Inducing regulation of any digraphs

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Abstract

For a given structure D (digraph, multidigraph, or pseudodigraph) and an integer r large enough, a smallest inducing r-regularization of D is constructed. This regularization is an r-regular superstructure of the smallest possible order with bounded arc multiplicity, and containing D as an induced substructure. Sharp upper bound on the number, ρ, of necessary new vertices among such superstructures for n-vertex general digraphs D is determined, ρ being called the inducing regulation number of D. For $\tilde{\Delta}(D)$ being the maximum among semi-degrees in D, simple n-vertex digraphs D with largest possible ρ are characterized if either $r \geq \tilde{\Delta}(D)$ or $r = \tilde{\Delta}(D)$ (where the case $r = \tilde{\Delta}$ is not a trivial subcase of $r \geq \tilde{\Delta}$).

1 Introduction

Different notions of regulation of a simple graph have been considered by a number of authors. *Regulation* of a graph consist in embedding the graph into a regular graph or a regular multigraph. We are interested in so-called *inducing regulation* in which a given structure (a general graph or a general digraph) is embedded as an induced substructure.

Let G be a given structure. A result of regulation of G is called a *regularization* of G. For an integer r large enough, an r-regular superstructure containing G as an induced substructure is called an *inducing r-regularization* (or simply *inducing regularization*) of G. Thus inducing regularization involves additional new vertices such that each new *line* (edge or arc) is necessarily incident to a new vertex. This contrasts with Berge's notion of *regularisable graphs* [4, 5, 6] (see also Jaeger and Payan [14]) wherein only old edges can be multiplied (replaced by parallel edges) in order to produce a regular spanning supermultigraph.

Another variant of regulation of a simple graph G with maximum degree Δ is studied in [1, 2, 8]. In this case not only new vertices but also new edges joining old nonadjacent vertices are allowed in order to construct a smallest Δ-regular simple supergraph of G. Then the number $r(G)$ of new vertices which must be added to G is studied under the name of *regulation number* of G in Akiyama and Harary [2]. An algorithm for determining $r(G)$ is also mentioned there. The sharp upper bound on $r(G)$ is found in Akiyama *et al.* [1]. Namely, for graphs G with given $\Delta = \Delta(G)$, either $r(G) \leq \Delta + 2$ if Δ is odd or $r(G) \leq \Delta + 1$ if Δ is even. Algorithmic aspects of determining both $r(G)$ and the smallest Δ-regular supergraph of G are presented in Bodlaender *et al.* [8] (without quoting [2]). If $r(G) = 0$ then G is called Δ-*regularizable* [8] and this notion is in opposition to regularisable graphs in the sense of Berge. The two notions are totally unrelated (no one or its negation implies or excludes the other). For example, the path P_3 is 2-regularizable and is not regularisable, the join $2K_1 \star (K_1 \cup K_2)$ is regularisable and is not 3-regularizable, the complete bipartite graph $K_{2,3}$ is neither regularisable nor 3-regularizable, and the graph obtained from the cycle C_6 by adding a diagonal is both regularisable and 3-regularizable.

When dealing with regulation, one can impose extra requirements on regular superstructure. In case of an inducing regulation of a given simple graph it is especially restrictive to require that a regular superstructure be strongly regular. This kind of superstructure (namely, *inducing strong regularization* in our language) is studied in several articles (e.g. [7, 15]) and Jajcay and Mesner [15] have succeeded in providing a construction which is polynomial for all given simple graphs. Namely, inducing strong regularization based on Desarguesian affine plane geometries in [15] has $O(n^4)$ vertices for any n-vertex graph. Moreover, it is noted in [15] that a smaller inducing strong regularization, if any, can be obtained only by using another family of strongly regular graphs.

Inducing regulation of multigraphs is originated by König [16, 17] as early as 1916. Specification for simple graphs appears in Chartrand and Lesniak's book [9]. Optimal (i.e., the smallest) inducing graphical regularization (within simple graphs) with a fixed maximum degree is characterized by Erdős and Kelly's theorem [10, 11], found about 40 years ago. Their result is a straightforward corollary in our paper [12]. The most general results on optimal inducing regulation of graphs, multigraphs and pseudographs are presented in [12].

In this paper it is proved that the smallest number of new vertices among all inducing r-regularizations of a general n-vertex digraph D with a fixed upper bound p on arc multiplicity is at most $\max\{\lceil r/p\rceil, n\}$. Moreover, the smallest inducing r-regularization $F(D)$, i.e. inducing r-regularization with minimum number of vertices, is constructed. Next all n-vertex simple digraphs D with largest number $n+\max\{r,n\}$ of vertices in the smallest inducing r-regularization of D are characterized. Similarly, all of digraphs D are characterized in case when a smallest regularization preserves the maximum, $\tilde{\Delta}$, among semi-degrees in D and the number of new vertices necessarily is the largest possible (and is n or $n-1$ depending on $\tilde{\Delta}$). The latter characterization (for $r=\tilde{\Delta}$) is thus not a special case of the former one (for $r \geq \tilde{\Delta}$), which rather trivially is not a surprise.

2 Inducing regulation number

Digraphs are finite and simple, *multidigraphs* without loops, and *pseudodigraphs* may contain loops and multiple arcs. Similarly we differentiate between graphs, multigraphs, and pseudographs. Our special symbol K_n^o stands for a complete pseudograph on n vertices (with edge multiplicity 1). For undefined terminology and notation we refer to Chartrand and Lesniak [9].

Speaking about a general structure (general graph or general digraph), we have in mind a $\mathcal{P}$-structure (*pseudostructure*, with loops allowed) or an $\mathcal{M}$-structure (*multistructure*, loopless). Therefore we possibly speak about an X-structure where X is a variable, $X \in \{\mathcal{M}, \mathcal{P}\}$. In fact, we shall use names X-graph and X-digraph. Define X_r^p to be the class of X-structures F where r is an upper bound on maximum degree (in case of graphs F) and maximum semi-degree (in case of digraphs F), p being the upper bound on line multiplicity in F. Hence X_r^p stands for $\mathcal{M}_r^p$ or $\mathcal{P}_r^p$. Given an X-structure G and large enough integers r and p, an r-regular X_r^p-structure F is called *X_r^p-regularization* (or *inducing regularization within X_r^p*) of G if F contains G as an induced substructure. We define the *X_r^p-regulation number* (or *inducing regulation number*) of G, in symbols $\rho(G, X_r^p)$, to be the smallest number of new vertices among all X_r^p-regularizations of G. We have tacitly assumed that lines of G, if exist, determine whether X_r^p-regularization of G is to be graphical or digraphical. If $G=\overline{K_n}$, the notion of X_r^p-regularization of G in case $r>0$ is ambiguous. Therefore we then use symbols $\vec{\rho}$ and ρ in order to differentiate between respectively digraphical and graphical regularization of $\overline{K_n}$, e.g., $\vec{\rho}(\overline{K_{2n}}, \mathcal{M}_{2n+1}^1) = 2n+1$ and $\rho(\overline{K_{2n}}, \mathcal{M}_{2n+1}^1) = 2n+2$ can be seen. Using the variable symbol X enables us to state our results jointly on $\mathcal{M}$-structures and on $\mathcal{P}$-structures.

Our new results are very much like the former ones on undirected general graphs.

Theorem 0 ([12]) *Given an X-graph G on n vertices with minimum and maximum degrees δ and Δ, let p and r be integers such that $r \geq \Delta$ and p is an upper bound on the maximum edge multiplicity in G. The inducing regulation number $\rho(G, X_r^p)$ of G is the least nonnegative integer t such that, for $\sigma = \sum_{v\in V(G)} (r - \deg_G(v))$,* (i‘) $tr \geq \sigma$;

(ii‘) $pt \geq r - \delta$;

(iii‘) $(t+n)r$ *is an even integer;*

(iv‘) *either* $pt^2 - (r+p)t + \sigma \geq 0$ *if* $X = \mathcal{M}$ *or* $pt^2 - (r-p)t + \sigma \geq 0$ *if* $X = \mathcal{P}$.

$$\textit{Moreover,} \quad \rho(G, X_r^p) \leq \begin{cases} \lceil r/p \rceil + 1 & \textit{if both } r \textit{ and } \lceil r/p \rceil + n \textit{ are odd,} \\ & \lceil r/p \rceil > n, \textit{ and } \delta < r + p - p\lceil r/p \rceil, \\ \max\{\lceil r/p \rceil, n\} & \textit{otherwise.} \end{cases}$$

The *order* and *size* of an X-digraph are the number of vertices and that of arcs, respectively. The ordered pair $(\mathrm{od}_D(v), \mathrm{id}_D(v))$ of *semi-degrees* of a vertex v (the outdegree, $\mathrm{od}_D(v)$, followed by the indegree, $\mathrm{id}_D(v)$) is called the *degree pair* of v. Note that the sum of outdegrees and that of indegrees over all vertices of an X-digraph coincide. An X-digraph is called *diregular* or r-*diregular* if all its vertices have the same degree pair (r,r) $(r = 0, 1, 2, \ldots)$. The number of arcs from vertex v to vertex u is called the *arc multiplicity* of the ordered vertex pair (v, u). Notice that if an r-diregular X-digraph with any arc multiplicity bound p has n vertices then $r \leq p(n-1)$ if $X = \mathcal{M}$ and $r \leq pn$ if $X = \mathcal{P}$. Let ${}^p\mathcal{D}K_n$ (resp. ${}^p\mathcal{D}K_n^o$) be the complete n-vertex $\mathcal{M}$-digraph ($\mathcal{P}$-digraph) with arc multiplicity p; $\mathcal{D}K_n$ and $\mathcal{D}K_n^o$ stand for ${}^1\mathcal{D}K_n$ and ${}^1\mathcal{D}K_n^o$, respectively.

Let D be an X-digraph of order n such that the maximum and minimum among semi-degrees in D are $\tilde{\Delta}$ and $\tilde{\delta}$, respectively. Assume that $V = \{v_1, v_2, \ldots, v_n\}$ is the vertex set of D. Given any integer $r \geq \tilde{\Delta}$, the differences

$$a_i^+ := r - \mathrm{od}_D(v_i) \qquad \text{and} \qquad a_i^- := r - \mathrm{id}_D(v_i)$$

are the r-*semi-deficiencies* (called r-*out-deficiency* and r-*in-deficiency*, respectively) of the ith vertex v_i in D. Then $r - \tilde{\delta}$ is the *maximum* r-*semi-deficiency* among vertices of D. Let

$$\sigma^+ = \sum_i a_i^+$$

be the sum of r-out-deficiencies, which clearly is the sum, say σ^-, of r-in-deficiencies; $\sigma^+ = \sigma^-$. Given a parameter ξ of D (e.g., $\xi = \tilde{\delta}, \tilde{\Delta}, \sigma^+$) we replace ξ by $\xi(D)$ in case we want to avoid ambiguity.

Theorem 1 *Let D be an X-digraph of order n, with vertex semi-degrees at most r $(r \geq \tilde{\Delta})$ and with arc multiplicity at most p. The inducing regulation number $\rho(D, X_r^p)$ of D is the least nonnegative integer t such that*

(i) $tr \geq \sigma^+$;

(ii) $pt \geq r - \tilde{\delta}$

(iii) *either*

(M) $pt^2 - (r+p)t + \sigma^+ \geq 0$ *if* $X = \mathcal{M}$ *or*

(P) $pt^2 - rt + \sigma^+ \geq 0$ *if* $X = \mathcal{P}$

where $\tilde{\delta} = \tilde{\delta}(D)$, $\sigma^+ = \sigma^+(D)$. The following bound is sharp.

$$\rho(D, X_r^p) \leq \max\{\lceil r/p \rceil, n\}.$$

3 Proof of the main result

Proof of Theorem 1. Let F be a smallest X_r^p-regularization of D, $D \subseteq F$. Then $F = D$ if D is r-diregular, that is, if $\sigma^+ = 0$. On the other hand, for $\sigma^+ = 0$, $t = 0$ satisfies all requirements of Theorem 1. Consider the case $\sigma^+ > 0$. Let $H = F - V(D)$ and let $t = |V(H)|$.

Necessity. The total number of arcs in F with terminal vertices in the sub-X-digraph H is tr and cannot be smaller than the number of arcs, σ^+, from D to H, i.e., (i) follows. On the other hand, the number tr does not exceed $\sigma^+ + t(t-1)p$ for $X = \mathcal{M}$ and $\sigma^+ + t^2p$ for $X = \mathcal{P}$ where $t(t-1)p$ and t^2p are the maximum numbers possible of arcs in H itself for $X = \mathcal{M}$ and for $X = \mathcal{P}$, respectively. This gives (M) and (P). The largest semi-deficiency at a vertex of D, which is equal to $r - \tilde{\delta}$, forces condition (ii). So all three conditions are necessary.

Sufficiency. Let t be the least positive integer satisfying conditions (i)–(iii). Assume that $V = \{v_1, v_2, \ldots, v_n\} = V(D)$. Let U be a set of t extra vertices $u_1, u_2, \ldots, u_t$. Let B be a bipartite V–U multidigraph which, for each v_i, comprises a_i^+ arcs from v_i to U as well as a_i^- arcs from the set U to the vertex v_i. Thus $2\sigma^+$ is the number of all V–U arcs. Define a digraph F to be the arc-disjoint union of X-digraphs D, B and H where H is induced by the set U. Assume that V–U arcs make up a sequence $\mathcal{A}$ such that each arc directed from V precedes each arc directed to V. Moreover, for each i $(\leq n-1)$, arcs directed from v_{i+1} follow all those from v_i, and similarly, arcs directed to v_i precede all those to v_{i+1}. In order to establish incidence of V–U arcs with vertices of U, consider the auxiliary sequence of vertices $(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_t, \ldots, \tilde{u}_{2\sigma^+})$, where $\tilde{u}_j = u_k$ if $j \equiv k \pmod{t}$ for $j = 1, \ldots, 2\sigma^+$ and $k = 1, \ldots, t$. Assume that the jth vertex $\tilde{u}_j$ is made incident to the jth arc of $\mathcal{A}$. Thus arcs of B both those from any $v \in V$ to U and ones oppositely directed, are evenly distributed among vertices in U. This fact together with condition (ii) imply that arc multiplicities in B do not exceed p. Furthermore, $F = D \cup B$ if equality holds in condition (i). For instance, this is the case if both $t = 1$ and $X = \mathcal{M}$.

Otherwise strong inequality holds in (i), $tr > \sigma^+$, and additionally $t > 1$ if $X = \mathcal{M}$. Then H must contain some U–U arcs. If $t = 1$ and $X = \mathcal{P}$, then we add $r - \sigma^+$ $(\leq p$ by $(P))$ loops to the vertex u_1 in order to get a required H. Assume that $t \geq 2$ and $X \in \{\mathcal{M}, \mathcal{P}\}$. We define nonnegative integers h and s,

$$h := \left\lfloor \frac{\sigma^+}{t} \right\rfloor, \qquad s := \sigma^+ - ht$$

whence $\sigma^+/t = h + s/t$ where $s < t$. Hence $h < r$ because strong inequality in (i) is assumed. Then $\mathrm{id}_B(u_j) = h+1$ for $j = 1, 2, \ldots, s$; otherwise $\mathrm{id}_B(u_j) = h$. On the other hand, $\mathrm{od}_B(u_k) = h+1$ for all s consecutive vertices in the auxiliary sequence $\tilde{u}_{ht+s+1}$ $(= u_{s+1})$, $\tilde{u}_{ht+s+2}, \ldots, \tilde{u}_{ht+2s}$. This is clearly the s-subsequence $u_{s+1}, u_{s+2}, \ldots, u_{2s}$ of $(u_k)_{k=1,\ldots,t}$ if $s \leq t/2$. Otherwise, if $2s > t$, those vertices u_k make up the terminal $(t-s)$-section $u_{s+1}, \ldots, u_t$ of (u_k) and the initial $(2s-t)$-section $u_1, u_2, \ldots, u_{2s-t}$. For the remaining vertices u_k, $\mathrm{od}_B(u_k) = h$.

Table 1: Distribution of degree pairs in B among vertices u_k in U

degree pair	for $2s > t$	for $2s \leq t$
$(h+1, h+1)$	$k = 1, \ldots, 2s-t$	$k \in \emptyset$
$(h, h+1)$	$k = 2s-t+1, \ldots, s$	$k = 1, \ldots, s$
$(h+1, h)$	$k = s+1, \ldots, t$	$k = s+1, \ldots, 2s$
(h, h)	$k \in \emptyset$	$k = 2s+1, \ldots, t$

In order to construct a required X-digraph H we shall refer to Table 1. By definitions of h and s, and by condition (iii),

$$r - h - s/t = r - \sigma^+/t \leq \begin{cases} (t-1)p & \text{if} \quad X = \mathcal{M}, \\ tp & \text{if} \quad X = \mathcal{P}. \end{cases} \tag{1}$$

Because $0 \le s/t < 1$ and remaining terms in (1) are integers, one has $r-h \le (t-1)p$ if $X = \mathcal{M}$ and $r-h \le tp$ if $X = \mathcal{P}$. Hence the greatest remaining semi-deficiency $r-h$ among vertices u_k in B can be covered up in H.

Recall that the complete symmetric loopless digraph $\mathcal{D}K_t$ can be decomposed into $t-1$ (1,1)-factors of which all are Hamilton dicycles unless t is even and $t \ge 4$, and then precisely one (1,1)-factor is $(t/2)\ \mathcal{D}K_2$. The existence of such a decomposition follows easily from the well-known solution to Kirkman's problem on packing Hamilton cycles into complete (undirected) graph K_t, cf. Berge [3]. The (rotational) solution presented in Berge appears already in Lucas [18] of 1883 wherein no reference to Kirkman is made. Recall that the packing in question is an edge decomposition of K_t if t is odd and that of K_t minus a perfect matching if t is even. Then, while passing on from K_t to $\mathcal{D}K_t$, (1,1)-factors of the digraph arise if each nonloop edge is split into two opposite arcs and each Hamilton cycle of K_t is split into the union of two arc-disjoint Hamilton dicycles of $\mathcal{D}K_t$. Moreover, the involved perfect matching of K_t for even t is transformed into (1,1)-factor $(t/2)\ \mathcal{D}K_2$. Additionally, replacing each loop in the complete pseudograph, denoted K_t^o, by a single directed loop transforms each 2-factor made up of all loops in K_t^o into a (1,1)-factor made up of all directed loops in $\mathcal{D}K_t^o$.

The following decompositions will be useful.

$$^p\mathcal{D}K_t = L_1 \oplus \ldots \oplus L_p, \qquad ^p\mathcal{D}K_t^o = L_1^o \oplus \ldots \oplus L_p^o$$

where L_i and L_i^o are isomorphic to $\mathcal{D}K_t$ and $\mathcal{D}K_t^o$, respectively, $i = 1, 2, \ldots, p$. Define

$$a := \begin{cases} \left\lfloor \frac{r-h}{t-1} \right\rfloor & \text{for} \quad X = \mathcal{M}, \\ \left\lfloor \frac{r-h}{t} \right\rfloor & \text{for} \quad X = \mathcal{P}, \end{cases} \qquad b := \begin{cases} r-h \bmod t-1 & \text{for} \quad X = \mathcal{M}, \\ r-h \bmod t & \text{for} \quad X = \mathcal{P}. \end{cases}$$

Hence, by (1), $a \le p$ if $b = 0$ and $a + 1 \le p$ if $b > 0$.

Case $s = 0$. Then all vertices u_k have the same degree pair (h, h) in the digraph B, see Table 1. If $b = 0$, we put

$$H = \begin{cases} L_1 \oplus \ldots \oplus L_a & \text{for} \quad X = \mathcal{M}, \\ L_1^o \oplus \ldots \oplus L_a^o & \text{for} \quad X = \mathcal{P}. \end{cases}$$

Otherwise we let W_j and W_j^o be arc-disjoint (1,1)-factors of L_{a+1} and L_{a+1}^o, respectively, $i = 1, \ldots, b$. That many (1,1)-factors, namely b, exist because b is small enough. Then we put

$$H = \begin{cases} L_1 \oplus \ldots \oplus L_a \oplus W_1 \oplus \ldots \oplus W_b & \text{for} \quad X = \mathcal{M}, \\ L_1^o \oplus \ldots \oplus L_a^o \oplus W_1^o \oplus \ldots \oplus W_b^o & \text{for} \quad X = \mathcal{P}. \end{cases}$$

Case $s > 0$. Let $\tilde{H}$ stand for H constructed above in Case $s = 0$. Thus $\tilde{H}$ comprises $r-h$ arc-disjoint (1,1)-factors of $^p\mathcal{D}K_t$ or $^p\mathcal{D}K_t^o$ (depending on X). Therefore each semi-degree $h+1$ of any u_k listed in Table 1 becomes $r+1$ in $\tilde{F} := D \cup B \cup \tilde{H}$. We can assume that one of the (1,1)-factors in question is a specified Hamilton dicycle, say $\tilde{W}$. We claim that the required digraph H is obtainable from $\tilde{H}$ by removal of s arcs of a $\tilde{W}$. We specify $\tilde{W}$ as follows. In case $2s > t$, we only assume that $\tilde{W}$ contains a matching comprising $t-s$ arcs $u_{2s-t+j} \to u_{s+j}$ with $j = 1, 2, \ldots, t-s$. Let $2s \le t$. Then we assume that $\tilde{W}$ includes both a matching comprising $s-1$ arcs $u_j \to u_{s+j}$ with $j = 1, 2, \ldots, s-1$ and the disjoint dipath (with $t-2s+1$ arcs) $u_s \to u_{2s+1} \to u_{2s+2} \to \cdots \to u_t \to u_{2s}$, the dipath reduces to the single arc $u_s \to u_{2s}$ if $t = 2s$. Due to Table 1, removing from $\tilde{H}$ all s arcs of $\tilde{W}$ which are not specified above really gives the required H.

The upper bound and its sharpness. Recall the assumption $\sigma^+ > 0$. Then, for $n = 1$, $t = \lceil r/p \rceil \ge 1$. Let $n \ge 2$.

Claim 1 *Condition (M) holds for $t \ge n$ if $r \le 2pn$, otherwise for $t \ge \lceil r/p \rceil + 1 - n$.*

Proof. Let $L(t)$ stand for the left-hand side of the inequality (M). Then $L(n) \geq 0$ because $L(n) = (\sigma^+ - n(r - \tilde{\Delta})) + n((n-1)p - \tilde{\Delta})$ is the sum of two nonnegative summands . Hence (M) holds for $t = n$. Assume that $r \leq 2pn$ and note that $L(t)$ is the quadratic trinomial in t, which attains its minimum at $\tau := (r/p + 1)/2 \leq n + 1/2$. Therefore for any integer $t \geq n + 1$, $L(t) > L(n) \geq 0$. Otherwise $r > 2pn$ whence $\tau > n + 1/2$. Therefore $L(t) \geq 0$ for $t \leq n$. Hence, since $L(t)$ is symmetrical with respect to $t = \tau > n$, $L(t) \geq 0$ for each $t \geq n + 2(\tau - n) = r/p + 1 - n$. $\square$

Claim 2 *Conditions* (i)*–*(iii) *hold for* $t \geq \max\{\lceil r/p \rceil, n\}$.

Proof. Since $\sigma^+ \leq nr$, the condition (i) holds for $t \geq n$. Moreover, both conditions (ii) and (P) hold for $t \geq \lceil r/p \rceil$. This together with Claim 1 proves Claim 2. $\square$

It remains to prove sharpness. The following n-vertex X-digraphs D show that the upper bound on X_r^p-regulation number is sharp. For any X and any p, let D be an X-digraph with at most $r - 1$ arcs if $r/p \leq n$, otherwise let $\tilde{\delta}(D) < r + p - p\lceil r/p \rceil$. Then $\rho(D, X_r^p) = n$ if $r/p \leq n$, otherwise $\rho(D, X_r^p) = \lceil r/p \rceil$ if $r/p > n$. The proof is complete. $\square$

Remark 1 Given an X-digraph D, one of the smallest X_r^p-regularizations of D is constructed in the sufficiency part of the above proof.

4 Digraphs with largest inducing regulation number

Throughout this section, D stands for a simple digraph. Let $\mathcal{D}K_n^{-m}$ denote a digraph obtained from the complete symmetric digraph $\mathcal{D}K_n$ by removal of any m arcs. Recall that $\tilde{\Delta}$ and $\tilde{\delta}$ stand for the maximum and minimum semi-degrees in D.

Theorem 2 *Use notation of Theorem* 1 *with exception that* D *is a simple digraph (on* n *vertices). Then, for* $r > 0$*, the (inducing)* $\mathcal{M}_r^1$*-regulation number of* D *is the largest possible,* $\rho(D, \mathcal{M}_r^1) = \max\{r, n\}$*, if and only if any of the following four conditions* (α)*–*(δ) *holds.*

(α) $n \leq r$ *and* $\tilde{\delta} = 0$.

(β) $n = r \geq 4$, $\tilde{\delta} > 0$, *and* $D = \mathcal{D}K_n^{-m}$ *where* $1 \leq m \leq n - 3$ $(\tilde{\Delta} = n - 1)$.

(γ) $n = r + 1 \geq 3$ *and* $D = \mathcal{D}K_n^{-m}$ *where* $1 \leq m \leq n - 2$ $(\tilde{\Delta} = n - 1)$.

(δ) $n > r > \tilde{\Delta}$ *and* D *has at most* $r - 1$ *arcs.*

Proof. We refer to conditions (i), (ii), and (M) in Theorem 1, with $p = 1$.

Sufficiency. If (α) holds then $t \geq r \geq n$ by (ii). If (δ) holds then $\sigma^+ > (n-1)r$ and therefore $t \geq n$ by (i).

Claim A *Condition* (M) *with* $p = 1$ *holds for* $t \geq r + 1 - j$ *as well as for* $t \leq j$ *and does not hold for* $t \in [j + 1, \cdots, r - j]$ *if and only if*

$$j(r + 1 - j) \leq \sigma^+ < (j + 1)(r - j) \tag{2}$$

where $r > 0$, $j = 0, 1, \cdots, \lceil \frac{r}{2} \rceil - 1$.

Proof. Notice that condition (M), $t(t - r - 1) + \sigma^+ \geq 0$, is satisfied for $t = r + 1 - j$ if and only if it is satisfied for $t = j$. Moreover, (M) is satisfied for $t = r + 1 - j$ and is not satisfied for $t = r - j$ exactly if inequalities (2) hold. Since (M) is a quadratic inequality in t, the rest of the proof is easily seen. $\square$

If (β) holds then $r + 1 \leq \sigma^+ < 2r - 2$. Hence $t \geq 2$ by (i) and then condition (M) forces $t \geq r = n$ due to Claim A (with $j = 1$). If (γ) holds then $0 < \sigma^+ < r$ and therefore $t \geq r + 1 = n$ is forced by (M) due to Claim A (with $j = 0$).

Necessity. Let D be a simple digraph with $\mathcal{M}_r^1$-regulation number $\rho = \max\{r, n\}$. If $\tilde{\delta} = 0$ and $r \geq n$ then condition (ii) forces $t \geq r$ which implies (α). Suppose that $r > n \geq 2$ and $\tilde{\delta} > 0$. This leads to $\rho < r$, a contradiction, because conditions (i), (ii), and (M) (due to Claim 1 in proof of Theorem 1) are satisfied for $t \geq r - 1$ $(\geq n)$.

It remains to assume that $n \geq 2$, $r \leq n$, and $\tilde{\delta} > 0$ if $r = n$. Notice that condition (i) forces $t \geq n$ if and only if $\sigma^+ > (n-1)r$ or—equivalently—D is a digraph with at most $r - 1$ arcs. Hence $\tilde{\Delta} \leq r - 1$. Moreover, since D has less than n arcs, $\tilde{\delta} = 0$ whence $n > r$. Therefore condition (δ) is satisfied. It remains to identify requirements under which $t \geq n$ is forced by (M).

Claim B *If* $n \geq r$ *and* $\sigma^+ > 0$, *then condition* (M) *with* $p = 1$ *holds for*

$$t \geq \begin{cases} r & \text{if } \tilde{\Delta} < r, \\ \tilde{\Delta} + 1 & \text{otherwise } (r = \tilde{\Delta}). \end{cases}$$

Proof. For $r > \tilde{\Delta}$,

$$\sigma^+ \geq (r - \tilde{\Delta})n \geq r - \tilde{\Delta} + n - 1$$

whence $\sigma^+ \geq r$. Therefore condition (M) can be seen to hold for $t = r$. By Claim A (with $j = 0$) condition (M) is satisfied for $t \geq r + 1$. Hence, if $r = \tilde{\Delta}$ then (M) holds for $t \geq \tilde{\Delta} + 1$. □

Due to Claim B, condition (M) can force $t \geq n$ only when either $r = n$ (and $\tilde{\delta} > 0$) or $r = n - 1 = \tilde{\Delta}$. Hence, due to Claim A (with $j = 0, 1$), $t \geq n$ can be forced whenever $n = r \leq \sigma^+ < 2r - 2$ and $\tilde{\delta} > 0$ or $\sigma^+ \leq n - 2$ $(< r)$ and $r = n - 1 = \tilde{\Delta}$. Since $\sigma^+ > 0$ necessarily holds, the last case is equivalent to (γ) because $\sigma^+(\mathcal{D}K_n^{-m}) = m$. Consider the former case and assume that $r = n$, $\tilde{\delta} > 0$, and $n \leq \sigma^+ < 2n - 2$. Then $\sigma^+ > n$ because otherwise $D = \mathcal{D}K_n$ and all three conditions (i), (ii) and (M) are satisfied for $t = 1$. Moreover, if $\sigma^+ > n$ then $t \neq 1$ by condition (i). Thus condition (β) holds. □

Let $R_r(n)$ denote the *largest inducing regulation number* $\rho(D, \mathcal{M}_r^1)$ among n-vertex simple digraphs D with $\tilde{\Delta}(D) = r$.

Corollary 3 *For* $n \geq 2$ *and* $r > 0$, $R_r(n) \leq n$. *The only* n*-vertex simple digraphs* D *with* $\tilde{\Delta}(D) = r$ *and* $\rho(D, \mathcal{M}_r^1) = n$ *are those satisfying condition* (γ) *above, that is,* $D = \mathcal{D}K_n^{-m}$ *where* $n \geq 3$ *and* $1 \leq m \leq n - 2$ *whence* $\tilde{\Delta}(D) = n - 1$. □

Note that, for $n = 2$, $R_1(2) = \rho(\vec{P_2}, \mathcal{M}_1^1) = 1$.

Theorem 4 *Assume that* $n \geq 3$ *and* $0 < r \leq n - 2$. *Then* $R_r(n) = n - 1$. *Moreover,* D *is an* (*extremal*) n*-vertex digraph with* $\tilde{\Delta}(D) = r$ *and* $\rho(D, \mathcal{M}_r^1) = n - 1$ *if and only if either*

(α^*) *D has m arcs where $r \leq m \leq 2r - 1$ and there are r and not more than r of those arcs which are directed either to or from a vertex of D or*

(β^*) *$D = \mathcal{D}K_n^{-m}$ where $n \geq 4$, $n + 1 \leq m \leq 2n - 3$, and $(1,1)$-factor of $\mathcal{D}K_n$ is formed of n from among m removed arcs.*

Proof. We refer to conditions (i), (ii), and (M) with $p = 1$ in Theorem 1.

Necessity. Notice that condition (ii) holds for $t \geq n - 2 \geq r$. Hence $t \geq n - 1$ has to be forced by (i) and (M) only. Moreover, $\sigma^+ \leq (n-1)r$ for all digraphs D with $\tilde{\Delta}(D) = r$ whence condition (i) holds for $t \geq n - 1$. Then (i) forces $t \geq n - 1$ if and only if $\sigma^+ > (n-2)r$ or—equivalently—D is a digraph with at most $2r - 1$ (and at least r) arcs. Therefore condition (α^*) holds.

Due to Claim B, condition (M) forces $t \geq n - 1$ only when $r = n - 2$. Hence, due to Claim A (with $j = 0$), $t \geq n - 1$ can be forced by (M) if $\sigma^+ \leq n - 3$ $(< r)$. Since $\sigma^+ > 0$ necessarily holds, condition (β^*) is satisfied.

Sufficiency. If (β^*) holds then $\tilde{\Delta}(D) = r = n - 2$, $0 < \sigma^+ \leq n - 3 < r$, and therefore condition (M) forces $t \geq r + 1 = n - 1$ due to Claim A (with $j = 0$). If (α^*) holds then $(n-2)r < \sigma^+ \leq (n-1)r$ and $t \geq n - 1$ is forced by (i). Hence $R_r(n) = n - 1$ since, by Corollary 3, $R_r(n)$ is not larger. □

5 Concluding remarks

Some remarks on complexity of our main results, presented in Theorem 1 and Section 3, follow. We assume that a representation of an X_r^p-digraph D together with corresponding parameters $\tilde{\delta}$, $\tilde{\Delta}$, σ^+ which fit to constants r, p are included in the input. Therefore the complexity of determining the inducing regulation number $\rho = \rho(D, X_r^p)$ reduces to the time complexity of a few operations solving inequalities (i)–(iii). The only problem is that possibly large numbers, e.g. σ^+, are involved. However, some simplifications are possible. For instance, the inequality (iii) holds in case $\sigma^+ \geq (r+p)^2/4p$ or $\sigma^+ \geq r^2/4p$ (depending on $X = M$ or P).

Due to Remark 1, in order to estimate the time complexity of constructing a smallest (inducing) X_r^p-regularization, say F, of D we refer to the sufficiency part of the related proof, see Section 3. Recall that $F = D \cup B \cup H$. It can be seen that the number of steps which are required to construct digraphs B and H is proportional to $2\sigma^+$ (with $\sigma^+ \leq nr$) and $r\rho - \sigma^+$, respectively. Hence the construction can be completed in time proportional to $(n+\rho)r$, the size of the output.

The construction of the analogous X-structure H in case of the undirected X-graph G (see Theorem 0) is a bit simpler. Details are presented in [13].

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