Complete n-closure for pancyclism being disclosed

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Abstract

A short and detailed proof, found in 2000, shows that a condition for an n-vertex graph G to be pancyclic implies completeness of the Bondy-Chvátal's n-closure of G. This simplifies the original proof of hamiltonicity of G.

Theorem 1 (Flandrin et al. [4]) *Let $G = (V, E)$ be a 2-connected graph on n vertices with minimum degree δ and such that for any two vertices x and y if $\delta = d_G(x) < n/2$ and $d_G(y) < n/2$ then $xy \in E$. Then $K := \mathrm{cl}_n(G) = K_n$.*

Proof. Suppose that $K \neq K_n$. Then $2 \le \delta < n/2$ whence $n \ge 5$. Let X and Y be sets of vertices whose degrees in G are δ and in the interval $[\delta + 1, n/2)$, respectively. Let $|X| = i$ $(i > 0)$ and $|Y| = j$. Then $i + j \le \delta + 1$ because each vertex x of degree δ in G is adjacent to all vertices in $Y \cup X \setminus \{x\}$. Since, moreover, $\delta + 1 \le (n+1)/2 \le n - 2$, the complement of $Y \cup X$ in V comprises two or more vertices and induces in K a complete subgraph, say Q, whose all vertices z have degrees $d_Q(z) = n - 1 - i - j$. Hence $\delta \le i + j$ because otherwise $K = K_n$. Suppose $i + j = \delta$. Then the set $V \setminus X$ is a clique in K with vertex degrees $\ge n - 1 - i$ which are $\ge n - \delta$ if $j > 0$, whence $K = K_n$, a contradiction. Thus $i = \delta$ (≥ 2) and $j = 0$. Then each of i vertices x in X has in G exactly one neighbour belonging to $V \setminus X$. However, because of 2-connectivity, in G there are two (or more) neighbours z_p of the set X in the set $V \setminus X$, $p = 1, 2$. Then $d_K(z_p) \ge d_Q(z_p) + 1 = n - \delta$. Hence each $x \in X$ is adjacent in K to both z_p's, a contradiction.

Therefore $i + j = \delta + 1$. Then all neighbours in G of any $x \in X$ are in $X \cup Y$. Because of 2-connectivity of G, $|Y| = j \ge 2$ and there are two or more neighbours z_p of Y in the set $V \setminus (Y \cup X)$. Therefore $d_K(z_p) \ge d_Q(z_p) + 1 = n - 1 - \delta$ whence each z_p is adjacent in K to all vertices of Y. Hence $d_K(z_p) \ge d_Q(z_p) + |Y| \ge n - \delta$. Consequently, $xz_p \in E(K)$ for all z_p's and all $x \in X$ whence $d_K(x) \ge \delta + 2$. Therefore the set $V \setminus Y$ is a clique in K with minimum degree $\ge n - 1 - j \ge n - 1 - \delta$ whence $K = K_n$, a contradiction. ∎

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