Relaxation in shape optimization problems governed by hyperbolic equations

Maciej Smołka
Instytut Informatyki Uniwersytetu Jagiellońskiego, ul. Nawojki 11, 30-072 Kraków
email: smolka@ii.uj.edu.pl

Abstract *We consider the shape optimization problem*

$$\min_{G\in\mathcal{A}(\Omega)} J(G,u_G)$$

with u_G satisfying the hyperbolic equation

$$\begin{cases} u_G'' + Au_G = f \\ u_G(0) = u^0, \quad u_G'(0) = u^1 \\ u_G = 0 \quad \textit{on } (0,T)\times(\Omega\setminus G). \end{cases}$$

As the class of admissible shapes we take $\mathcal{A}(\Omega)$, i.e., the family of all open subsets of a fixed open and bounded $\Omega \subset \mathbf{R}^N$. It can be shown that such a problem in general does not admit a solution, so we need to apply the procedure of relaxation. This method consists of two parts. First we extend, in a sense, the class of admissible shapes and introduce the notion of relaxed hyperbolic problem. This extended admissible set is a suitable family of Borel measures on Ω. Then we find the relaxation of J on this greater space. The theory of relaxed functionals ensures the existence of solutions for such problems. Finally, we provide some necessary conditions for optimality for relaxed problems.

1. Introduction

We consider the problem of finding a minimum of a functional

$$J : Y \longrightarrow \overline{\mathbb{R}}$$

defined on a topological space Y. In our case the space Y is not compact, so even if J is lower semicontinuous, the existence of a minimum is not guaranteed. This forces us to apply the procedure of relaxation. Namely we take a larger space X which is compact and the inclusion

$$Y \subset X$$

is continuous and dense. Then we compute the **relaxed functional for** f, i.e. the greatest l.s.c. functional

$$\overline{J} : X \longrightarrow \overline{\mathbb{R}}$$

such that

$$\overline{J}|_Y \leq J.$$

It is well known that

$$\overline{J}(x) = \inf\left\{ \liminf_{n\to\infty} J(y_n) :\ y_n \in Y,\ y_n \xrightarrow{X} x \right\}$$

or, in other words, that the following two conditions are satisfied:

1. for every sequence (y_n) of elements of Y convergent to x in X
$$\overline{J}(x) \leq \liminf_{n\to\infty} J(y_n);$$
2. there exists a sequence (y_n) of elements of Y such that $y_n \to x$ in X and
$$\overline{J}(x) = \lim_{n\to\infty} J(y_n).$$

$\overline{J}$ is lower semicontinuous, so if X is compact the direct method of calculus of variations ensures the existence of a minimum.

The relaxed functional is useful for studying the asymptotic behaviour of minimizing sequences of J as we know that

$$\min_X \overline{J} = \inf_Y J;$$

moreover, every cluster point of a minimizing sequence for J is a minimum point of $\overline{J}$ and, conversely, every minimum point of $\overline{J}$ is the limit of a minimizing sequence for J.

In our case Y is the class of all open subsets of a fixed open and bounded set

$$\Omega \subset \mathbb{R}^N.$$

This class is also denoted by $\mathcal{A}(\Omega)$. The next step is to introduce a topology in $\mathcal{A}(\Omega)$.

Let $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ be a symmetric elliptic operator in the divergence form, i.e.

$$Av = -\sum_{i,j=1}^{N} D_i(a_{ij}(x)D_j v)$$

with suitable continuity and coercivity conditions. For

$$G \in \mathcal{A}(\Omega)$$

we define a function

$$w_G \in H_0^1(\Omega)$$

as the solution of the Dirichlet problem

$$\begin{cases} Aw_G = 1 & \text{in } G, \\ w_G = 0 & \text{in } \Omega \setminus G. \end{cases}$$

The topology in $\mathcal{A}(\Omega)$ is given as follows:

$$G_n \to G \iff w_{G_n} \to w_G$$

weakly in $H_0^1(\Omega)$.

As X we take the class $\mathcal{M}_0(\Omega)$ of nonnegative Borel measures μ on Ω satisfying, for any Borel subset B of Ω, the following two conditions:

- $\mu(B) = 0$ if $\text{cap}(B, \Omega) = 0$;
- $\mu(B) = \inf\{\mu(G) : B \subset G, G \text{ quasi open}\}$.

Here $\text{cap}(B, \Omega)$ denotes the harmonic capacity of B with respect to Ω.

$\mathcal{A}(\Omega)$ can be considered as a subspace of $\mathcal{M}_0(\Omega)$ by identifying an open set G with the measure

$$\infty_{\Omega \setminus G}(B) = \begin{cases} 0 & \text{if cap}(B \setminus G, \Omega) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

For $\mu \in \mathcal{M}_0(\Omega)$ we introduce a Hilbert space

$$V_\mu(\Omega) = H_0^1(\Omega) \cap L_\mu^2(\Omega)$$

with the scalar product

$$(u, v)_{V_\mu(\Omega)} = \int_\Omega (Du, Dv)dx + \int_\Omega uv d\mu.$$

Furthermore we define an element

$$w_\mu \in V_\mu(\Omega)$$

as the solution of the **relaxed Dirichlet problem**

$$\begin{cases} Aw_\mu + \mu w_\mu = 1 \\ w_\mu \in V_\mu(\Omega), \end{cases}$$

which is to be understood in the following way: we look for $w_\mu \in V_\mu(\Omega)$ such that

$$\int_\Omega \sum_{i,j=1}^{N} a_{ij} D_i w_\mu D_j v \, dx + \int_\Omega w_\mu v \, d\mu = \int_\Omega v \, dx$$

for every $v \in V_\mu(\Omega)$.

When

$$\mu = \infty_{\Omega \setminus G}$$

for an open G, we have

$$V_\mu(\Omega) = H_0^1(G)$$

(up to the extending a function from $H_0^1(G)$ by zero outside G in order to obtain an element of $H_0^1(\Omega)$) and

$$w_\mu = w_G.$$

So if we endow $\mathcal{M}_0(\Omega)$ with the topology given by the equivalence

$$\mu_n \to \mu \iff w_{\mu_n} \to w_\mu \text{ weakly in } H_0^1(\Omega),$$

then $\mathcal{A}(\Omega)$ becomes a topological subspace of $\mathcal{M}_0(\Omega)$. This topology is called **the topology of γ^A-convergence**.

Proposition 1. *$\mathcal{M}_0(\Omega)$ is a compact metrizable space and $\mathcal{A}(\Omega)$ is its dense subset.*

For a measure $\mu \in \mathcal{M}_0(\Omega)$ we define the following sets

$$\begin{aligned} A(\mu) &= \{x \in \Omega : \ w_\mu(x) > 0\} \\ S(\mu) &= \{x \in \Omega : \ w_\mu(x) = 0\} = \Omega \setminus A(\mu) \end{aligned}$$

called, respectively, the **regular** and the **singular** set for μ (both are defined up to a null-capacity set).

It can be shown that any function $v \in V_\mu(\Omega)$ vanishes quasi-everywhere outside $A(\mu)$.

2. Relaxed hyperbolic problems

Fix $0 < T < +\infty$ and denote

$$Q = (0,T) \times \Omega.$$

For $\mu \in \mathcal{M}_0(\Omega)$ we introduce the Gelfand-Lions triplet of Hilbert spaces

$$V_\mu(\Omega) \subset H_\mu(\Omega) \subset V'_\mu(\Omega)$$

where

$$H_\mu(\Omega) \text{ — the closure of } V_\mu(\Omega) \text{ in } L^2(\Omega).$$

It can be shown that

$$H_\mu(\Omega) = \left\{ v \in L^2(\Omega) : \ v = 0 \text{ a.e. in } S(\mu) \right\}.$$

The relaxed hyperbolic problem is the following evolution problem

$$\begin{cases} u'' + Au + \mu u = f \text{ in } L^2(0,T; V'_\mu(\Omega)) \\ u(0) = u^0 \\ u'(0) = u^1 \\ u \in C([0,T]; V_\mu(\Omega)) \cap C^1([0,T]; H_\mu(\Omega)) \end{cases}$$

For any $f \in L^1(0,T; L^2(\Omega))$, $u^0 \in V_\mu(\Omega)$ and $u^1 \in H_\mu(\Omega)$ this problem admits a unique solution. In case $\mu = \infty_{\Omega \setminus G}$ this solution is simply the extension by zero of the solution of the classical hyperbolic equation on an open set G with the homogeneous Dirichlet boundary conditions.

Consider now the sequence of relaxed hyperbolic problems

$$\begin{cases} u''_n + Au_n + \mu_n u_n = f_n \\ u_n(0) = u_n^0 \\ u'_n(0) = u_n^1 \\ u_n \in C([0,T]; V_{\mu_n}(\Omega)) \cap C^1([0,T]; H_{\mu_n}(\Omega)). \end{cases}$$

for some fixed $f_n \in L^1(0,T; L^2(\Omega))$, $u_n^0 \in V_{\mu_n}(\Omega)$ and $u_n^1 \in H_{\mu_n}(\Omega)$.

R. Toader (cf. [9]) proved the following theorem.

Theorem 2. *Let $C > 0$. Assume that*

$$\begin{aligned} &\mu_n \xrightarrow{\gamma^A} \mu, \\ &f_n \longrightarrow f \text{ weakly in } L^1(0,T; L^2(\Omega)), \\ &u_n^0 \longrightarrow u^0 \text{ weakly in } H_0^1(\Omega), \\ &\|u_n^0\|_{L^2_{\mu_n}(\Omega)} \le C, \quad \forall n \in \mathbb{N}, \\ &u_n^1 \longrightarrow u^1 \text{ weakly in } L^2(\Omega), \\ &u^1 \in H_\mu(\Omega). \end{aligned}$$

Then $u^0 \in V_\mu(\Omega)$ and

$$\begin{aligned} &u_n \longrightarrow u \text{ weakly * in } L^\infty(0,T; H_0^1(\Omega)) \\ &u'_n \longrightarrow u' \text{ weakly * in } L^\infty(0,T; L^2(\Omega)) \\ &\|u_n\|_{L^\infty(0,T; V_{\mu_n}(\Omega))} \le const. \end{aligned}$$

Moreover, for every $\theta \in H^{-1}(\Omega)$

$$\langle \theta, u_n(\cdot) \rangle \longrightarrow \langle \theta, u(\cdot) \rangle \text{ uniformly.}$$

Denote

$$E(v) = \int_\Omega \sum_{i,j=1}^N a_{ij} D_i v D_j v \, dx + \int_\Omega v^2 \, d\mu.$$

Theorem 3. *If we assume additionally that*

$$\begin{aligned} &f_n \longrightarrow f \text{ strongly in } L^1(0,T; L^2(\Omega)), \\ &E(u_n^0) \longrightarrow E(u^0), \\ &u_n^1 \longrightarrow u^1 \text{ strongly in } L^2(\Omega). \end{aligned}$$

Then

$$u'_n \longrightarrow u' \text{ strongly in } C([0,T]; L^2(\Omega))$$

and

$$E(u_n(\cdot)) \longrightarrow E(u(\cdot)) \text{ uniformly.}$$

3. Relaxation of optimal shape design problems

For $\mu \in \mathcal{M}_0(\Omega)$ consider two linear and continuous operators

$$\begin{array}{rcl} P_\mu : H_0^1(\Omega) & \longrightarrow & V_\mu(\Omega) \\ \mathcal{P}_\mu : L^\infty(\Omega) & \longrightarrow & H_\mu(\Omega) \end{array}$$

given by the formulae

$$\begin{array}{rcl} P_\mu v & = & v_\mu \\ \mathcal{P}_\mu v & = & w_\mu v \end{array}$$

where v_μ is the solution of the relaxed Dirichlet problem

$$\begin{cases} Av_\mu + \mu v_\mu = Av \\ v_\mu \in V_\mu(\Omega) \end{cases}$$

(w_μ is the solution of similar problem with 1 as the right hand side).
For $G \in \mathcal{A}(\Omega)$ we denote

$$P_G = P_{\infty_{\Omega\setminus G}}$$

and

$$\mathcal{P}_G = \mathcal{P}_{\infty_{\Omega\setminus G}}.$$

We consider the following shape optimization problems:
Find a minimum of the functional

$$\begin{array}{rcl} J(G) & = & \displaystyle\int_Q j(t,x,u_G,u_G')\,dtdx \\ & + & \displaystyle\int_\Omega k(x,u_G(T),u_G'(T))\,dx \end{array}$$

or

$$\begin{array}{rcl} J(G) & = & \displaystyle\int_0^T \int_G j(t,x,u_G,u_G')\,dxdt \\ & + & \displaystyle\int_G k(x,u_G(T),u_G'(T))\,dx \end{array}$$

over the class $\mathcal{A}(\Omega)$ where u_G is the solution of the hyperbolic equation

$$\begin{cases} u_G'' + Au_G = f \\ u_G(0) = P_G u_0 \\ u_G'(0) = \mathcal{P}_G u_1 \\ u_G = 0 \quad \text{on } (0,T)\times(\Omega\setminus G) \end{cases}$$

for some fixed $f \in L^2(Q)$, $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^\infty(\Omega)$.

In general, in neither of the two above mentioned cases we can guarantee the existence of a minimum, so the next target is to compute the explicit form of relaxed functionals.

The first case (without the explicit dependence on geometric domain G in J) is much simpler. Namely under some natural conditions (L^2-lower semicontinuity) imposed on J we can show that the form of the relaxed functional is

$$\begin{array}{rcl} \overline{J}(\mu) & = & \displaystyle\int_Q j(t,x,u_\mu,u_\mu')\,dtdx \\ & + & \displaystyle\int_\Omega k(x,u_\mu(T),u_\mu'(T))\,dx \end{array}$$

where $\mu \in \mathcal{M}_0(\Omega)$ and u_μ is the solution of a relaxed hyperbolic problem

$$\begin{cases} u_\mu'' + Au_\mu + \mu u = f \\ u_\mu(0) = P_\mu u_0 \\ u_\mu'(0) = \mathcal{P}_\mu u_1 \\ u_\mu \in C([0,T];V_\mu(\Omega)) \cap C^1([0,T];H_\mu(\Omega)) \end{cases} \qquad (RHP)$$

The second case requires a different treatment and stronger assumptions imposed on J (we require L^2-continuity). The form of the relaxed functional is, of course, not so simple as before. Namely, it is

$$\begin{array}{rcl} \overline{J}(\mu) & = & \displaystyle\int_0^T \int_{A(\mu)} j(t,x,u_\mu,u_\mu')\,dxdt \\ & + & \displaystyle\int_{A(\mu)} k(x,u_\mu(T),u_\mu'(T))\,dx \\ & + & \inf\Big\{ \displaystyle\int_{B\setminus A(\mu)} g(x)\,dx \ : \\ & & B \in \mathcal{B}(\Omega), A(\mu) \subset B \Big\} \end{array}$$

where

$$g(x) = \int_0^T j(t,x,0,0)\,dt + k(x,0,0).$$

and u_μ is the solution of (RHP). We use the convention $\inf\emptyset = +\infty$.

Remark. In the original problem we can impose an additional constraint on G

$$m \leq |G| \leq M$$

for some $0 \leq m \leq M \leq |\Omega|$ with $M > 0$. In this case the infimum in the relaxed fuctional should be taken over those Borel sets B which satisfy $m \leq |B| \leq M$.

4. Optimality conditions

For relaxed shape optimization problems we can obtain some necessary conditions for optimality.

Let the cost functional have the form

$$J(\mu) = \int_Q j(t, x, u_\mu(t, x))\, dtdx$$

and u_μ satisfy the problem

$$\begin{cases} u_\mu'' + Au_\mu + \mu u_\mu = f \\ u_\mu(0) = 0 \\ u_\mu'(0) = 0 \\ u = 0 \text{ on } (0, T) \times \partial\Omega. \end{cases}$$

Assume that the functional

$$u \longmapsto \int_Q j(t, x, u(t, x))\, dtdx$$

is C^1 with respect to the strong topology in $L^2(Q)$.

Theorem 4. *Let the above assumptions hold. If μ is a minimal point for J, then*

$$\int_0^T u_\mu p_\mu\, dt \leq 0$$

almost everywhere in Ω and

$$\int_Q u_\mu p_\mu\, dtd\mu \geq 0$$

where p_μ is the solution of the ***adjoint equation***

$$\begin{cases} p_\mu'' + Ap_\mu + \mu p_\mu = j_u(\cdot, \cdot, u_\mu(\cdot, \cdot)) \\ p_\mu(T) = 0 \\ p_\mu'(T) = 0 \\ p = 0 \text{ on } (0, T) \times \partial\Omega. \end{cases}$$

A similar result holds in case of final state observation, i.e. when the cost functional has the form

$$J(\mu) = \int_\Omega k(x, u(T, x), u'(T, x))\, dx,$$

but this time the adjoint equation is

$$\begin{cases} p_\mu'' + Ap_\mu + \mu p_\mu = 0 \\ p_\mu(T) = k_{u'}(\cdot, u_\mu(T, \cdot), u_\mu'(T, \cdot)) \\ p_\mu'(T) = -k_u(\cdot, u_\mu(T, \cdot), u_\mu'(T, \cdot)) \\ p = 0 \text{ on } (0, T) \times \partial\Omega. \end{cases}$$

Bibliography

[1] G. Buttazzo, *Semicontinuity, Relaxation and Integral Represantation in the Calculus of Variations*, Pitman Res. Notes Math. Ser. 207, Longman, Harlow, 1989.

[2] G. Buttazzo, G. Dal Maso, *Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions*, Appl. Math. Optim., 23 (1991), 17–49.

[3] G. Buttazzo, G. Dal Maso, *An existence result for a class of shape optimization problems*, Arch. Rat. Mech. Anal., 122 (1993), 183–195.

[4] G. Buttazzo, G. Dal Maso, A. Garroni, A. Malusa, *On the relaxed formulation of some shape optimization problems*, Adv. Math. Sci. Appl., 7 (1997), 1–24.

[5] D. Chenais, *On the existence of a solution in a domain identification problem*, J. Math. Anal. Appl., 52 (1975), 128–219.

[6] G. Dal Maso, *Problemi di semicontinuitá e rilassamento nel calcolo delle variazioni*, Equazioni Differenziali e Calcolo delle Variazioni, Quaderni U.M.I. 39, 145–196, Pitagora Editrice, Bologna, 1995.

[7] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer–Verlag, Berlin, 1971.

[8] V. Šverák, *On optimal shape design*, J. Math. Pures Appl., 72 (1993), 537–551.

[9] R. Toader, *Some Problems in the Asymptotic Analysis of Partial Differential Equations in Perforated Domains*, PhD. Thesis, SISSA, Trieste, 1997.