sequence $(z_n) \subset X$ a *uniformly separated sequence*, if for some $\delta > 0$ and for any $k, j \in \mathbb{N}$ one has $d(z_j, z_k) \ge \delta$. Clearly, any of its subsequences is also uniformly separated and it cannot satisfy Cauchy's condition. Therefore to prove that a set Z is not relatively compact it suffices to find a uniformly separated sequence of its points. Check that orthonormal sequences are uniformly separated and that completely bounded sets are bounded. Now the positive result:

Theorem 3.6. *(*Hausdorff*) A set* E *in a complete metric space is relatively compact if and only if it is totally bounded. In particular,* E *is compact iff* E *is closed and totally bounded.*

Using this characterisation it is relatively easy to describe the set $B_0(H_1, H_2)$ of compact operators. Recall that a *sequence* (x_n) *converges weakly* to x_0 in a Banach space X, if for any continuous linear functional $\phi \in X'$ the scalar sequence $\phi(x_n)$ converges to $\phi(x_0)$. In Hilbert spaces this is equivalent to $\lim\langle x_n - x_0, z\rangle = 0 \forall_{z\in X}$.

Theorem 3.7. *Let* $T \in \mathcal{B}(H_1, H_2)$ *be a bounded linear operator in a Hilbert space. Then TFAE:*

(1) *T is compact (in symbols, $T \in \mathcal{B}_0(H_1, H_2)$).*
(2) *If $(x_n) \subset H_1$ converges weakly to zero, then $\|Tx_n\| \to 0$.*
(3) *There exists a sequence of finite rank operators $T_n \in \mathcal{B}(H_1, H_2)$ such that $\|T_n - T\| \to 0$.*

The set $\mathcal{B}_0(H)$ of compact linear operators $S : H \to H$ is a two-sided closed ideal in $\mathcal{B}(H)$. In other words, the limit of norm-convergent sequence (S_n) of compact operators is compact and so is any linear combination of two compact operators. If S is compact and T is bounded, then both ST and TS are compact. (We prove this theorem and the above remark on tutorials.)

4. Compact operators

4.1. **Weak convergence.** The weak topology is a bit exotic, since it is non-metrizable in any $\infty-$ dimensional Banach space. Denote the weak convergence by $\rightharpoonup$. Recall, that a sequence of vectors x_j in a Banach space X *converges weakly to* $x_0 \in X$, a fact denoted by $x_j \rightharpoonup x_0$, if for any fixed continuous linear functional $\varphi \in X'$ the sequence of (real or complex) numbers $\varphi(x_j)$ converges to $\varphi(x_0)$. In Hilbert spaces we have $\varphi(x)$ uniquely represented as $\langle x, z\rangle$ for some $z \in H$, so the defining condition becomes simply

$$x_j \rightharpoonup x_0 \Leftrightarrow \forall_{z\in H} \lim_j \langle x_j, z\rangle = \langle x, z\rangle.$$

In any case this convergence comes from *the weak topology* defined as the weakest topology making all $\varphi \in X'$ continuous. This is the topology defined by a family of seminorms $\{p_\phi(\cdot) : \phi \in X'\}$, where $p_\phi(x) = |\phi(x)|$ for $x \in X$. Basis of weak neighbourhoods of zero is formed by the family of sets

$$W(\phi_1, \dots \phi_k) = \{x \in X : |\phi_1(x)| < 1, \dots |\phi_k(x)| < 1\}, \quad \textit{where } k \in \mathbb{N}, \phi_1, \dots \phi_k \in X'.$$

Hence as an indexing set we may take the family of all finite subsets of X'. Note that for families of seminorms p_j one should consider finite intersections of balls $\{x \in X : p_j(x) < r\}$ of radii $r > 0$. This is because for one neighbourhood of zero the minimum (taken over $j \in \{1, \dots, k\}$) of a finite set of radii r_j can be put instead of the radii $r_1, \dots, r_k$. In the case of the weak topology it simplifies further -we can take $r = 1$ throughout, since ϕ can be replaced by the functional $r^{-1}\phi$, since $|\frac{1}{r}\phi(x)| < 1 \Leftrightarrow |\phi(x)| < r$.

In any Hilbert space with orthonormal sequence (e_n) it follows from Bessel's inequality that $e_n \rightharpoonup 0$ as $n \to \infty$. The famous example due to von Neumann shows this: If $E = \{e_n + ne_m,\ n, m \in \mathbb{N}\}$, then the iterated weak limit is zero:

$$\lim_n(\lim_m e_n + ne_m) = \lim_n(e_n + n0) = 0,$$

hence zero belongs to the weak closure of E. But -unlike in the metric spaces - we have the following fact: there is **no sequence of vectors** $x_k \in E$ **weakly convergent to zero**. Indeed, by the Uniform Boundedness Principle (or directly from the Banach-Steinhaus Theorem), as a weakly convergent sequence, such a sequence should be bounded. As $x_k = e_{n_k} + n_k e_{m_k}$ for some $n_k, m_k \in \mathbb{N}$ and $\|x_k\|^2 = 1 + n_k^2$, the sequence n_k should be bounded, hence it should contain a constant subsequence n_{k_j} -equal to some n_0. But then either m_k is eventually constant (say, equal m_0 for sufficiently large k, leading to $x_{k_j} \rightharpoonup e_{n_0} + n_0 e_{m_0}$), or we should have $m_{k_j} \to \infty$, yielding $x_{k_j} \rightharpoonup e_{n_0} + 0$, a contradiction.

For general normed spaces X – the non-metrizability follows from the fact that any countable intersection of a sequence of weak neighbourhoods of zero "is" (to be precise: *contains the subset*) of the form $\bigcap_{n=1}^\infty W(\phi_n)$ and contains more than one point (show that codim $\bigcap_{n=1}^\infty \ker(\phi_n) \le \aleph_0$ – exercise). In metric spaces the balls centered at 0, of radii $\frac{1}{n}$ have one-point intersection.

Any ball in X with respect to the norm (and any bounded set) **has empty interior** in the weak topology.

Despite of these drawbacks, the weak convergence (of ordinary sequences) is in a sense quite natural:

In the sequence spaces $\ell^p, 1 < p < \infty$ a sequence $\mathbf{x}_n \in \ell^p$ converges weakly to $\mathbf{x}_0$ iff it is bounded ($\sup_n \|\mathbf{x}_n\|_p < \infty$) and each coordinate converges: $(\mathbf{x}_n)_k = e_k^*(\mathbf{x}_n) \to (\mathbf{x}_0)_k$ as $n \to \infty (\forall_k \in \mathbb{N})$. This is a particular case of the following general principle (with $D = \{e_k^* : k \in \mathbb{N}\}$):

Theorem 4.1. *If the linear span of a subset $D \subset X'$ is dense in X' (with its norm), then a sequence (x_n) converges weakly to x_0 iff it is bounded and $\phi(x_n) \to \phi(x_0)$ for any $\phi \in D$. If moreover such a "generating set" $D = \{\phi_j\}_{j\in\mathbb{N}}$ is countable, then the metric ρ defined by the formula $\rho(x, y) := \sum_{n=1}^\infty 2^{-j}|\phi_j(x) - \phi_j(y)|$ defines the weak topology on bounded subsets of X.*

A surprising result at $p = 1$ is due to I. Schur: *A sequence in* ℓ^1 *converges weakly if and only if it norm- converges to the same limit.* (Note that the two topologies are different!)

In Hilbert spaces any bounded sequence has weakly convergent subsequence and any closed ball is weakly compact. In separable Hilbert spaces the weak topology, when restricted to bounded subsets is metrizable! (but not on the whole space!) The **weak compactness of the closed unit ball** (and of any closed ball) takes place iff the space is **reflexive** (i.e. the canonical embedding in its second dual space

$$j : X \to X'',\ (j(\phi))(x) := \phi(x), \phi \in X', x \in X$$

is surjective -a fact denoted often as $X = X''$.

For convex sets their weak closures are equal to their norm-closures, due to Hahn-Banach theorem (precisely, from its corollary on separation of points from closed convex sets).

4.2. **Convergence of sequences of operators.** We say that a sequence (or a generalized sequence, i.e. a net) of operators $T_j \in \mathcal{B}(X,Y)$ acting between two Banach spaces converges to an operator $T \in \mathcal{B}(X,Y)$

- uniformly, or in norm (notation $T_j \to T$), if the operator norms $\|T_j - T\|$ converges to zero
- strongly (notation $T_j \to T(SOT)$), if for any $x \in X$ we have $\|T_jx - Tx\| \to 0$
- weakly (notation $T_j \to T$ (WOT)) if for any $x \in X$ we have $T_jx - Tx \rightharpoonup 0$

Clearly, the uniform convergence implies the strong convergence. The strong convergence implies the weak convergence. None of these implications is reversible.

The remaining part of this page can be omitted at the first reading, we are going to concentrate on ordinary (not generalised) sequences. The (SOT) convergence corresponds to the *strong operator topology* given by the family of seminorms $\{p_x : x \in X\}$, where $p_x(T) := \|Tx\|$. Similarly, the weak operator topology is defined by $p_{x\varphi}(T) := |\varphi(Tx)|$, a family of seminorms indexed by $(x, \varphi) \in X \times Y'$. Let us concentrate for a while on the latter two convergences in the Hilbert spaces case: Neither of these two convergences is metrizable (but their restrictions to bounded sets of operators are). Clearly, in the Hilbert space case for $T_j, T \in \mathcal{B}(H)$

- the weak convergence: $T_j \to T$ (WOT) means that $\langle T_jx, y\rangle \to \langle Tx, y\rangle\ \forall_{x,y\in H}$.

It turns out, that **the multiplication in $\mathcal{B}(H)$ is discontinuous -but is sequentially continuous!** Here the multiplication TS is just the composition: $(TS)(x) = T(Sx)$. The relevant example can be found in a separate file entitled "example1.pdf". The difference between nets and sequences is that for sequences the index j runs through the set $\mathbb{N}$ of natural numbers and only finitely many terms of a convergent sequence can stay outside a given neighbourhood of the limit. By Uniform Boundedness Principle, WOT- convergent sequences are bounded. The same cannot be asserted for convergent nets, where we have the index set J possibly uncountable, directed by some transitive relation $j \preceq i$ such that $\forall_{j,k\in J}\exists_{i\in J} j \preceq i$ and $k \preceq i$. Here the convergence of T_j to T means that for any neighbourhood W of T there exists $j_0 \in J$ such that $j \in J, j_0 \preceq j \Rightarrow T_j \in W$. Any accumulation point (or a point from the closure $\bar{E}$) of a set E is always a limit of some generalized sequence of elements of E. The example from the previous subsection on weak convergence with $E = \{e_n + ne_m,\ n,m \in \mathbb{N}\}$ shows, that the ordinary sequences can be inadequate for the latter purpose -even for countable sets E.

EXERCISE 1. Let us direct the set of integers $\mathbb{Z}$ by the ordinary relation $j \le k$. Is the net $(2^{-j})_{j\in\mathbb{Z}}$ bounded? Does it converge?

EXERCISE 2. If we direct $\mathbb{N} \times \mathbb{N}$ by $(j,k) \preceq (n,m)$ meaning that $\max(j,k) \le \min(n,m)$, is the convergence $\|x_n - x_m\|$ corresponding to this direction of $\mathbb{N}\times\mathbb{N}$ equivalent to the Cauchy condition (in a normed space X)?

EXERCISE 3. The diameter $\delta(\mathcal{T})$ of a partition $\mathcal{T} = (t_0 = a < t_1 < \cdots < t_n = b$ of an interval $[a,b]$ is defined as $\delta(\mathcal{T}) := \max\{t_j - t_{j-1} : j = 1, \dots n\}$. In the set of pairs $(\mathcal{T}, \Lambda)$, where $\Lambda = \{\lambda_1, \dots \lambda_k\}$ is a collection of intermediate points for $\mathcal{T}$, so that $\lambda_j \in [t_{j-1}, t_j]$ define a direction $(\mathcal{T}_1, \Lambda_1) \preceq (\mathcal{T}_2, \Lambda_2)$, if $\delta(\mathcal{T}_1) \le \delta(\mathcal{T}_2)$.

Now let $f : [a,b] \to \mathbb{R}$ be a given function. Define

$$S(f, \mathcal{T}, \Lambda) = \sum_{j=1}^{n} f(\lambda_j)(t_j - t_{j-1}).$$

If we treat $S(f, \mathcal{T}, \Lambda) \in \mathbb{R}$ as a generalised sequence indexed by such pairs $(\mathcal{T}, \Lambda)$, what is the meaning for its convergence to some limit $S \in \mathbb{R}$? (answer using a notion from basic calculus)

EXERCISE 4. If we have a neighbourhood basis $(W_j)_{j\in J}$ of a point x_0 in some topological space, so that for any neighbourhood U of x_0 there exists $j \in J$ such that $W_j \subset U$, let us pick arbitrary points $x_j \in W_j$ and direct J by $j \preceq k$ meaning $W_j \supset W_k$ (the reverse inclusion!). Show that the net x_j converges to x_0. (Using this one can show that the continuity at a point x_0 of a mapping F between two topological spaces takes place iff for any generalised sequence x_j converging to x_0 the values $F(x_j)$ converge to x_0.)

EXERCISE 5. Let Ω be the set of all continuous functions x on $[0,1]$, taking values $x(t) \in [0,1]$. Consider the topology of pointwise convergence defined on $C[0,1]$ by the seminorms $p_t(x) := |x(t)|$, where $t \in [0,1], x \in C[0,1]$. Show that the functional $F : \Omega \ni x \to F(x) : \int_0^1 x(t)\,dt$ **is sequentially continuous**, i.e. it satisfies the Heine condition: for all ordinary sequences $x_n \in \Omega$ convergent pointwise to x_0 one has $F(x_n) \to F(x_0)$. On the other hand, any neighbourgood of zero in Ω contains a basic neighbourghood of the form $W_{t_1,\dots,t_k} := \{x \in \Omega : |x(t_1)| < \epsilon, \dots |x(t_k)| < \epsilon\}$ corresponding to some $\epsilon > 0$ and a finite set of points $t_1, \dots t_k \in [0,1]$. We can pick a function $z \in W_{t_1,\dots,t_k}$ (e.g. vanishing at these finite set of points -then any $\epsilon > 0$ will be "good") such that $\int_0^1 z(t)\,dt > \frac{1}{2}$ (draw a picture of a piecewise-linear function having these properties). This shows **the discontinuity** of F on Ω.

WE HAVE PROVED THIS: If a sequence of operators $P_n \to P$ (SOT) and K is a compact operator, then $P_n K \to PK$ uniformly. If a Schauder basis exists in X, then $\exists P_n$ of finite rank, (SOT)-convergent to I (identity). Compact operators are then uniform limits of finite rank operators. Continuous finite rank operators and uniform limits of compact operators are compact.

Lemma 4.2. *Let $T : X \to Y$ be a linear operator between normed spaces X, Y. The quotient space $X/\ker(T)$ has the same dimension as the range space $T(X)$ and if the latter is finite ($rk(T) < \infty$), then T is continuous* iff $\ker(T)$ *is closed and if the latter takes place, T is a compact operator.*

To prove this, one uses the so called "canonical factorisation": $T = \tilde{T} \circ \pi$, where $\pi : X \ni X \to [x] := \{z \in X : x - z \in \ker T\} \in X/ker(T)$ is the canonical surjection onto the quotient space (with the (semi-)norm $\|[x]\| := \inf\{\|z\| : z \in [x]\}$) and $\tilde{T} : X/\ker(T) \to Y$ is defined by $\tilde{T}([x]) := Tx$ (which does not depend on the choice of the representative x of the equivalence class $[x]$). Now $\tilde{T}$ is a linear isomorphism onto the range space $T(X)$, preserving the dimensions. Continuity of π is obvious. Any linear mapping on a normed space of finite dimension is continuous. Hence $\tilde{T}$ will be continuous, if only $\|[x]\|$ is a norm, not just a seminorm. But $\mathrm{dist}(x, \ker T) = 0$ if and only if x belongs to the closure of $\ker(T)$, while $[x] = 0 \Leftrightarrow x \in \ker(T)$. These two conditions are equivalent iff $\ker T$ is closed. In the latter case any bounded sequence (x_n) in X is mapped by π into a bounded sequence $[x_n]$. All finitely dimensional normed spaces are isomorphic to the Euclidean space $\mathbb{K}^n$, where each bounded sequence has a convergent subsequence. Hence $[x_{n_k}]$ is convergent for some subsequence (n_k) of $\mathbb{N}$. By continuity of $\tilde{T}$, the subsequence $\tilde{T}[x_{n_k}]$ converges in Y. But as $\tilde{T}[x_{n_k}] = Tx_{n_k}$, the compactness of T follows. In the opposite direction, compactness implies continuity of T, which in turn implies that $\ker T$ is closed.

4.3. **Hilbert-Schmidt operators.** Hilbert-Schmidt operators (shortly, H-S operators) are defined by the condition

$$|||T|||^2 := \sum_n \|Te_n\|^2 < \infty,$$

where $T : H_1 \to H_2$ is linear and (e_n) is any orthonormal basis in H_1. Using Parseval Identity for some other orthonormal basis (f_k) in H_2 we represent $|||T|||^2$ as the double sum

$$|||T|||^2 = \sum_n \sum_k |\langle Te_n, f_k\rangle|^2 = \sum_n \sum_k |\langle T^* f_k, e_n\rangle|^2, \tag{11}$$

which is $\sum_k \|T^* f_k\|^2$ after interchanging the order of summation and applying the Parseval identity for $T^* f_k$ in the basis (e_n). At this stage any different ONB $(\tilde{e}_n)$ in H_1 can be applied.

Theorem 4.3. *H-S operators are bounded with $\|T\| \le |||T|||$. Moreover, H-S operators are compact*

The reasoning can be this:

(1) For arbitrarily chosen $x \in H$ we can construct an orthonormal basis (e_n) in H with $e_1 = x$. Then the Hilbert-Schmidt norm $|||T||| = (\sum_n \|Te_n\|^2)^{\frac{1}{2}}$ is $\ge \|Te_1\|$ Passing to the supremum over all such x we get $\|T\| \le |||T|||$.

(2) If P_n is the orthoprojection onto the linear span of the first n vectors of an orthonormal basis $e_1, \dots, e_n$, then TP_n has finite rank (at most n). The squared H-S -norm of $T - T_n$ equals

$$|||T - TP_n|||^2 = \sum_{j=n+1}^{\infty} \|Te_j\|^2$$

-converging to zero as $n \to \infty$. By (11), we also have $\|T - TP_n\| \to \infty$ and since $rk(TP_n) < \infty$, the compactness follows from lemma 4.2..

4.4. **integral operators of H-S type.** The integral operator T_K corresponding to a kernel function $K : X \times Y \to \mathbb{C}$ is defined for two (complete) measure spaces $(X, \mu), (Y, \nu)$, where the respective σ-algebras of measurable sets are hidden to simplify the notation. Even further simplification is obtained if one writes $\int f(x)\,dx$ in place of $\int f(x) d\mu(x)$ and similarly one replaces $d\nu(y)$ with dy. The inner product on $L^2(\mu)$ is $\langle f, g\rangle = \int f(x)\overline{g(x)}\,dx$. Here (and in what follows) the bar is used to denote complex conjugation. The product measure $\mu \times \nu$ is defined on the smallest σ-algebra on $X \times Y$ containing all the products of the form $E \times F$ with E -a measurable subset of X, F- measurable subset of Y. We put $\mu\times\nu(E\times F):=\mu(E)\nu(F)$ and extend this to a complete measure space (e.g. by Caratheodory's construction). Using Fubuni -Tonelli's theorem we can establish the following basic fact:

Lemma 4.4. *If $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mu)$ and $(f_k)_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(\nu)$, then*
(1) also $(\bar{f}_k)_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(\nu)$ and*
(2) the functions $e_n\times\bar{f}_k : X\times Y\ni(x,y)\to e_n(x)\overline{f_k(y)}$ with (n,k) running through the product $\mathbb{N}\times\mathbb{N}$ generate an othonormal basis in $L^2(\mu\times\nu)$.*

Definition For $f\in L^2(\nu)$ and for a measurable function $K:X\times Y\to\mathbb{C}$ define

$$(T_Kf)(x)=\int K(x,y)f(y)d\nu(y)$$

whenever the latter integral makes sense and then it is the inner product $\langle K(x,\cdot),\overline{f}\rangle$.

If $K\in L^2(\mu\times\nu)$, one shows (a direct consequence of Tonelli's theorem) that
(3*) for almost all x we have $K(x,\cdot)\in L^2(\nu)$ and $\int\|K(x,\cdot)\|^2\,dx=\|K\|^2<\infty$.

Theorem 4.5. *For $K\in L^2(\mu\times\nu)$ the operator $T_K : L^2(\nu)\to L^2(\mu)$ is H-S and $|||T_K|||=\|K\|_{L^2(\mu\times\nu)}$. Conversely, any H-S operator $T:L^2(\nu)\to L^2(\mu)$ is of the form $T=T_K$ as above.*

Indeed, with $Tf_k=\langle K(x,\cdot),\overline{f_k}\rangle$ and $\|Tf_k\|^2=\int|\langle K(x,\cdot),\overline{f_k}\rangle|^2dx$ we get

$$\sum_k\|Tf_k\|^2=\sum_k\int|\langle K(x,\cdot),\overline{f_k}\rangle|^2dx=\int\sum_k|\langle K(x,\cdot),\overline{f_k}\rangle|^2dx.$$

We can interchange the summation and integration, because of the nonnegativity. Now by (1*) and from the Parseval Identity, $\sum_k|\langle K(x,\cdot),\overline{f_k}\rangle|^2=\|K(x,\cdot)\|^2$, where the norm of $K(x,\cdot):Y\ni y\to K(x,y)$ is computed in $L^2(\nu)$. From (*) we get the needed equality $|||T_K|||^2=\|K\|^2$.

Conversely, if $|||T|||<\infty$, this implies by (11) the square-summability of the double-indexed sequence $c_{(n,k)}:=\langle Tf_k,e_n\rangle$ and this impies that the Fourier series with such coefficients converges in $L^2(\mu\times\nu)$ to some $K\in L^2(\mu\times\nu)$, namely, $K(x,y):=\sum_{(n,k)}c_{(n,k)}e_n(x)\overline{f_k(y)}$. By comparing the coefficients in the basis (e_n), we get $Tf_k=T_Kf_k$. The equality extends to finite linear combinations, by linearity of both operators. By continuity, the equality holds also on the closure of the linear span of the basic vectors (f_k) -which is the entire space $L^2(\nu)$. In particular, if both X,Y have finite measures, than any bounded measurable $K:X\times Y\to\mathbb{C}$ yields a compact (even H-S) integral operator. For Lebesgue measure compactness of $X\subset\mathbb{R}^d, Y\subset\mathbb{R}^p$ and continuity of K will be sufficient.

5. Spectral theory of compact selfadjoint operators

5.1. **Spectrum and its parts.** Recall that *the spectrum* of an operator $T\in\mathcal{B}(H)$ is the set $\sigma(T)$ of all $\lambda\in\mathbb{C}$ such that the operator $\lambda I-T$ (denoted for this paragraph by T_λ) has a bounded inverse. Its complement, $\mathbb{C}\setminus\sigma(T)$ is called *the resolvent set* for T.

One of the possible reasons for the non-invertibility is that $\ker(\lambda I-T)\neq\{0\}$ so that T_λ is non-injective. Any nonzero vector $x\in\ker T_\lambda$ is called an *eigenvector* and the corresponding λ -an *eigenvalue* of T. The set of all eigenvalues is called the *point spectrum* of T, denoted $\sigma_p(T)$.

For $\lambda\in\mathbb{C}\setminus\sigma_p(T)$ the injective operator T_λ has an inverse $R_\lambda:=(T_\lambda)^{-1}$ but defined only on the range space $\mathcal{R}(T):=T(H)$. If the latter is closed (actually, if and only if) then R_λ is bounded. If $\lambda\notin\sigma_p(T)$ then either $\mathcal{R}(T)=H$, so that T_λ is bijective and then its bounded (by Banach's Inverse Mapping Theorem) inverse R_λ is called the resolvent operator for T. The name can be explained by the fact that $x:=R_\lambda y$ provides the unique solution x of the inhomogeneous equation $Tx-\lambda x=y$.

Another reason for non-invertibility is the possible unboundedness of R_λ (or non-injectivity). This is equivalent to the lower bound of $\|T_\lambda x\|$ over the unit sphere $\{x\in H:\|x\|=1\}$ being equal to zero and defines the approximate point spectrum, $\sigma_a(T)$. More precisely -we say that $\lambda\in\mathbb{C}$ is in the *approximate point spectrum* of T, denoted $\sigma_a(T)$, if there exists a sequence (x_n) in H such that $\|x_n\|=1$ and $\|Tx_n-\lambda x_n\|\to 0$. (Check, that for any such a sequence if (x_n) contains a convergent subsequence than its limit is an eigenvector of T with eigenvalue λ, and that $\sigma_p(T)\subset\sigma_a(T)$.)

We have checked that if $T\in\mathcal{B}(H)$ is compact and $\lambda\in\sigma_a(T)\setminus\{0\}$, then $\lambda\in\sigma_p(T)$. For normal (e.g. for selfadjoint) operators $N\in\mathcal{B}(H)$ we have proved that the spectrum satisfies $\sigma(N)=\sigma_a(T)$. Also the eigenvectors corresponding to different eigenvalues are orthogonal. In other words,

$$\text{if } \mathrm{NN}^*=\mathrm{N}^*\mathrm{N}, \text{ then } \ker(\mathrm{N}_\lambda)=\ker((\mathrm{N}^*)_{\bar\lambda}) \text{ and } \forall_{\lambda\neq\mu}\ker(\mathrm{N}_\lambda)\perp\ker(\mathrm{N}_\mu). \tag{12}$$

For compact operators T nonzero eigenvalues have finite multiplicity (*multiplicity of λ, denoted $n(\lambda)$:=the dimension of* $\ker T_\lambda$) and zero is their only possible accumulation point. As a finite rank operator, the orthogonal projection P_λ onto $\ker(T_\lambda)$ for $\lambda\in\sigma_p(T)\setminus\{0\}$ is of the form

$$H\ni x\to P_\lambda x=\sum_{j=1}^{n(\lambda)}\langle x,e_j(\lambda)\rangle e_j(\lambda),$$

where $\{e_1(\lambda), \dots, e_{n(\lambda)}(\lambda)\}$ is any orthonormal basis of $\ker(T_\lambda)$. To get the eigenvalue decomposition for T on the whole space H we need to control the operator norm by means of the spectrum. The easiest case of selfadjoint operators will be addressed first in the next paragraph, using the numerical range of T.

(Some problems for tutorials)

(1) Suppose that a sequence (x_n) in a Hilbert space H converges weakly to x_0. (A) Then show that :

$$\|x_0\| \leq \liminf \|x_n\|. \tag{13}$$

(B) Moreover, some sequence (z_k) of convex combinations of the x_n's norm- converges to x_0.(i.e. $z_n = \alpha_1 x_{j_1} + \dots + \alpha_k x_{j_k}$, where $\alpha_j > 0, \alpha_1 + \dots + \alpha_n = 1$. (Hint: the convex hull of the set $\{x_n : n \in \mathbb{N}\}$ has equal weak- and norm- closures, by Separation Theorem in functional analysis).

(2) Show that if $x_n \rightharpoonup x_0$ and $\|x_n\| \to \|x_0\|$, then $\|x_n - x_0\|^2 \to 0$. (just compute)

(3) Show that for $T \in \mathcal{B}(H, K)$ where H, K are Hilbert spaces, T is compact iff for any weakly convergent to zero sequence (x_n) in H we have $\|Tx_n\| \to 0$. (Hints: We may assume that H is separable and then use the weak-compactness of its closed unit ball).

(4) Using the results of two previous problems show that the compactness of T^*T implies that of T. Deduce that the adjoint of compact operator in a Hilbert space is compact

(5) Let M be the bound of the quadratic form $Q_T(x)$ corresponding to the sesquilinear form $q_T(x, y)$ defined by a selfadjoint operator $T \in \mathcal{B}(H)$, i.e.

$$q_T(x, y) := \langle Tx, y \rangle, \quad Q_T(x) := q_T(x, x), \quad M := \sup\{|Q_T(x)| : x \in H, \|x\| = 1\}.$$

Show that $|Q_T(x)| \leq M\|x\|^2$ for any $x \in H$. Using the polarisation formula valid if $q(y, x) = \overline{q(x, y)}$ obtain analogous bounds for $|q_T(x, y)|$ and finally deduce that $M = \|T\|$.

(6) **The numerical range** of $T \in \mathcal{B}(H)$ is defined as the set

$$W(T) := \{\langle Tx, x \rangle : \|x\| = 1\}.$$

Let $a = \inf W(T)$, $b = \sup W(T)$. If $T = T^*$, then clearly $W(T) \subset \mathbb{R}$ and by the previous result, $M = \|T\|$. But $M = \max(|A|, |B|)$. The selfadjoint operator S is said to be nonnegative, if $Q_S(x) \geq 0$ for any $x \in H$. For nonnegative operators we have the Cauchy-Schwarz inequality

$$|q_S(x, y)|^2 \leq Q_S(x) Q_S(y).$$

Show that the operators $T - aI$ and $S := bI - T$ are nonnegative. If for some unit vectors $\|x_n\| = 1, x_n \in H$ we have $Q_T(x_n) \to b$, so that $Q_S(x_n) \to 0$ deduce that $\|Sx_n\| \to 0$, hence $b \in \sigma_a(T)$. Similarly, $a \in \sigma_a(T)$. (hint: $\|Sx\| = \sup\{|q_S(x, y)| : \|y\| = 1\}$.)

(7) If $S = S^*$ is non-negative, for any $x \in H$ show that $\|Sx\|^2 \leq \|S\| Q_S(x)$.

In this manner we have shown that the spectrum of T - a selfadjoint operator (= here the approximate point spectrum) contains a point (here either a, or b) of modulus equal to $\|T\|$. The same can be proved for normal operators. For compact normal operators if $\|T\| > 0$ we already know that this point actually is an eigenvalue!

8. If $S \in \mathcal{B}(H)$ is selfadjoint and $M \subset H$ is its *invariant subspace*, which means $S(M) \subset M$, then show that $S|_M : M \to M$ is alaso a selfadjoint operator on M. Moreover, $M^\perp = H \ominus M$ is also invariant for M.
9. Check that if M is an invariant subspace for a bounded linear operator $T : H \to H$, then for $T|_M \in \mathcal{B}(M)$ its adjoint (as an operator in $\mathcal{B}(M)$) satisfies $(T|_M)^* = P_M T^*|_M$. Deduce that if T is normal, then its restriction to an invariant subspace M is normal if (and only if) M is also invariant for T^*. We call such a subspace *a reducing subspace for* T. It is equivalent to the invariance of both M and $M^\perp$ for T.
10. Let M be a closed linear span of some eigenspaces $\ker(T - \lambda_j I)$ (taken over $j \in J$ for some index set J). Show that if T is a normal operator then M is a reducing subspace. (We shall use this only for finite sets J).

We already know that compact normal operators have at most countable spectrum, with 0 as the only possible accumulation point.

If $\delta > 0$ and

$$M = \operatorname{span} \bigcup \{\ker(T - \lambda I) : |\lambda| \geq \delta\},$$

then $\dim M < \infty$, by compactness of T. The restriction $S_\delta := T|_{M^\perp}$ of T to the orthocomplement of M is normal. In the case of selfadjoint T, also S_δ is selfadjoint, compact and $\|S_\delta\|$ is either zero, or is the maximum of the moduli of the eigenvalues of S_δ. But any eigenvalue λ of S_δ satisfies $S_\delta x_0 = \lambda x_0$ for some $x_0 \in H, x_0 \perp M, x_0 \neq 0$ and is also an eigenvalue of T. We cannot have $|\lambda| \geq \delta$, since otherwise we would have $x_0 \in M$, hence $x_0 \in M \cap M^\perp = \{0\}$ –contrary to our setting. Hence $\|S_\delta\| = \max\{|\lambda| : \lambda \in \sigma(T_\delta)\} \leq \delta$. Since $T|_{\ker(T - \lambda I)}$ is of the form λP_λ, where P_λ is the projection onto the eigenspace $\ker(T - \lambda I)$, we have proved the following

SPECTRAL THEOREM FOR COMPACT SELFADJOINT OPERATORS T:

$$T = \sum_{\lambda \in \sigma(T) \setminus \{0\}} \lambda P_\lambda.$$

In our class instead of restricting to a subspace, we have subtracted from T its "part" corresponding to the eigenvalues $|\lambda| \geq \delta$: Let $R_\delta = T - \sum_{|\lambda| \geq \delta} \lambda P_\lambda$. For M as above we have $R_\delta | M = 0$ -which is easily seen on each of the (generating M) subspaces $\ker(T-\lambda I)$. If $x \in H \setminus \{0\}$ is some eigenvector of R_δ, say $R_\delta x = \mu x$, then decompose x orthogonally to $x = x_1 + x_2$, where $x_2 \in M$, $x_1 \perp M$. Then $R_\delta x_2 = 0$, hence $R_\delta x = R_\delta x_1 = \mu x_1 + \mu x_2$. Since R_δ is selfadjoint and M is invariant, also its orthocomplement is invariant for R_δ and $R_\delta x_1 \perp M$, forcing μx_2 to be orthogonal to M, hence zero. But $R_\delta x_1 = T x_1$, since $P_\lambda x_1 = 0$, as $x_1 \perp \ker(T-\lambda I)$ for any $|\lambda| \geq \delta$. This shows that μ is an eigenvalue of T and $|\mu| < \delta$. We have shown that $\sigma(R_\delta) \subset \{\lambda \in \mathbb{C} : |\lambda| < \delta$. This estimates the norm : $\|R_\delta\| < \delta$, so that the series $\sum \lambda P_\lambda$ converges to T in operator norm .

——————— (added December 08 2021 :)———————-

6. Bounded self-adjoint operators, projections

Theorem 6.1. *Let x_n be an orthogonal sequence in a Hilbert space H and let $S_k := \sum_{j=1}^k x_j$. Then the following are equivalent:*

1. *The series $\sum_{j=1}^\infty x_j$ is convergent in norm (i.e. $\exists_{S\in H} \lim \|S - S_k\| = 0$),*
2. *The series converges weakly (i.e. $\exists_{S\in H}\forall_{y\in H} \lim\langle S - S_k, y\rangle = 0,$)*
3. *The series is bounded (i.e. $\sup_n \|S_n\| < \infty$),*
4. $\sum_{j=1}^\infty \|x_j\|^2 < +\infty$.

Proof. Note that (3)$\Leftrightarrow$ (4), since $\|S_k\|^2 = \sum_{j=1}^k \|x_j\|^2$ (by Pythagorean Theorem). Also (2) $\Rightarrow$ (3), by Uniform Boundedness Principle. Now for $m > k$ we have $\|S_m - S_k\|^2 = \sum_{j=k+1}^m \|x_j\|^2$, hence (4) implies the Cauchy condition for our series, yielding (1). The rest is easy. □

Recall that for self-adjoint operators $T_1, T_2 \in \mathcal{B}(H)$ the inequality

$$T_1 \leq T_2 \quad \text{means that} \quad \forall_{x\in H}\ \langle T_1 x, x\rangle \leq \langle T_2 x, x\rangle.$$

Let us now consider a sequence of orthogonal projections P_n, where P_n project onto subspaces $M_n \subset H$.

Lemma 6.2. *M_n is the range, $\mathcal{R}(P_n)$ and $M_n = \mathcal{N}(I - P_n)$. Moreover*

$$P_0 \leq P_1 \Leftrightarrow M_0 \subset M_1, \qquad \textit{and then } \ P_1 - P_0 \textit{ is a projection onto } M_1 \ominus M_0 := M_1 \cap M_0^\perp,$$

$$P_1 + P_2 \quad \textit{is a projection} \quad \Leftrightarrow P_1P_2 = 0 \Leftrightarrow M_1 \cap M_2 = \{0\} \ (\textit{and then } \mathcal{R}(P_1+P_2) = M_1 \oplus M_2\,).$$

Moreover, if P_1, P_2 are two commuting projections, then P_1P_2 projects onto $M_1 \cap M_2$.
Any monotone sequence of projections P_n has a strong limit $P := sot - \lim P_n$ (but not a norm limit!) which is again a projection. For non-decreasing sequences P is the projection onto the closure of $\bigcup_n M_n$. For nonincreasing P_n their limit projects onto $\bigcap M_n$.

(Proof will be at tutorials)

As we know[1], for bounded monotone sequences of self-adjoint operators A_n, the strong limits exist, are self-adjoint and $(\lim A_n)^2 = \lim(A_n^2)$ (in strong topology). Since $P_n^2 = P_n$, we see that if $A_n = P_n$ are ortho-projections, then so is P- their strong limit.

We have constructed the "continuous functional calculus" in a self-adjoint bounded operator $T = T^*$: If $\Omega := \sigma(T)$ is the spectrum of T, then $\Omega \subset \mathbb{R}$ and there is an isometric homomorphism $\Phi : C(\Omega) \ni f \mapsto f(T) \in \mathcal{B}(H)$ of Banach algebras with unit, preserving the involution: $\Phi(\bar{f}) = (\Phi(f))^*$. Here *isometry* means that $\|\Phi(f)\| = \|f\|_\Omega := \sup\{|f(t)| : t \in \Omega\}$.

This homomorphism Φ agrees on the polynomials $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ with the natural functional calculus : $p(T) := a_0 I + a_1 T + \cdots + a_n T^n = \Phi(p)$ and it preserves the positivity: ($f \geq 0$ on $\sigma(T)$) $\Rightarrow \Phi(f) \geq 0$. If we fix $x \in H$, we have the "elementary measures" μ_x, denoted also $\mu_{x,x}$ (i.e. nonnegative Borel measures on Ω such that $\int f\, d\mu_x = \langle \Phi(f)x, x\rangle$ for all $f \in C(\Omega)$). We also have for any $x, y \in H$ the countably additive set functions (signed measures attaining complex values) $\mu_{x,y}$ such that

$$\int f d\mu_{x,y} = \langle \Phi(f)x, y\rangle, \quad f \in C(\Omega).$$

Here we may use polarisation formula to obtain $\mu_{x,y}$ from these elementary measures:

$$\mu_{x,y}(\Delta) = \frac{1}{4}\sum_{k=1}^4 i^k \mu_{x+i^k y, x+i^k y}(\Delta).$$

For a given Borel set $\Delta \subset \Omega$ the sesquilinear form $(x, y) \mapsto \mu_{x,y}(\Delta)$ is equal to $\langle E(\Delta)x, y\rangle$ for some bounded self-adjoint operator $E(\Delta) \in \mathcal{B}(H)$.

Linearity and self-adjointness of $E(\Delta)$ is easy to obtain, but some work is needed to establish the identity $E(\Delta)^2 = E(\Delta)$.

In this purpose we may extend the continuous functional calculus Φ to define $\Psi(f) = \lim \Phi(f_n)$ in the case, when there exists a monotone sequence of continuous real-valued functions f_n converging to f.

[1] For any $x \in H$ the monotone sequence $a_n(x) := \langle A_n x, x\rangle$ has a limit: call this limit $a(x)$. Then passing with the Polarisation Formula (cf. Theorem 2.1) to the limit we obtain a bounded sesquilinear form $\alpha(x, y) := \frac{1}{4}(a(x+y) - a(x-y) - ia(x-iy) + ia(x+iy))$. It corresponds to a bounded selfadjoint operator A. The strong convergence $A_n \to A$ follows from the Schwarz inequality (cf. Theorem 2.1) for the quadratic form of $A - A_n$ if $A_n \leq A$. Indeed, $|\langle Ax - A_n x, y\rangle| \leq \langle Ax - A_n x, x\rangle\langle Ay - A_n y, y\rangle$ and passing to supremum over $\|y\| \leq 1$ we obtain $\|Ax - A_n x\| \leq \langle Ax - A_n x, x\rangle \cdot (\|A\| + \|A_n\|)$, where the first factor equal $a(x) - a_n(x)$ converges to zero.

Then The sequence of self-adjoint operators $\Phi(f_n)$ is monotone and strongly converges. The set of so obtainable functions f is not so easy to describe, but it contains indicator (= characteristic) functions of all closed subsets of Ω. Indeed, if $\Delta = \bar{\Delta} \subset \Omega$, then there exists a sequence of $h_n \in C(\Omega)$ such that $0 \leq h_n \leq 1, h_n(t) = 1$ for $t \in \Delta$, while $h_n(s) = 0$ if $\mathrm{dist}(s, \Delta) \geq \frac{1}{n}$. Then h_n converge pointwise to the indicator function χ_Δ. By replacing each h_k with $\min(h_1, h_2, \ldots, h_k)$ -still continuous, we may also get the monotone convergence.

The following lemma is very important

Lemma 6.3. *(a)If $(f_n), (g_n)$ are two monotone sequences in $C(\Omega)$ converging to the same function f, then* $\lim \Phi(f_n) = \lim \Phi(g_n)$.
(b) If $f_n \to \chi_\Delta$ in a monotone way, then $\Psi(\chi_\Delta) := \lim \Phi(f_n)$ is an idempotent, i.e. $(\Psi(\chi_\Delta))^2 = \Psi(\chi_\Delta)$.
(c) Moreover, in this case $\Psi(\chi_\Delta) = E(\Delta)$, hence $E(\Delta)^2 = E(\Delta)$.

Proof. If we fix a vector $x \in H$, then $\langle \Phi(f_n)x, x\rangle = \int f_n \, d\mu_x \to \int f \, d\mu_x$ and $\langle \Phi(g_n)x, x\rangle = \int g_n \, d\mu_x \to \int f \, d\mu_x$ by the Monotone Convergence Theorem. On the other hand, we have strong limits, say $T = SOT - \lim \Phi(f_n)$ and $S = SOT - \lim \Phi(g_n)$ of these monotone sequences of operators. But strong (and even WOT)-convergence implies that $\langle Tx, x\rangle = \lim \langle \Phi(f_n)x, x\rangle$ and $\langle Sx, x\rangle = \lim \langle \Phi(g_n)x, x\rangle$ From previous equalities we deduce that $\langle Tx, x\rangle = \langle Sx, x\rangle$. Now, since $x \in H$ was arbitrary, (after applying polarisation) we deduce that $S = T$, which proves (a).

Now for f_n as in (b) we may additionally assume that $f_n \geq 0$ (after replacing f_n with $\max(f_n, 0)$ if needed. Then apply part (a) for $g_n = f_n^2$ converging in a monotone way to $\chi_\Delta^2 = \chi_\Delta$ and hence these limits must be equal. But the sequential continuity of operator multiplications in SOT topology implies that $\lim \Phi(f_n^2) = \lim(\Phi(f_n))^2 = (\lim \Phi(f_n))^2$, which gives us part (b). If we have a monotone sequence of functions $f_n \in C(\Omega)$ converging pointwise to χ_Δ, then the first equality in (c) follows from Monotone Convergence Theorem, since $\langle \Psi(\chi_\Delta)x, x\rangle = \lim \langle \Phi(f_n)x, x\rangle = \lim \int f_n(t), d\mu_x(t) = \int \chi_\Delta(t) \, d\mu)t) = \mu_x(\Delta) = \langle E(\Delta)x, x\rangle$. The rest follows from part (b). ☐

The operators $E(\Delta)$ commute, as limits of commuting operators, hence all parts of Lemma 6.2 are applicable. Let us define the set

$$\mathfrak{M} := \{E \in B(\Omega) : E(\Delta)^2 = E(\Delta)\}.$$

Theorem 6.4. *This family $\mathfrak{M}$ is an algebra of sets, containing all closed subsets of Ω and a monotone class. Hence $\mathfrak{M}$ is a sigma-field, containing all Borel subsets $E(\Delta)$ are orthogonal projections.*

Proof. Firs we show (using Lemma 6.2 that $\mathfrak{M}$ is an algebra. Simce $E(\Omega) = I$ and for any projection P also $I - P$ is a projection, while $E(\Omega \setminus \Delta) = \Psi(1 - \chi_\Delta)$, $\mathfrak{M}$ is closed under taking complements. For $\Delta = \Delta_1 \cap \Delta_2$ the characteristic function s satisfy $\chi_\Delta = \chi_{\Delta_1}\chi_{\Delta_2}$ and Ψ preserve products. But products of commuting idempotents is also idempotent, hence $\mathfrak{M}$ is closed under finite intersections. By additivity, it is closed under finite unions of disjoint sets and by taking intersections and finite intersections, we convert finite unions into finite unions of pairwise disjoint sets. Therefore $\mathfrak{M}$ is an algebra of sets. The monotone class property follows from the Lemma on projections, as for increasing sequences of sets the corresponding $E(\Delta_n)$ form a monotone, strongly convergent sequences of projections and passing to strong limits we preserve the idempotency. Now the result follows from Monotone Class Theorem. This theorem says that the smallest monotone class containing an algebra of sets is the sigma-algebra generated by this algebra. (This is a standard tool in the proof of Fubini's Theorem) ☐

The sigma- additivity of $\mu_{x,y}$ implies directly the sigma-additivity of E in the weak operator topology, but if Δ_j are pairwise disjoint, then for any $x \in H$ the sequence $x_n = E(\Delta_n)x$ is orthogonal and Theorem 6.1 implies the norm convergence of $\sum_{n=1}^{\infty} x_n$, which gives the strong operator topology convergence of $\sum(\Delta_n)$. Hence the orthogonal projections $E(\Delta)$ form a spectral measure. For $g_1(t) = t$ we obtain (looking back to the formulae defining $\mu_{x,y}$ and the equality $g_1(T) = T$ that

$$\int g_1(t) d\mu_{x,y} = \langle Tx, y\rangle.$$

This is THE SPECTRAL THEOREM FOR T, since the last equality is meaning that

$$T = \int_{\sigma(T)} t \, E(dt). \tag{14}$$

Similarly,

$$\Phi(f) = \int f(t) \, E(dt), \quad f \in C(\Omega).$$

Moreover, an operator S commutes with T (i.e. $ST = TS$) iff S commutes with every $E(\Delta)$. Indeed, then S commutes with any polynomial of T and with uniform limits of such polynomials, i.e. with $\Phi(T)$. The set of all operators $S \in \mathcal{B}(H)$ which commute with T is called the commutant of T and denoted $\{T\}'$. This is a subalgebra of $\mathcal{B}(H)$ closed under SOT-limits (or even WOT-limits). Hence it contains the closed subsets of Ω and all the monotone class containing the algebra of sets generated by such closed sets (hence it contains all Borel sets). Conversely, if we approximate the identity function (our t in (14)) by simple functions, their spectral integrals are linear combinations of some $E(\Delta_j)$. Hence if S commutes with any $E(\Delta)$ =the spectral measure for T, it must commute also with $T = \int t \, E(dt)$, as this spectral integral is a limit (in WOT-topology) of integrals of simple functions converging to the identity function $f(t) = t$ on $\Omega = \sigma(T)$.

—————————— (added January 05 2022 :)——————————-

7. UNBOUNDED OPERATORS, GRAPHS, ADJOINTS

We consider only densely defined ("d.d.") linear operators. This means that the domain $D(T)$ is a dense linear subspace of a Hilbert space H. I will denote this by

$$T : D(T) \subset H \to H.$$

(Some authors write $T : H \to H$ specifying later that $D(T) \neq H$.) Along with T we consider its *graph*

$$\Gamma_T := \{(x, Tx) \in H \times H : x \in D(T)\},$$

which is a linear subspace of $H \times H$. This Cartesian product is often denoted by $H \oplus H$, since these spaces are isometrically identified. We also need two surjective isometries:

$$U, W : H \times H \to H \times H, \quad U(f, g) := (g, f) \quad W(f, g) := (-g, f). \tag{15}$$

(Recall that $\|(f, g)\| := \left(\|f\|^2 + \|g\|^2\right)^{\frac{1}{2}}$ and note that $W^2 = -I, U^2 = I$ - *identity*.) For example, $\Big(U(\Gamma_T)$ is a graph of some operator (namely, of T^{-1})$\Big) \Leftrightarrow \Big($ operator T is injective $\Big)$:

$$\texttt{A linear operator } T \texttt{ is injective } \Leftrightarrow \mathcal{N}(T) := \{x \in D(T) : Tx = 0\} = \{0\}.$$

Definition.

We say that *a linear operator T is closed*, if its graph is closed. If the closure, $\overline{\Gamma_T}$ is a graph of some operator (denoted by $\bar{T}$) then we say that T *is closable* and $\bar{T}$ is then called *the closure of* T. We say that operator S *is an extension* of T, in symbols: $S \supset T$, if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $Sx = Tx$ for any $x \in \mathcal{D}(T)$.

By the Closed Graph Theorem, if $D(T)$ is a closed subspace in H and T is closed, then T is bounded. An easy exercise shows that the converse is also true. This results from the following Lemma applied to $S : \Gamma_T \ni (f, Tf) \mapsto f \in D(T)$ (what is the range, $\mathcal{R}(S)$ -how it relates to T in this case?).

Lemma 7.1. *If S is a closed operator bounded below: $\exists_{c>0} \forall_{x \in \mathcal{D}(S)} \|Sx\| \geq c\|x\|$, then $\mathcal{R}(S)$ is closed.*

Proof. If a sequence of $Sx_n \in \mathcal{R}(S)$ converges to some $y \in H$ we have to show that $y = Sx$ for some $x \in \mathcal{D}(S)$. But (x_n) is a Cauchy sequence, since $\|x_n - x_k\| \leq \frac{1}{c}\|S(x_n - x_k)\| \to 0$ as $n, k \to \infty$. Now the sequence of pairs: (x_n, Sx_n) converges in $H \times H$. By the closedness, $x := \lim \in \mathcal{D}(T)$ and $Sx = y$. □

EXAMPLES:(1) If $\varphi : X \to \mathbb{C}$ is a measurable, possibly unbounded function, the multiplication operator

$$M_\varphi : \mathcal{D}(M_\varphi) \ni f \mapsto \varphi f \in L^2(\mu)$$

is closed on its *maximal domain* $\mathcal{D}(M_\varphi) := \{f \in L^2(\mu) : \varphi f \in L^2(\mu)\}$. (easy exercise). In particular, if this domain equals $L^2(\mu)$, we must have T_φ bounded and in the case of σ-finite μ, we deduce that $\varphi \in L^\infty(\mu)$. Let for example $X = \bigcup_{n<\infty} X_n$, where $\mu(X_n) < \infty$, $X_n \subset X_{n+1}$. Since Define the sets

$$A_n := \{x \in X_n : |\varphi(x)| \leq n\}, \qquad ,\mathcal{F}_n := \{f \in L^2(\mu) : f = 0 \texttt{ outside } A_n\}.$$

Then clearly $\forall_n \mathcal{F}_n \subset \mathcal{D}(M_\varphi)$, hence this domain is dense. In fact, for any $f \in L^2(\mu)$ the sequence defined by $f_n(x) = f(x)$ for $x \in A_n$ and $f_n(x) = 0$ if $x \notin A_n$ satisfies $f_n \in \mathcal{F}_n$, $f_n \to f$ (pointwise). Since $|f_n - f| \leq |f|$, by Lebesgue's Dominated Convergence Theorem we obtain $\int |f - f_n|^2 d\mu \to 0$ as $n \to \infty$, showing the density of $\mathcal{D}(M_\varphi)$ in $L^2(\mu)$.

(2) In Banach space $C[a, b]$ (with sup-norm) the differential operator $\frac{d}{dt}$ with domain $C^1[a, b]$ is closed.

(3)The same operator with the same domain, but on the Hilbert space $L^2[a, b]$ is not closed. The domain of its closure is the Sobolev space $H^1[a, b] = W^{1,2}[a, b]$. This space may be defined as the completion of $C^1[a, b]$ under the graph norm defined below, but also can be defined as the set of absolutely continuous functions whose (existing almost everywhere) derivative belong to $L^2[a, b]$. Third definition uses the notion of weak derivative (and, unlike the previous one, is applicable also in multi-dimensional setting) -this will be exactly the domain of the adjoint operator to $\frac{d}{dx}$. (cf. infra)

(4) Example of non-closable operator is $T : H^1[0, 1] \ni f \mapsto f(0) \cdot \mathbf{1}$, where $\mathbf{1}$ is the constant function equal 1. Indeed, it is easy to see that $T : \mathcal{D}(T) \to H$ is closable iff for $x_n \in \mathcal{D}(T)$ with $\|x_n\| \to 0$ there must be $Tx_n \to 0$. Here take $x_n(t) := (1 - t)^n$, so $T(x_n) = \mathbf{1}$, while $x_n \to 0$.

We often use the so called *graph norm* on $\mathcal{D}(T)$ defined by

$$\|x\|_T := \|x\| + \|Tx\|.$$

This is not a Hilbertian norm, but is clearly equivalent to such a norm, namely to $\|x\|_{(T)} := \left(\|x\|^2 + \|Tx\|^2\right)^{\frac{1}{2}}$.

Theorem 7.2. *A densely defined operator $T : \mathcal{D}(T) \to H$ is closed iff $(\mathcal{D}(T), \|\cdot\|_T)$ is a complete normed space.*

Proof. The mapping $\mathcal{D}(T) \ni x \mapsto (x, Tx) \in \Gamma_T$ is an isometric bijection and the completness of either of the spaces: $(\mathcal{D}(T), \|\cdot\|_{(T)}), (\Gamma_T, \|\cdot\|)$ with respect to their corresponding norms implies the completeness of the other. Here we may use the sum of the norms of coordinates of the pair (x, Tx), equivalent to the Hilbertian norm on $H \times H$. Completeness of Γ_T under these (both) norms is equivalent to its closedness. □

The most important example of a closed operator is *the adjoint operator* T^* defined for a densely defined operator T as follows

$$\mathcal{D}(T^*) := \{y \in H : \exists_{z \in H} \forall_{x \in \mathcal{D}(T)} \langle Tx, y\rangle = \langle x, z\rangle\} \texttt{ with } T^* y := z.$$

The uniqueness of z satisfying the above condition results from the density of $\mathcal{D}(T)$. Another formulation of the definition of $\mathcal{D}(T^*)$ is to postulate the continuity of the linear functional $\psi : \mathcal{D}(T) \ni x \mapsto \langle Tx, y\rangle \in \mathbb{C}$. As a continuous functional, it extends uniquely to a bounded linear functional on H, generated by some z as above: $\psi(x) = \langle x, z\rangle$. On the last meeting in 2021 we have verified that T^* is always a closed operator.

Examples: (a) If φ is a measurable function, then the multiplication operator M_φ from Example (1) is closed, its adjoint has the same domain as the (maximal) domain of M_φ and is the multiplication by the complex conjugate function $\bar{\varphi}$, like in the bounded case.

(b) In the notation of Example (3), we consider first the adjoint of $T_\bullet := \frac{d}{dt}$ on $L^2[0,1]$ having the domain $D_\bullet := \{f \in C^1[0,1] : \exists_{\delta>0} \text{supp}(f) \subset (\delta, 1-\delta)\}$. Then for any $f, g \in D_\bullet$ the integration by part gives

$$\langle T_\bullet f, g\rangle = \int_0^1 f'(t)\bar{g}(t)\,dt = f\bar{g}|_0^1 - \int_0^1 f(t)\overline{g'(t)}\,dt = \langle f, -T_\bullet g\rangle. \tag{16}$$

Hence $T_\bullet^* \supset -T_\bullet$. The containment is, in fact, proper: As long as the integration by part applies, any $g \in L^2[0,1]$ such that g' exists almost everywhere and $g' \in L^2[0,1]$ -will satisfy this equality. Here $f\bar{g}|_0^1 = f(1)\bar{g}(1) - f(0)\bar{g}(0) = 0$ for any g, since $f(0) = f(1) = 0$ and we are using the fact that $\frac{d}{dt}\bar{g}(t)$ is the complex conjugate of $\frac{d}{dt}g(t)$. We may take g from the Sobolev space $H^1[0,1]$.

Note that if $g' \in L^2[0,1]$ and the integration by parts applies with $f = \mathbf{1}$, then $|g(t) - g(s)| = |\int_s^t g'(x)\,dx| \le \sqrt{|t-s|}\int_t^s |g'(x)|^2$. Moreover if $[s_j, t_j]$ is a finite collection of pairwise disjoint subintervals of $[0,1]$ with $\delta := \sum_{j=1}^k |t_j - s_j|$ sufficiently small, then $\sum_{j=1}^k |g(t_j) - g(s_j)|$ will be arbitrarily small. This is, by definition, the absolute continuity of g. When $k = 1$ it is just the uniform continuity, but there exist uniformly continuous functions that are not absolutely continuous. An example is the "Devil's staircase function" γ: nondecreasing, constant in any interval Δ deleted from $[0,1]$ during the construction of Cantor's ternary set. Its derivative is equal zero on any such Δ, hence almost everywhere, so the Lebesgue integral $\int_0^1 \gamma'(x)\,dx = 0 \ne 1 = \gamma(1) - \gamma(0)$. For absolutely continuous functions g we have $g' \in L^1[a,b]$ and $\int_a^1 g'(b)\,dx = g(b) - g(a)$. For unbounded domains (e.g. if $g \in L^2(\mathbb{R})$), one assumes the absolute continuity of g on any bounded interval contained in this domain of g.

We call a linear densely defined operator T symmetric, if $T \subset T^*$ -which is the same as saying that

$$\forall_{f,g \in \mathcal{D}(T)} \langle Tf, g\rangle = \langle f, Tg\rangle.$$

Unlike in the case of bounded operators, this is not the same, as the self-adjointness condition (s-a -for short): $T = T^*$. For example, if we multiply our operator $T_\bullet$ by the imaginary unit i, we obtain a symmetric, but not s-a operator, since the domain of $iT_\bullet^*$ is the same as $\mathcal{D}(T_\bullet^*) = H^1[a,b]$. This operator is even not essentially self-adjoint. It can be shown that the closure $i\bar{T}_\bullet$ of $iT_\bullet$ is the operator acting as $i\frac{d}{dt}$ on $H^1_{00}(0,1) := \{f \in H^1[0,1] : f(0) = 0 = f(1)\}$. This closure has the same adjoint (in general, always $\bar{T}^* = T^*$).

There exist (even infinitely many) different self-adjoint extensions of this symmetric operator:

$$T_\alpha := i\frac{d}{dt} \quad \texttt{with the domain } D_\alpha := \{f \in H^1[0,1] : f(1) = \alpha f(0)\},\ \textit{where } \alpha \in \mathbb{C}, |\alpha| = 1.$$

Then while integrating by parts as in (16), we only have to remember that $f\bar{g}|_0^1 = \alpha f(0)\overline{\alpha g(0)} - f(0)g(0) = 0$, since $\alpha\bar{\alpha} = |\alpha|^2 = 1$, by our assumption. Usually one takes ony $\alpha = 1$. Then for the sake of simplicity we may assume that the functions are real-valued. One defines $h \in L^2[a,b]$ to be a *weak derivative of* $f \in L^2$, if for any C^1 function g whose support is a compact subset of (a,b) [a fact is denoted: $g \in C^1_c(a,b)$], we have

$$\int_a^b f(t)g'(t)\,dt = -\int_a^b h(t)g(t).$$

Clearly, if $f \in C^1[a,b]$, its weak and strong derivatives are the same: $h = f'$. The existence of such $h \in L^2[a,b]$ for f is evidently equivalent to the fact, that f is in the domain of the adjoint operator to $\left\{\frac{d}{dt} \text{ acting on this space } C^1_c(a,b)\right\}$. From this one can deduce that this domain of the adjoint is exactly the Sobolev space.

We have also considered the integral operators. The adjoint operators to T_K, where $(T_K f)(x) = \int f(y)K(x,y)d\mu(y)$ is the integral operator corresponding to the "kernel function" K^*, where $K^*(x,y) = \overline{K(y,x)}$ with bar denoting the complex conjugate. In most applications, however, these integral operators will be bounded, everywhere defined. They appear in formulae for solutions of certain differential equations.

One important result is the following characterisation of closability, proved by J.von Neumann:

Theorem 7.3. *For a densely defined linear operator $T : \mathcal{D}(T) \subset H \to H$ we have $\Gamma_{T^*} = \big(W(\Gamma_T)\big)^{\perp}$, where W is defined as in* (15). *Moreover, T is closable iff $\mathcal{D}(T^*)$ is a dense subspace in H. In such case the closure is $\bar{T} = T^{**}$.*

Proof. If $(x, y) \in H \times H$, then $(x, y) \perp W(\Gamma_T)$ means that for all $f \in \mathcal{D}(T)$ we have $0 = \langle(-Tf, f), (x, y)\rangle$. The latter inner product is equal to $\langle -Tf, x\rangle + \langle f, y\rangle$. Hence this orthogonality holds iff $\langle Tf, x\rangle = \langle f, y\rangle$. Since $\mathcal{D}(T)$ is dense, T^* is a well-defined operator and then $(x, y) \in \big(W(\Gamma_T)\big)^{\perp}$ iff $x \in \mathcal{D}(T^*), y = T^*x$, so that $(x, y) \in \Gamma_{T^*}$. The mapping W, as a linear bijective isometry, can be interchanged with orthogonal complement sign $^{\perp}$, which in case of $\mathcal{D}(T^*)$ dense- gives

$$\Gamma_{T^{**}} = \big(W(\Gamma^*_T)\big)^{\perp} = \big(W(\big(W(\Gamma_T)\big)^{\perp})\big)^{\perp} = \big(W \circ W(\Gamma_T)\big)^{\perp\perp}$$

. Since $W \circ W = -I$ and Γ_T is a linear subspace, we finally have $\Gamma_{T^{**}} = (\Gamma_T)^{\perp\perp}$. The double ortho-complement is the closure, hence T is then closable, with its closure equal T^{**}. It remains to prove that conversely, if T is closable, then $\mathcal{D}(T^*)$ must be dense. We have seen that the closure of Γ_T -which we now assume to be a graph of some operator- is equal to $\big(W(\Gamma^*_T)\big)^{\perp}$. Replacing now T^* with S it is enough to apply the following Lemma 7.4. □

Lemma 7.4. *If $S : \mathcal{D}(S) \subset H \to H$ is a linear operator then $\big(W(\Gamma_S)\big)^{\perp}$ is a graph of some operator iff $\mathcal{D}(S)$ is dense in H.*

Proof. A linear subspace $L \subset H \times H$ is a graph of some mapping iff $(0, h) \in L \Rightarrow h = 0$, for if $(x, y_1) \in L$ and $(x, y_2) \in L$, their difference equal to $(0, y_1 - y_2)$ also belongs to L. Note also that

$$(0, h) \perp W(\Gamma_S) \Leftrightarrow h \perp \mathcal{D}(S).$$

Indeed, $\langle(0, h), (-Tx, x)\rangle = \langle h, x\rangle$. Now $\big\{(\forall_{x\in\mathcal{D}(S)}\, h \perp x) \Rightarrow h = 0\big\}$ holds iff $\mathcal{D}(S)$ is dense in H. □

There is a sharp difference between some properties of unbounded operators comparing to the bounded case. For example, the sum of two closed operators may not be closed (take $T + (-T)$, where T is closed, the sum is the zero operator on $\mathcal{D}(T)$, different from its closure for unbounded T). In the case of sums some additional assumption guarantees closedness:

We say that S is *T-bounded* if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and there exist $a, b \geq 0$ such that

$$\forall_{x\in\mathcal{D}(T)} \|Sx\| \leq a\|x\| + b\|Tx\|. \tag{17}$$

This actually is the continuity of S in either of the two graph norms on $\mathcal{D}(T)$. If the above constants can be chosen so that $b < 1$, then we say that S *is strongly dominated by* T.

Theorem 7.5. *Let S be a strongly dominated*[2] *operator by a densely defined operator T. If T is closed, then $S + T$ is closed on $\mathcal{D}(T)$. If moreover S^* is also strongly dominated by T^*, then $\mathcal{D}(T^* + S^*) = \mathcal{D}(T^*)$ and*

$$(T + S)^* = T^* + S^*. \tag{18}$$

Proof. If T is closed, replacing it with some $\epsilon \cdot T$ and S with ϵS we may assume that also $a < 1$ in condition (7.5). Then for some $\alpha \in (0, 1)$ we have $\|Sx\| \leq \alpha(\|x\| + \|Tx\|)$, from where we deduce that

$$(1 - \alpha)\|x\|_T \leq \|x\|_{T+S} \leq (1 + \alpha)\|x\|_T \text{ for all } x \in \mathcal{D}(T).$$

The graph norms on $\mathcal{D}(T)$ of T and of $T + S$ are therefore equivalent and the completeness of these two norms is also equivalent. Since T is closed, so is $T + S$.

Operators $S + T$ defined on $\mathcal{D}(T)$ and $S^* + T^*$ on $\mathcal{D}(T^*)$ are closed by the above arguments. Let $y \in \mathcal{D}(T^*)$. Then $y \in \mathcal{D}(S^*)$ and

$$\langle Sx + Tx, y\rangle = \langle x, S^*y\rangle + \langle x, T^*y\rangle = \langle x, S^*y + T^*y\rangle,\ \forall_{x\in\mathcal{D}(T)}.$$

Hence $T^* + S^* \subset (T + S)^*$ and in $H \times H$ the orthogonality $W(\Gamma_{T+S}) \perp \Gamma_{T^*+S^*}$ takes place. By Theorem 7.3 the equality (18) will follow, once we show that $W(\Gamma_{T+S}) \oplus \Gamma_{T^*+S^*} = H \times H$. As above, we may assume that $a, b < 1$, the strong domination taking the form

$$\|Sx\|^2 \leq a^2(\|x\|^2 + \|Tx\|^2),\ x \in \mathcal{D}(T), \|S^*x\|^2 \leq a^2(\|x\|^2 + \|T^*x\|^2),\ x \in \mathcal{D}(T^*).$$

Define a linear operator Q on $(f, g) \in H \times H$ as follows. From the decomposition $H \times H = W(\Gamma_T) \oplus \Gamma_{T^*}$, there exist $x \in \mathcal{D}(T), y \in \mathcal{D}(S)$ such that $f = -Tx + y,\ g = x + T * y$. Put Q(f, g) = (-Sx, S*y). It follows from the above "domination inequalities that

$$\|Q(f, g)\|^2 = \|Sx\|^2 + \|S^*x\|^2 \leq a^2(\|f\|^2 + \|g\|^2),$$

which means that $\|Q\| \leq a < 1$. Hence $I + Q$ is invertible in $\mathcal{B}(H \times H)$ Hence by our choice of (x, y), we have

$$(I + Q)(f, g) = (-Tx + y, x + T^*y) + (-Sx, S^*y) = W(x, Tx + Sx) + (y, T^*y + S^*y) \in W(\Gamma_{T+S}) \oplus \Gamma_{T^*+S^*}.$$

This shows the desired orthogonal decomposition of $H \times H$. □

The following implication from the domains inclusion is very useful.

[2]In some books instead of strong domination the "graph T-norm of S is < 1" appears, which is equivalent in such purposes (after replacing S, T by ϵS, ϵT an in the present proof)

Theorem 7.6. *If T is closed and S is closable, then the inclusion $\mathcal{D}(T) \subset \mathcal{D}(S)$ implies T -boundedness of S.*

Proof. Denote by S_T the restriction of S to $\mathcal{D}(T)$, where the latter domain is considered with $\|\cdot\|_T$ -norm (hence it is a complete space). By the Closed Graph Theorem, it remains to show that S_T is a closed operator. Take $x, x_n \in \mathcal{D}(T)$ such that simultaneously $\|x_n - x\|_T \to 0$ and $\|Sx_n - y\| \to 0$. Then $\|x_n - x\| \to 0$ (c0nvergence in H). Now since S is closable and $x \in \mathcal{D}(T) \subset \mathcal{D}(S)$, we have $y = Sx$. $\square$