/15/ Skupień, Z., A short proof of the theorem on the structure of maximal graphs without 1-factors, Prace Nauk. Inst. Mat. Fiz. Teoret. Politech. Wrocław. Nr 9, Ser. Studia i mater., Nr 9, Grafy i hipergrafy, 9-14 (1973).

/16/ Skupień, Z., Partitions and counting isomorphism types of certain maximal graphs without 1-factors, Demonstratio Math., 8, 447-463 (1975).

/17/ Skupień, Z., Homogeneously traceable and Hamiltonian connected graphs, preprint, 1976. Demonstratio Math. 17(1984) 1051-1067.

/18/ Skupień, Z., Maximum degree among vertices of a non-Hamiltonian homogeneously traceable graph, preprint, 1978. Lect. Notes in Math. 885, Springer 1981, 496-500.

/19/ Skupień, Z. and Wojda, A. P., Extremal non-(p,q)-Hamiltonian graphs, preprint, 1974. Studia Sci. Math. Hung. 10 (1975) 323-328.

/20/ Vrba, A., Counting hamiltonian circuits, in: Borowiecki, M., Skupień, Z. and Szamkołowicz, L. (Eds.), Graphs, Hypergraphs and Block Systems, (Proc. Symp. held in Zielona Góra, 1976), 319-323, Zielona Góra 1976.

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On Maximal non-Hamiltonian Graphs

m.n.-H.(maximal non-Hamiltonian) graphs are partially characterized and generating those graphs of small orders is reported.

1. Introduction

The problem of finding both elegant and useful characterization of Hamiltonian (or non-Hamiltonian) graphs seems to be the most exciting open problem in graph theory, more particularly as the Four Colour Hypothesis has been announced /1/, /2/ to be confirmed. However, despite efforts of many specialists, the problem is far from the satisfactory solution. For instance, the Dirac's condition "$\delta(G) \geq n/2$" is only sufficient for a graph G to be Hamiltonian. Note, however, that there are two different formulas for the number, say h(G), of Hamiltonian circuits in G, due to Lihtenbaum /11/ and Vrba /20/, respectively. Both of those formulas involve adjacency matrix A of G. Obviously, both corresponding conditions "$h(G) \neq 0$" can be regarded as non-trivial necessary and sufficient conditions for a graph G to be Hamiltonian. Nevertheless, each of them is too complicated and therefore useless in general case.

Therefore it would be interesting to find a characterization [or the lists] of m.n.-H. graphs [of order n for small n at least]. Notice also that having a description of m.n.-H. graphs one could be in position to formulate and to test hypotheses about Hamiltonian graphs.

This note only contributes to the problem of characterizing m.n.-H. graphs. Namely, the class M_n of m.n.-H. graphs G_n of order n is partitioned into three (possibly empty) subclasses. One of those subclasses, I_n, consists of m.n.-H. graphs G_n

with the scattering number $s(G_n) = 1$. Those graphs are precisely described and counted. In particular, $|I_n|$ is expressed by means of numbers of partitions of some integers into certain numbers of parts.

Remaining m.n.-H. graphs are partially characterized. Namely, the second subclass, T_n, consists of m.n.-H. homogeneously traceable graphs G_n which are non-trivially non-Hamiltonian, i.e., with $n \geq 3$ vertices. Then $T_n \neq \emptyset$ iff $n \geq 9$, T_n contains many hypohamiltonian graphs and if $G \in T_n$ then $\Delta(G) \leq n-4$ and $s(G) \leq 0$. Now remaining m.n.-H. graphs G_n form the subclass J_n, have $\Delta(G) = n-1$, $s(G) \leq 0$ and exist iff either $n \leq 2$ or $n \geq 7$. Moreover, a method of generating all 2-connected m.n.-H. graphs G_n from the set of all minimal blocks of order n is sketched, all m.n.-H. graphs of order $n \leq 7$ are listed, and using a computer for generating m.n.-H. graphs of order n with $n \leq 10$ is reported.

2. Notation and terminology

For the sake of completeness, we recall basic names and denotation. Throughout the note, $G = G_n = (V,E)$ denotes a simple graph with the vertex set $V(G) = V$ and the edge set $E(G) = E$ of order $|V| = n$ and size $|E|$. The degree of a vertex $x \in V$ is denoted by $d(x,G)$. The symbols $\delta(G), \Delta(G), \varkappa(G)$, and $k(G)$ denote the minimum degree and the maximum degree among vertices of G, the connectivity, and the number of components of G, respectively. A factor and a counterfactor of G are a subgraph and a supergraph, respectively, both with the same vertex set $V(G)$. Symbols $K_n (n \geq 1)$, $\overline{K}_n (n \geq 2)$, $C_n (n \geq 3)$, and $P_n (n \geq 1)$ denote the complete graph, the totally disconnected graph, the circuit, and the path of order n each. K_G stands for the complete counterfactor of G. G is called Hamiltonian if C_n is a factor of G. Then C_n is a Hamiltonian circuit of G.

A graph G is called m.n.-H. (maximal non-Hamiltonian) graph if it is non-Hamiltonian but, for any new edge e, i.e., $\forall e \in E(K_G) - E(G)$, the graph $G \cup e$ is Hamiltonian. Thus, m.n.-H. graphs can be also defined as maximal elements in the class of

non-Hamiltonian graphs, ordered by the relation "is a factor of", where two graphs are considered equal if they are isomorphic. M_n will stand for the class of n-vertex m.n.-H. graphs. Following Jung /9/, the scattering number $s(G)$ of G is defined as follows.

$$s(G) = \max \{k(G-S) - |S| : S \subseteq V(G) \text{ and } k(G-S) \neq 1\}.$$

For instance,

$$s(K_n) = -n, \text{ and if } n \geq 2 \text{ then } s(\overline{K}_n) = n. \qquad (2.1)$$

The star $*$ denotes the operation of join on disjoint graphs with the convention that

$$F * G * H = F * G \cup G * H.$$

Following Skupień /17/, a graph G is called homogeneously traceable if, for each vertex x, there is a Hamiltonian path beginning at x.

3. General properties of m.n.-H. graphs

The m.n.-H. graphs, named "ready for a Hamiltonian cycle" by Bondy /3/ and "hypertortuous" by Nash-Williams /12/, have the following property which follows easily from Theorem 1 of Ore /13/ (see also /12/).

Theorem 3.1: If G_n is an n-vertex m.n.-H. graph then the sum of degrees of any two non-adjacent vertices is less than n.

The following result, formulated by Ore /14/, was proved by Bondy /3/ and next by Chvátal /5/ (a generalization can be found in /19/).

Theorem 3.2: If G_n with $n \geq 2$ is a m.n.-H. graph of maximum size then either $G_n = K_1 * K_1 * K_{n-2}$ or additionally $G_n = \overline{K}_3 * K_2$ if $n = 5$.

Observe that each of those extremal graphs G_n has the scattering number $s(G_n) = 1$. Note also that if, for an $S \subseteq V(G)$, $k(G-S) > |S| + 1$ then adding to G an edge connecting different components of G–S gives a counterfactor of G which is still non-Hamiltonian. Hence we have

Lemma 3.1: If G is an n-vertex m.n.-H. graph then $s(G) \le 1$.

Hence, each m.n.-H. graph G is connected. Moreover, we have

Lemma 3.2: If G is an n-vertex m.n.-H. graph then it is connected and, for $n \ge 3$, $1 \le \varkappa(G) \le (n-1)/2$.

Note that the given upper bound for $\varkappa(G)$ is implied by Dirac's sufficient condition for G to be a Hamiltonian graph.
In what follows the class M_n of n-vertex m.n.-H. graphs is divided into three mutually disjoint (possibly empty) subclasses I_n, T_n, and J_n where

$$I_n = \{G \in M_n : \ s(G) = 1\},$$

$$T_n = \begin{cases} \{G \in M_n : \ G \text{ is homogeneously traceable}\} & \text{for } n \ge 3, \\ \emptyset & \text{for } n \le 2, \end{cases}$$

$$J_n = M_n - I_n - T_n.$$

4. Characterizing and counting graphs of the class I_n

Recall that I_n consists of m.n.-H. graphs G of order n with the scattering number $s(G) = 1$. It is clear that K_1 and K_2 are the only m.n.-H. graphs of order 1 and 2, respectively. By (2.1), their scattering numbers are both negative. Therefore,

$$I_n = \emptyset \quad \text{for} \quad n \le 2.$$

Definition 4.1: We let $I_n(\varkappa)$ denote the subclass of I_n consisting of graphs G with connectivity $\varkappa(G) = \varkappa$.

For any $n \ge 3$, graphs belonging to I_n can be easily listed. For instance,

$$\left.\begin{aligned} I_3 &= I_3(1) = \{P_3 = K_1 * K_1 * K_1\}, \\ I_4 &= I_4(1) = \{K_2 * K_1 * K_1\}, \\ I_5 &= \{K_3 * K_1 * K_1, \ K_2 * K_1 * K_2, \ \overline{K}_3 * K_2\}, \\ I_6 &= \{K_4 * K_1 * K_1, \ K_3 * K_1 * K_2, \ K_2 * K_2 * \overline{K}_2\}. \end{aligned}\right\} \qquad (4.1)$$

$$I_7 = I_7(1) \cup I_7(2) \cup I_7(3) = \{K_5 * K_1 * K_1, \ K_4 * K_1 * K_2, \ K_3 * K_1 * K_3\} \cup \{K_3 * K_2 * \overline{K}_2, \ K_2 * \underset{\overset{*}{K_1}}{K_2} * K_2\} \cup \{\overline{K}_4 * K_3\}. \qquad (4.2)$$

The structure of graphs belonging to I_n is fully described in the following theorem.

Theorem 4.1: Assume that $n \ge 3$. Then $G \in I_n$ iff $\exists \varkappa : G \in I_n(\varkappa)$ with $1 \le \varkappa \le (n-1)/2$, or equivalently, there is an integer $\varkappa$ with $1 \le \varkappa \le (n-1)/2$, there are $\varkappa+1$ positive integers n_i where $n_1 \ge n_2 \ge \ldots \ge n_{\varkappa+1} \ge 1$, and there are $\varkappa+2$ disjoint complete graphs $K^o_\varkappa$ and $K^i_{n_i}$ $(i = 1,2,\ldots,\varkappa+1)$ such that the nonincreasing sequence (n_i) is a partition of $n-\varkappa$ into $\varkappa+1$ parts and

$$G = K^o_\varkappa * \sum_{i=1}^{\varkappa+1} K^i_{n_i} \qquad (4.3)$$

where $\sum$ denotes the union of disjoint graphs.

Proof: Sufficiency. If (4.3) is true then clearly $\varkappa(G) = \varkappa$ and $s(G) = 1$. Consequently, G is non-Hamiltonian since Hamiltonian graphs have non-positive scattering numbers. Note that if $e \in E(K_G) - E(G)$ then end-vertices of e belong to two different graphs K_{n_i} with $i \ge 1$. Hence $G \cup e$ has a Hamiltonian circuit which includes $\varkappa$ paths forming a factor of $e \cup \sum_{i=1}^{\varkappa+1} K^i_{n_i}$. Consequently, $G \in I_n(\varkappa)$, q.e.d.

Necessity. Assume that $G \in I_n$. Hence $s(G)=1$. Consequently, there is $S \subseteq V(G)$ with $|S| =: \varkappa$ such that $1 \ne k(G-S) = \varkappa+1$. Let $K^i_{n_i}$ $(1 \le i \le \varkappa+1)$ be complete counterfactors of components of $G - S$, ordered in such a way that (n_i) is a nonincreasing sequence and let

$$H = K^o_\varkappa * \sum_{i=1}^{\varkappa+1} K^i_{n_i} \quad \text{where } V(K^o_\varkappa) = S.$$

Thus G is a factor of H and the graph H is non-Hamiltonian because clearly $s(H) = 1$. Hence $G = H$ because of the maximality of G. Moreover, $\varkappa$ is the connectivity of G. Hence, by Lemma 3.2, $\varkappa$ satisfies the desired inequalities, q.e.d.

Remark: Theorem 4.1 resembles those of /15/ and /16/. The above proof is very similar to that given in /15/. Consequently, the cardinality $|I_n|$ of I_n can be found by a similar method as that used in /16/.

Recall (cf. /6/) the following

Definition 4.2: A partition of a positive integer s into r parts is a nonincreasing sequence of r positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r (\geq 1)$ whose sum is s. Let p(s,r) denote the number of those partitions. Moreover, p(0,0): = 1.

Corollary 4.1: There is a one-to-one correspondence between $I_n(\varkappa)$ and the class of partitions of $n-\varkappa$ into $\varkappa+1$ parts. Consequently,

$$|I_n(\varkappa)| = p(n-\varkappa, \varkappa+1) \quad \text{for} \quad n \geq 3 \quad \text{and} \quad 1 \leq \varkappa \leq (n-1)/2. \tag{4.4}$$

Theorem 4.2 (counting graphs belonging to I_n): For positive integers n and $\varkappa$ with $n \geq 3$,

$$|I_n| = \sum_{\varkappa=1}^{\lfloor (n-1)/2 \rfloor} |I_n(\varkappa)|, \tag{4.5}$$

$$|I_n(1)| = \lfloor (n-1)/2 \rfloor, \tag{4.6}$$

$$|I_n(\varkappa)| = |I_{n-2}(\varkappa-1)| + |I_{n-\varkappa-1}(\varkappa)| \quad \text{if } \varkappa \geq 2, \tag{4.7}$$

$$|I_n(\varkappa)| = 0 \text{ if } \varkappa > (n-1)/2,\ n \geq 0. \tag{4.8}$$

Proof: (4.5) follows from Lemma 3.2. (4.6) is an easy consequence of (4.4) and Definition 4.2. As it is well-known from the additive number theory (cf. /6/),

$$p(s,r) = p(s-1, r-1) + p(s-r, r)$$

with the initial conditions p(s,r) = 1 if s = 0 = r and

$$p(s,r) = 0 \text{ if neither } s = 0 = r \text{ nor } 1 \leq r \leq s.$$

Hence, by (4.4), both (4.7) and (4.8) follow, q.e.d.

5. m.n.-H. graphs with non-positive scattering number

m.n.-H. graphs are precisely those graphs which are non-Hamiltonian and contain Hamiltonian paths between each pair of non-adjacent vertices. Consequently, homogeneously traceable graphs among m.n.-H. graphs G_n with $n \geq 3$ are precisely those G_n with $\Delta(G_n) < n-1$. Hence we obtain

Lemma 5.1: $G \in T_n$ iff $G \in M_n$ and $\Delta(G) < n-1$.

Theorem 5.1: If $G \in T_n$ then $\Delta(G) \leq n-4$.

A more general result is proved independently in /4/ and /18/. The simple proof given in /18/ is suggested by Lemma presented in /10/. Also the following theorem follows from a result found independently by the three authors of /4/ and the present author.

Theorem 5.2: $T_n \neq \emptyset$ iff $n \geq 9$.

Skupień proved that $|T_9| = 1$. The unique element of T_9 consists of three mutually disjoint triangles with 6 new edges which form two disjoint triangles. So if $G \in T_9$ then $\Delta(G) = 4$. Theorem 5.1 can be improved.

Theorem 5.3 (Skupień): There is $G \in T_n$ with $\Delta(G) = n-4$ iff $n \geq 10$.

From Theorem 4.1 it follows that $\Delta(G) = n-1$ if $G \in I_n$. Hence, owing to Lemmas 5.1 and 3.1, and definitions of I_n, T_n, and J_n, we obtain

Theorem 5.4: Classes I_n, T_n, and J_n are mutually disjoint. Moreover, if $G \in M_n$ then $\Delta(G) = n-1$ iff $G \in I_n \cup J_n$ as well as $s(G) \leq 0$ iff $G \in T_n \cup J_n$. Furthermore,

$$J_n = \{G \in M_n:\ s(G) \leq 0 \text{ and } \Delta(G) = n-1\}.$$

Definition 5.1: Let K_n^{+3} denote the complete graph K_n together with 3 independent hanging edges ($n \geq 3$).

Lemma 5.2: If $n \geq 7$, the graph $F_n: = K_1 * K_{n-4}^{+3}$ belongs to J_n.

In fact, it is easily seen that $F_n \in M_n$, $\Delta(F_n) = n-1$, and $F_n \notin I_n$.

6. Generating 2-connected m.n.-H. graphs

Lemma 6.1: *If G is a 1-connected m.n.-H. graph and $G \neq K_2$ then $G \in I_n$.*

Proof: In fact, then $s(G) \geq 1$.

Since graphs of the class I_n have been completely described, only 2-connected m.n.-H. graphs (which do not belong to I_n) are of interest. Obviously, if $\chi(G) \geq 2$ then G contains a minimal block as a factor. Consequently, all 2-connected m.n.-H. graphs G_n can be obtained from the catalog of all minimal non-Hamiltonian blocks of order n by adding new edges. Since such catalogs do exist /7/, therefore this idea led the present author to the following results which together with formulas (4.1), (4.2) and Definition 5.1 gave lists of $G \in M_n$ with $n \leq 7$.

$$M_n = J_n = \{K_n\} \quad \text{if} \quad 1 \leq n \leq 2,$$
$$M_n = I_n \quad \text{if} \quad 3 \leq n \leq 6,$$
$$M_7 = I_7 \cup J_7 \quad \text{with } J_7 = \{K_1 * K_3^{+3}\}.$$

To produce such lists for bigger n, a computer was used /8/. Results are summarized in the following table. The numbers marked by the star * are due to /8/.

n	$\lvert M_n\rvert$	$\lvert I_n\rvert$	$\lvert T_n\rvert$	$\lvert J_n\rvert$
1	1	-	-	1
2	1	-	-	1
3	1	1	-	-
4	1	1	-	-
5	3	3	-	-
6	3	3	-	-
7	7	6	-	1
8	9*	7	-	2*
9	18*	11	1	6*
10	31*	13	4*	14*

Literature

/1/ Appel, K. and Haken, W., Every planar map is four colorable. Part I: discharging, Illinois J. Math., 21, 429-490 (1977).

/2/ Appel, K., Haken, W., and Koch J., Every planar map is four colorable. Part II: reducibility, ibid., 491-567 (1977).

/3/ Bondy, J. A., Variations on the Hamiltonian theme, Canad. Math. Bull., 15, 57-62 (1972).

/4/ Chartrand, G., Gould, R.J. and Kapoor, S. F., On homogeneously traceable nonhamiltonian graphs, preprint.

/5/ Chvátal, V., New direction in Hamiltonian graph theory. In: Harary, F. (Ed.), New Directions in the Theory of Graphs, 65-95 (p. 74), New York and London, 1973.

/6/ Hardy, G. H. and Wright, E. M., An Introduction to the Theory of Numbers, London and New York, 1960.

/7/ Hobbs, A. M., A catalog of minimal blocks, J. Res. Nat. Bur. Standards, 77 B, 53-60 (1972).

/8/ Jamrozik, J., Kalinowski, R., and Skupień, Z., A catalog of small maximal non-Hamiltonian graphs, preprint, 1978. DM 39 (1982) 229-234.

/9/ Jung, H. A., On a class of posets and the corresponding comparability graphs, J. Combinat. Theory, B24, 125-133 (1978).

/10/ Lesniak-Foster, L., On critically hamiltonian graphs, Acta Math. Acad. Sci. Hungar. 29, 255-258 (1977).

/11/ Lihtenbaum, L. M., On Hamiltonian cycles of a connected graph [Russian], Kibernetika, Kiev, 1968, No 4, 72-75 (1978).

/12/ Nash-Williams, C. St. J. A., Valency sequences which force graphs to have Hamiltonian circuits: Interim report, Univ. of Waterloo, Canada.

/13/ Ore, O., Note on Hamilton circuits, Amer. Math. Monthly 67, 55 (1960).

/14/ Ore, O., Arc covering of graphs, Ann. Mat. Pura Appl., 55, 315-321 (1961).

/15/ Skupień, Z., A short proof of the theorem on the structure of maximal graphs without 1-factors, Prace Nauk. Inst. Mat. Fiz. Teoret. Politech. Wrocław. Nr 9, Ser. Studia i mater., Nr 9, Grafy i hipergrafy, 9-14 (1973).

/16/ Skupień, Z., Partitions and counting isomorphism types of certain maximal graphs without 1-factors, Demonstratio Math., 8, 447-463 (1975).

/17/ Skupień, Z., Homogeneously traceable and Hamiltonian connected graphs, preprint, 1976. Demonstratio Math. 17(1984) 1051-1067.

/18/ Skupień, Z., Maximum degree among vertices of a non-Hamiltonian homogeneously traceable graph, preprint, 1978. Lect. Notes in Math. 885, Springer 1981, 496-500.

/19/ Skupień, Z. and Wojda, A. P., Extremal non-(p,q)-Hamiltonian graphs, preprint, 1974. Studia Sci. Math. Hung. 10 (1975) 323-328.

/20/ Vrba, A., Counting hamiltonian circuits, in: Borowiecki, M., Skupień, Z. and Szamkołowicz, L. (Eds.), Graphs, Hypergraphs and Block Systems, (Proc. Symp. held in Zielona Góra, 1976), 319-323, Zielona Góra 1976.

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