DETERMINATION OF DEATH COEFFICIENT AND STABILITY IN POPULATION-TYPE EQUATION

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ABSTRACT

The inverse problem of determination of the death-like function λ in a general population-type equation $\rho_t + c\rho_a + \lambda\rho = 0$ is considered. Two cases are treated: the linear case when λ is a function of age and the nonlinear one when λ depends on the total population. The employed methods provide us with the uniqueness of the solution of the inverse problem. In autonomous case, the stability result is obtained.

INTRODUCTION

In this paper we consider the problem of determining the unknown coefficient λ (which in the linear case is of the form $\lambda(a)$ and in the nonlinear case is of the form $\lambda(t, a, \rho)$) in the first order hyperbolic equation

$$\rho_t(t,a) + c(t,a)\rho_a(t,a) + \lambda\rho(t,a) = 0. \tag{1}$$

Such an equation is ubiquitous, but we can have in mind particular applications. If $c(t, a)$ is a constant function, then the density $\rho(t, a)$ denotes the number of individuals of age a at a particular time t. In this case λ denotes a *death function* (see [PR], [ERS] and [R] for details). We can also consider the density ρ of the population with respect to the maturation level of individuals rather than to age. In this case the appearance of an additional, not necessarily constant term $c(t, a)$ in front of ρ_a is justified. It stands for the velocity of the maturation of individuals at time t and at maturation level a. For some applications of this model to the description of maturation-proliferation process of the erythroid cells in the blood, we refer to [L] and [LMWC] where similar equation is studied. The first paper shows the existence and uniqueness of the solutions of the problem as well as some additional conditions which guarantee stable or chaotic behaviour of solutions. The second paper shows an application of the studied model in the therapy of anemia.

Yet another interpretation of (1) in the material theory could be that $\rho(t,a)$ is the 'density' of parts of age a still in use at time t and $c(t,a)$ is the age-dependent rate of failures. Equation (1) is also treated in papers [DL] and [D] where some existence and uniqueness results are obtained.

In this paper we will assume that the term $c(t,a)$ appearing in (1) is of the form

$$c(t,a) = g(t)h(a) \tag{2}$$

for two appropriate functions g and h.

RECOVERY OF $\lambda = \lambda(a)$ (LINEAR CASE)

We consider the following inverse problem

$$(IP_1)\begin{cases} \text{Find a pair of functions } (\rho,\lambda) \text{ satisfying} \\ (DP_1)\begin{cases} \rho_t + c(t,a)\rho_a + \lambda(a)\rho = 0 & \\ \rho(0,a) = \varphi(a) & \forall a \in [0,L] \\ \rho(t,0) = B(t) & \forall t \in [0,T] \end{cases} \\ \rho(T,a) = \psi(a) \qquad\qquad\qquad\qquad \forall a \in [0,L] \end{cases}$$

The above problem consists of two parts: the direct problem (DP_1), containing the population-type equation, initial condition $\varphi(a)$ and boundary condition $B(t)$ and the so called overposed condition containing ψ, which corresponds to the value of $\rho(t,a)$ at the final time T. We will assume that c is given by (2) and that g, h, φ, ψ, B are positive functions, of class C^1 if necessary. By integrating the first equation in (DP_1) along characteristic coordinates, we obtain

$$\rho(t,a) = \begin{cases} B(\overline{G}(t,a)) \exp\left(-\displaystyle\int_0^a \frac{\lambda(s)}{g(\Gamma(s,t,a))h(s)}\, ds\right) & \text{if } a < H^{-1}(G(t)) \\ \varphi(\overline{H}(t,a)) \exp\left(-\displaystyle\int_{\overline{H}(t,a)}^a \frac{\lambda(s)}{g(\Gamma(s,t,a))h(s)}\, ds\right) & \text{if } a \geq H^{-1}(G(t)), \end{cases}$$

where

$$\begin{aligned} G(t) &= \int_0^t g(s)ds \\ H(a) &= \int_0^a \frac{ds}{h(s)} \\ \overline{G}(t,a) &= G^{-1}(G(t) - H(a)) \\ \overline{H}(t,a) &= H^{-1}(H(a) - G(t)) \\ \Gamma(s,t,a) &= G^{-1}(H(s) - H(a) + G(t)) \end{aligned}$$

for $t \in [0,T]$ and $a \in [0,L]$. Now we can recover λ explicitly. If $T < L$ we obtain

$$\lambda(a) = \begin{cases} g(T)h(a)\frac{d}{da}\ln\frac{B(\overline{G}(T,a))}{\psi(a)} & \text{if } a < H^{-1}(G(T)) \\ \frac{g(T)h(a)}{g(0)h(\overline{H}(T,a))}\lambda(\overline{H}(T,a))\frac{d}{da}\ln\frac{\varphi(\overline{H}(T,a))}{\psi(a)} & \text{if } a \geq H^{-1}(G(T)), \end{cases}$$

and if $T \geq L$ only the first of above equations is required.

RECOVERY OF $\lambda = \lambda(t, a, \rho)$ (NONLINEAR CASE)

Now we will look for a λ in the following form

$$\lambda(t, a, \rho) = \lambda_N(a)[1 + f(P(t))]$$

where λ_N is a known function and

$$P(t) = \int_0^L \rho(t, a)\, da$$

denotes the total number of individuals at time t. Since the unknown function f appears only in the form $f(P(t))$, we will make the change of variable $\widetilde{f}(t) = f(P(t))$. The assumptions on P will suffice to recover f uniquely from $\widetilde{f}$. Now we can write the following inverse problem

$$(IP_2) \begin{cases} \text{Find } (\rho, \widetilde{f}) \in C^1((0,T) \times (0,L)) \times C([0,T];[0,M]), \text{ such that} \\ (DP_2) \begin{cases} \rho_t + c(t,a)\rho_a + \lambda(t,a,\rho)\rho = 0 \\ \rho(0,a) = \varphi(a) & \forall a \in [0,L] \\ \rho(t,0) = B(t) & \forall t \in [0,T] \end{cases} \\ P(t) = \int_0^L \rho(t,a)\, da & \forall t \in [0,T]. \end{cases}$$

We will assume the following hypotheses:

$\boxed{H(\lambda)}$ $\lambda : [0,T] \times [0,L) \times \mathbb{R} \longrightarrow \mathbb{R}_+$ is defined by

$$\lambda(t, a, \rho) \stackrel{df}{=} \lambda_N(a)[1 + \widetilde{f}(t)],$$

where $\lambda_N : [0,L) \longrightarrow \mathbb{R}_+$ is a continuous function, such that

$$\int_0^L \lambda_N(a)\, da = +\infty$$

and $\widetilde{f}(t) = f(P(t))$.

$\boxed{H(c)}$ $c : [0,T] \times [0,L] \longrightarrow \mathbb{R}_+$ is defined by (2) where $g : [0,T] \longrightarrow \mathbb{R}_+$ is a continuous function and $h : [0,L] \longrightarrow \mathbb{R}_+$ is a decreasing C^1-function.

$\boxed{H(\varphi)}$ $\varphi : [0,L] \longrightarrow \mathbb{R}_+ \cup \{0\}$ is a continuous function, such that the function

$$\varphi(a) \exp\left(\int_0^a \lambda_N(s)\, ds \right)$$

is uniformly bounded for $a \in [0,L]$.

$\boxed{H(B)}$ $B : [0,T] \longrightarrow \mathbb{R}_+$ is a continuous function, such that $B(0) = \varphi(0)$.

$\boxed{H(P)}$ $P : [0,T] \longrightarrow \mathbb{R}_+$ is a strictly monotone C^1-function, for which there exists $M > 0$, such that

$$g(t)h(0)B(t) - g(t)E(t;M) - (M+1)D(t;M) \le P'(t) \le g(t)h(0)B(t) - g(t)E(t;0) - D(t;0)$$

for all $t \in [0,T]$, where

$$D(t;\widetilde{f}) \stackrel{df}{=} \int_0^L \lambda_N(a)\rho(t,a;\widetilde{f})\, da$$

$$E(t;\widetilde{f}) \stackrel{df}{=} -\int_0^L h'(a)\rho(t,a;\widetilde{f})\, da$$

and $\rho(t,a;\widetilde{f})$ denotes the solution of (DP_2) for a particular function $\widetilde{f}$.

Remark. Given $\widetilde{f}$ we can solve (DP_2) obtaining ρ and, as a consequence, P, which allows us to verify assumption $H(P)$.

Theorem 1. *If assumptions $H(\lambda)$, $H(c)$, $H(\varphi)$, $H(B)$ and $H(P)$ hold then there exists a unique solution $(\rho,\widetilde{f}) \in C^1((0,T)\times(0,L)) \times C([0,T];[0,M])$ of (IP_2).*

Proof. By differentiation of $P(t)$ we easily obtain that $\widetilde{f}$ is a solution of (IP_2) iff it is a fixed point of the map $\mathbb{T}$, defined by

$$\mathbb{T}(\widetilde{f}) \stackrel{df}{=} \frac{g(\cdot)h(0)B(\cdot) - E(\cdot;\widetilde{f}) - P'(\cdot)}{D(\cdot;\widetilde{f})}.$$

On the other hand, integrating (DP_2) along characteristic coordinates, we obtain

$$\rho(t,a) = \begin{cases} B(\overline{G}(t,a))\exp\Big(-\displaystyle\int_0^a \frac{\lambda_N(s)[1+\widetilde{f}(\Gamma(s,t,a))]}{h(s)g(\Gamma(s,t,a))}ds\Big) & \text{if } a < H^{-1}(G(t)) \\ \varphi(\overline{H}(t,a))\exp\Big(-\displaystyle\int_{\overline{H}(t,a)}^a \frac{\lambda_N(s)[1+\widetilde{f}(\Gamma(s,t,a))]}{h(s)g(\Gamma(s,t,a))}ds\Big) & \text{if } a \ge H^{-1}(G(t)). \end{cases}$$

So, we see that for all $(t,a) \in [0,T]\times[0,L]$ and $\widetilde{f} \ge 0$, we have

$$\rho(t,a;\widetilde{f}) \le \max\{\|B\|_\infty, \|\varphi\|_\infty\}.$$

Moreover, if $\widetilde{f_1}(t) \ge \widetilde{f_2}(t)$ for all $t \in [0,T]$, then $\rho(t,a;\widetilde{f_1}) \le \rho(t,a;\widetilde{f_2})$ for all $(t,a) \in [0,T]\times[0,L]$, $D(t;\widetilde{f_1}) \le D(t;\widetilde{f_2})$ and $E(t;\widetilde{f_1}) \le E(t;\widetilde{f_2})$ for all $t \in [0,T]$, and finally

$$\mathbb{T}(\widetilde{f_1}) \ge \mathbb{T}(\widetilde{f_2}).$$

Applying hypothesis $H(P)$ we obtain

$$0 \le \mathbb{T}(\widetilde{f})(t) \le M,$$

so we have

$$\mathbb{T} : C([0,T];[0,M]) \longrightarrow C([0,T];[0,M]).$$

Now, after some calculations, for all $\widetilde{f}_1,\ \widetilde{f}_2 \in C([0,T];[0,M])$ and $t \in [0,T]$, we obtain

$$|\mathbb{T}(\widetilde{f}_1)(t) - \mathbb{T}(\widetilde{f}_2)(t)| \leq \alpha(t, \lambda_N, g, h, B, \varphi)\|\widetilde{f}_1 - \widetilde{f}_2\|_{C([0,T])},$$

where $\alpha \geq 0$ does not depend on $\widetilde{f}_1$ and $\widetilde{f}_2$ and is a continuous function of variable t such that $\alpha(t) \longrightarrow 0$ as $t \to 0$. So we can choose a small enough $0 < \tau \leq T$, such that

$$\|\mathbb{T}(\widetilde{f}_1) - \mathbb{T}(\widetilde{f}_2)\|_{C([0,\tau])} \leq \alpha_0\|\widetilde{f}_1 - \widetilde{f}_2\|_{C([0,T])},$$

with some $0 < \alpha_0 < 1$. Now, Banach Fixed Point Theorem gives us a unique solution on the interval $[0,\tau]$. By the bootstrap procedure we may extend this solution to all $t \in [0,T]$ by progressing in increments of τ. $\square$

Remark. As the unique $\widetilde{f}$ solving (IP_2) is a fixed point of a contraction it can be computed as the limit of an iteration process $\widetilde{f}_{n+1} = \mathbb{T}(\widetilde{f}_n)$ starting from any $\widetilde{f}_0 \in C([0,T];[0,M])$, e.g. $\widetilde{f}_0 = 0$ or $\widetilde{f}_0 = M$.

STABILITY OF SOLUTIONS IN AUTONOMOUS CASE

Here we deal with a direct problem in which c and λ depend only on a and time interval is $[0,+\infty)$, namely

$$(DP_3) \begin{cases} \rho_t + c(a)\rho_a + \lambda(a)\rho = 0 & \text{for } (t,a) \in (0,+\infty) \times (0,L) \\ \rho(0,a) = \varphi(a) & \forall a \in [0,L] \\ \rho(t,0) = B(t) & \forall t \in [0,+\infty). \end{cases}$$

We have the following result concerning the behaviour of the solution $\rho(t,a)$ as $t \to +\infty$.

Theorem 2. *If $c, \lambda, \varphi : [0,L] \longrightarrow \mathbb{R}_+$ and $B : [0,+\infty) \longrightarrow \mathbb{R}_+$ are continuous functions and*

$$\lim_{t \to +\infty} B(t) = b_0,$$

then any solution $\rho(t,a)$ of (DP_3) satisfies

$$\lim_{t \to +\infty} \rho(t,a) = b_0 w(a) \ \textit{uniformly on } [0,L],$$

where $w(a) = \exp\left(-\int_0^a \frac{\lambda(s)}{c(s)}\, ds\right)$.

Proof. Let us define

$$\overline{\rho}(t,a) \overset{df}{=} \rho(t,a) \exp\left(\int_0^a \frac{\lambda(s)}{c(s)}\, ds\right).$$

An easy calculation shows that $\overline{\rho}$ is a solution of the problem

$$(DP_3')\begin{cases}\overline{\rho}_t + c(a)\overline{\rho}_a = 0 & \text{for } (t,a)\in(0,+\infty)\times(0,L)\\ \overline{\rho}(0,a) = \frac{\varphi(a)}{w(a)} & \forall a\in[0,L]\\ \overline{\rho}(t,0) = B(t) & \forall t\in[0,+\infty).\end{cases}$$

In order to finish our proof it is enough to show that

$$\lim_{t\to+\infty}\overline{\rho}(t,a) = b_0.$$

But the explicit solution to (DP_3') is

$$\overline{\rho}(t,a) = \begin{cases} B(t-C(a)) & \text{if } C(a) < t\\ \varphi(C^{-1}(C(a)-t))\exp\left(-\int_0^{C^{-1}(C(a)-t)}\frac{\lambda(s)}{c(s)}\,ds\right) & \text{if } C(a)\geq t,\end{cases}$$

where

$$C(a) = \int_0^a \frac{ds}{c(s)},$$

so we see that $\lim_{t\to+\infty}\overline{\rho}(t,a) = b_0$. □

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