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An Existence Theorem for Wave–Type Hyperbolic Hemivariational Inequalities

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Abstract. In this paper we prove the existence of solutions for a hyperbolic hemivariational inequality of the form

$$u'' + Bu + \partial j(u) \ni f$$

where B is a linear elliptic operator and ∂j is the Clarke subdifferential of a locally Lipschitz function j. Our result is based on the parabolic regularization method.

1. Introduction

The theory of variational inequalities provides us with an appropriate mathematical model to describe many physical problems (compare G. Duvaut and J. L. Lions [8]). It was started in 60s by C. Baiocchi, H. Brezis, G. Duvaut, G. Fichera, D. Kinderlehrer, J. L. Lions, G. Stampacchia and many others. In 80s, P. D. Panagiotopoulos introduced so called hemivariational inequalities (see [21, 22, 23]), using the notion of Clarke subdifferential (see F. H. Clarke [7]), which can be defined for locally Lipschitz functions. An existence theorem for elliptic hemivariational inequalities can be found in the book of Z. Naniewicz and P. D. Panagiotopoulos [19] (see Theorem 4.25, p. 120), the proof of which exploits so called surjectivity theorem for pseudomonotone operators (see [19, Theorem 2.6, p. 47]). As for the parabolic case, an existence result was obtained e. g. by M. Miettinen (see [16, Theorem 1.1, p. 727]) by means of an approximation method. Hyperbolic hemivariational inequalities were studied by P. D. Panagiotopoulos (see [25] and [26]) and, most recently, by L. Gasiński (see [11] and [12]). The latter obtained an existence theorem for the following problem:

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Find $u \in C([0,T];V)$ with $u' \in \mathcal{W}$ and $\chi \in \mathcal{H}'$, such that

$$(1.1)\qquad \begin{cases} u''(t) + A(t,u'(t)) + Bu(t) + \chi(t) \;=\; f(t) \quad \text{for a.a.} \quad t \in (0,T)\,, \\ u(0) \;=\; \psi_0\,, \quad u'(0) \;=\; \psi_1 \quad \text{in } \Omega\,, \\ \chi(t,x) \in \partial j(g(u(t,x),u'(t,x))) \quad \text{a.e. in} \quad (0,T)\times\Omega\,, \end{cases}$$

where $A : (0,T)\times V \to V'$ is a nonlinear, pseudomonotone, locally bounded and coercive operator, $B \in \mathcal{L}(V,V')$ is a monotone operator, $j : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $\psi_0, \psi_1 : \Omega \to \mathbb{R}$ and $f : (0,T) \to V'$ are given functions.

Because of the coercivity of A it is not possible to put $A = 0$ in the above problem. The aim of this paper is to prove an existence result for hyperbolic hemivariational inequality (1.1) without a term depending on the time derivative of the unknown function. A similar problem (with $B = -\Delta$) is mentioned (without details) by J. Rauch in [30]. Similar problems (i. e. with $A = 0$) are also considered by P. D. Panagiotopoulos in [25] and [26] (in his case the multivalued law ∂j acts only on u'). However, his assumptions on the right–hand side f are much stronger than ours (he requires f to belong to $L^2((0,T)\times\Omega)$ together with its first and second time derivative). To prove the existence of solutions he uses an approximation by finite–dimensional problems containing some kind of regularizations of ∂j, putting additional assumptions on these regularizations.

In our proof we shall use the *parabolic regularization* method from the book of J. L. Lions and E. Magenes (see [15]), namely we shall approximate the solution of our problem by a sequence of solutions of (1.1), with operator A replaced by operators εB vanishing as $\varepsilon \to 0$.

Many applications of hemivariational inequalities can be found in the above mentioned monography of Z. Naniewicz and P. D. Panagiotopoulos [19] as well as in the book [22] of P. D. Panagiotopoulos. For particular application of hyperbolic hemivariational inequalities in mechanics (e. g. plane linear elastic body with nonmonotone skin effects) we refer to P.D. Panagiotopoulos [25] and [26]. Note, however, that the multivalued reaction–velocity laws presented in [26, Figs. 1, b–f] do not satisfy assumption (2.20) of the existence theorem [26, Proposition 2.1]. Our hypotheses H(j) cover all those cases if considered as reaction–displacement relations.

2. Preliminaries

Let X be a Banach space with a norm $\|\cdot\|_X$ and X' its topological dual. By $\langle\,\cdot\,,\,\cdot\,\rangle_{X'\times X}$ we shall denote the duality brackets for the pair (X,X'). If X is in addition a Hilbert space, then by $(\,\cdot\,,\,\cdot\,)_X$ we shall denote the scalar product in X.

In the formulation of our hemivariational inequality the crucial role will be played by the notion of Clarke subdifferential of a locally Lipschitz function. A function $j : X \to \mathbb{R}$ is said to be *locally Lipschitz* if for every $x \in X$ there exists a neighbourhood U of x and a constant $k_x > 0$ depending on U such that $|j(z)-j(y)| \le k_x\|z-y\|_X$ for all $z,y \in U$. From convex analysis it is well–known (see e. g. I. Ekeland and R. Temam [9]) that a proper, convex and lower semicontinuous function $g : X \to \mathbb{R}\cup\{+\infty\}$ is locally

Lipschitz in the interior of its (proper) domain $\operatorname{dom} g \overset{df}{=} \{x \in X : g(x) < +\infty\}$. In analogy with the directional derivative of a convex function, we define *the generalized directional derivative* of a locally Lipschitz function j at $x \in X$ in the direction $h \in X$, by

$$j^0(x;h) \overset{df}{=} \limsup_{\substack{x' \to 0 \\ t \searrow 0}} \frac{j(x+x'+th) - j(x+x')}{t}.$$

It is easy to check that the function $X \ni h \longmapsto j^0(x;h) \in \mathbb{R}$ is sublinear and continuous and that $\left|j^0(x;h)\right| \leq k_x \|h\|_X$. Hence by the Hahn–Banach theorem $j^0(x;\,\cdot\,)$ is the support function of nonempty, convex and w^*–compact set

$$\partial j(x) \overset{df}{=} \left\{x^* \in X' : \langle x^*, h\rangle_{X' \times X} \leq j^0(x;h) \text{ for all } h \in X\right\},$$

known as the *Clarke subdifferential* of j at x. Note that for every $x^* \in \partial j(x)$ we have $\|x^*\|_{X'} \leq k_x$. We have also that if $j, g : X \to \mathbb{R}$ are locally Lipschitz functions, then $\partial(j+g)(x) \subset \partial j(x) + \partial g(x)$ and $\partial(tj)(x) = t\partial j(x)$ for all $t \in \mathbb{R}$. Moreover, if $j : X \to \mathbb{R}$ is also convex then the subdifferential of j in the sense of convex analysis coincides with the generalized subdifferential introduced above. Finally, if j is strictly differentiable at x (in particular if j is continuously Gateaux differentiable at x), then $\partial j(x) = \{j'(x)\}$.

Let us introduce the following spaces, needed in the sequel:

$$\begin{aligned} H &= L^2(\Omega), \\ V &= H^1(\Omega) = \left\{v : v \in L^2(\Omega),\ D^\alpha v \in L^2(\Omega) \text{ for } 0 \leq |\alpha| \leq 1\right\}, \\ V' &= V' = \left[H^1(\Omega)\right]'. \end{aligned}$$

It is well–known that $V \subset H \subset V'$ form an evolution triple.

In our evolution case, we will also make use of the following spaces:

$$\begin{aligned} \mathcal{H} &= L^2(0,T;H) = L^2((0,T) \times \Omega), \\ \mathcal{V} &= L^2(0,T;V), \\ \mathcal{W} &= \{v : v \in \mathcal{V},\ v' \in \mathcal{V}'\}. \end{aligned}$$

3. Hyperbolic hemivariational inequality

Let $T > 0$ be any positive real number and let $N \geq 1$. By $\Omega \subset \mathbb{R}^N$ we will denote any open and bounded set. We consider the following hyperbolic hemivariational inequality:

Find $u \in C([0,T];V) \cap C^1([0,T];H)$ with $u'' \in \mathcal{V}'$ and $\chi \in \mathcal{H}$, such that

$$(HVI) \qquad \begin{cases} u''(t) + Bu(t) + \chi(t) = f(t) \quad \text{in } V' \text{ for a.a. } t \in (0,T), \\ u(0) = \psi_0, \quad u'(0) = \psi_1 \text{ in } \Omega, \\ \chi(t,x) \in \partial j(u(t,x)) \text{ for a.a. } (t,x) \in (0,T) \times \Omega, \end{cases}$$

where $B \in \mathcal{L}(V, V')$, $j : \mathbb{R} \to \mathbb{R}$, $\psi_0, \psi_1 : \Omega \to \mathbb{R}$ and $f : (0,T) \to V'$ are given.

For our existence result, we will need the following assumptions:

H(j) $j : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, such that

(i) $j(\xi) = \int_0^\xi \beta(s)\,ds$, where $\beta \in L^\infty_{\mathrm{loc}}(\mathbb{R})$;
(ii) for every $\xi \in \mathbb{R}$ there exist limits $\lim_{\zeta \to \xi^\pm} \beta(\zeta)$;
(iii) for every $\xi \in \mathbb{R}$, we have $|\beta(\xi)| \le c_0(1 + |\xi|^r)$, with some $c_0 > 0$ and $0 \le r < 1$.

H(B) $B : V \to V'$ is a linear operator, such that

(i) B is continuous, i. e. there exists $\alpha_B > 0$, such that for all $v, u \in V$, we have $\langle Bu, v\rangle_{V'\times V} \le \alpha_B \|u\|_V \|v\|_V$;
(ii) B is coercive, i. e. there exists $\beta_B > 0$, such that for all $v \in V$, we have $\langle Bv, v\rangle_{V'\times V} \ge \beta_B \|v\|_V^2$;
(iii) B is symmetric, i.e. for all $v, w \in V$, we have $\langle Bv, w\rangle_{V'\times V} = \langle Bw, v\rangle_{V'\times V}$.

H(f,ψ) $f \in \mathcal{H}$, $\psi_0 \in V$, $\psi_1 \in H$.

Now we can state our main result.

Theorem 3.1. *If hypotheses* H(j), H(B) *and* H(f,ψ) *hold, then* (HVI) *admits a solution.*

First, for any $\varepsilon > 0$ we consider the following regularized hyperbolic hemivariational inequality:

Find $u_\varepsilon \in C([0,T];V)$ with $u'_\varepsilon \in \mathcal{W}$ and $\chi_\varepsilon \in \mathcal{H}$, such that

$$(HVI_\varepsilon) \qquad \begin{cases} u''_\varepsilon(t) + \varepsilon B u'_\varepsilon(t) + B u_\varepsilon(t) + \chi_\varepsilon(t) \;=\; f(t)\,, \\ u_\varepsilon(0) \;=\; \psi_0\,, \quad u'_\varepsilon(0) \;=\; \psi_1\,, \\ \chi_\varepsilon(t,x) \in \partial j(u_\varepsilon(t,x))\,. \end{cases}$$

Lemma 3.2. *If hypotheses* H(j), H(B) *and* H(f,ψ) *hold, then for any $\varepsilon > 0$ there exists at least one solution u_ε of* (HVI_ε).

Proof. This is a consequence of the result of L. Gasiński (see [11] or [12]). Note that our hypotheses suffice to obtain such a solution (see [12, Theorem 3.1]). It is worth mentioning that the method exploited in [11, 12] does not in general work with $r = 1$ in hypothesis H(j)(iii) (see [12, Remark 3.3]). □

In the next lemma we show an estimate on selections of $\partial j(u)$.

Lemma 3.3. *If hypotheses* H(j) *hold and $u \in C([0,T];V)$ with $u' \in \mathcal{W}$ and $\eta \in \mathcal{H}$ are such that $\eta(t,x) \in \partial j(u(t,x))$ for almost all $(t,x) \in (0,T)\times\Omega$ then*

$$\|\eta\|_{\mathcal{H}} \;\le\; \bar{c}\,(1 + \|u\|_{\mathcal{H}})\,, \tag{3.1}$$

with some constant $\bar{c} = \bar{c}\,(\Omega, T, c_0) > 0$, not depending on u, η and r.

Proof. Using hypothesis H(j)(iii), we obtain

$$\begin{aligned} \|\eta\|_{\mathcal{H}}^2 &= \int_0^T \|\eta(t)\|_H^2\,dt \\ &= \int_0^T\int_\Omega |\eta(t,x)|^2\,dx\,dt \\ &\le \int_0^T\int_\Omega 4c_0^2\,(1+|u(t,x)|)^2\,dx\,dt \\ &\le 8\,c_0^2\int_0^T \big(|\Omega| + \|u(t)\|_H^2\big)\,dt \\ &\le 8\,c_0^2\big(T\,|\Omega| + \|u\|_{\mathcal{H}}^2\big)\,, \end{aligned}$$

so estimate (3.1) holds, with $\bar{c} \overset{df}{=} c_0 2\sqrt{2}\max\big\{\sqrt{T\,|\Omega|}\,,1\big\}$. □

The following lemma gives some estimates on the solutions of (HVI_ε).

Lemma 3.4. *If hypotheses* H(j), H(B), H(f,ψ) *hold and* u_ε *is a solution of* (HVI_ε), *then for any* $\varepsilon\in(0,1)$, *we have*

$$\begin{aligned} &\max_{t\in[0,T]}\left(\|u_\varepsilon(t)\|_V + \|u_\varepsilon'(t)\|_H\right) + \sqrt{\varepsilon}\,\|u_\varepsilon'\|_{\mathcal{V}} + \|u_\varepsilon''\|_{\mathcal{V}'} \\ &\le\ \bar{\bar{c}}\,(1+\|\psi_0\|_V + \|\psi_1\|_H + \|f\|_{\mathcal{H}})\,, \end{aligned} \tag{3.2}$$

where $\bar{\bar{c}} = \bar{\bar{c}}(\Omega,T,c_0,\alpha_B,\beta_B) > 0$ *is a constant not depending on* ε, ψ_0, ψ_1, B, f *and* j.

Proof. As $u_\varepsilon, u_\varepsilon' \in \mathcal{V}$, so in particular u_ε is an absolutely continuous function and

$$u_\varepsilon(t) \;=\; \int_0^t u_\varepsilon'(s)\,ds + \psi_0 \quad \text{for all} \quad t\in(0,T)$$

(see Barbu [3, Theorem 2.2, p. 19]). Thus for any $s\in(0,T)$, we have

$$\|u_\varepsilon(s)\|_H^2 \;\le\; 2\,T\int_0^s \|u_\varepsilon'(\tau)\|_H^2\,d\tau + 2\,\|\psi_0\|_H^2\,. \tag{3.3}$$

From the equality in (HVI_ε), taking the duality brackets on $u_\varepsilon'(s)$ and integrating over interval $(0,t)$, for any $t\in(0,T)$, we obtain

$$\begin{aligned} &\int_0^t \big\langle u_\varepsilon''(s), u_\varepsilon'(s)\big\rangle_{V'\times V}\,ds \;+\; \varepsilon\int_0^t \big\langle Bu_\varepsilon'(s), u_\varepsilon'(s)\big\rangle_{V'\times V}\,ds \\ &+ \int_0^t \big\langle Bu_\varepsilon(s), u_\varepsilon'(s)\big\rangle_{V'\times V}\,ds \;+\; \int_0^t \big\langle \chi_\varepsilon(s), u_\varepsilon'(s)\big\rangle_{V'\times V}\,ds \\ &= \int_0^t \big\langle f(s), u_\varepsilon'(s)\big\rangle_{V'\times V}\,ds\,. \end{aligned} \tag{3.4}$$

We will estimate separately each term in (3.4). First, we have

$$\int_0^t \big\langle u_\varepsilon''(s), u_\varepsilon'(s)\big\rangle_{V'\times V}\,ds \;=\; \frac{1}{2}\,\|u_\varepsilon'(t)\|_H^2 - \frac{1}{2}\,\|u_\varepsilon'(0)\|_H^2 \;=\; \frac{1}{2}\,\|u_\varepsilon'(t)\|_H^2 - \frac{1}{2}\,\|\psi_1\|_H^2$$

(compare ZEIDLER [31, Proposition 23.23(iv), pp. 422 – 423]).

Next, hypothesis H(B) (ii) implies

$$\varepsilon \int_0^t \langle Bu_\varepsilon'(s), u_\varepsilon'(s)\rangle_{V'\times V}\, ds \;\ge\; \varepsilon\beta_B \int_0^t \|u_\varepsilon'(s)\|_V^2\, ds\,.$$

Using the differentiation formula (see ZEIDLER [31, Proof of Theorem 32.E(III), p. 881]) and hypotheses H(B)(i) and (ii), we obtain

$$\begin{aligned}\int_0^t \langle Bu_\varepsilon(s), u_\varepsilon'(s)\rangle_{V'\times V}\, ds &= \frac{1}{2}\int_0^t \frac{d}{ds}\langle Bu_\varepsilon(s), u_\varepsilon(s)\rangle_{V'\times V}\, ds\\ &= \frac{1}{2}\langle Bu_\varepsilon(t), u_\varepsilon(t)\rangle_{V'\times V} - \frac{1}{2}\langle Bu_\varepsilon(0), u_\varepsilon(0)\rangle_{V'\times V}\\ &\ge \frac{\beta_B}{2}\|u_\varepsilon(t)\|_V^2 - \frac{\alpha_B}{2}\|\psi_0\|_V^2\,.\end{aligned}$$

Next, using hypothesis H(j)(iii), estimate (3.3) and the continuity of the embedding $V \subset H$, for all $t \in (0,T)$ we have

$$\begin{aligned}\int_0^t \langle \chi_\varepsilon(s), u_\varepsilon'(s)\rangle_{V'\times V}\, ds &= \int_0^t (\chi(s), u_\varepsilon'(s))_H\, ds\\ &\ge -\int_0^t \|\chi(s)\|_H\, \|u_\varepsilon'(s)\|_H\, ds\\ &\ge -\frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\, ds - \frac{1}{2}\int_0^t\!\!\int_\Omega c_0^2\,(1+|u_\varepsilon(s,x)|)^2\, dx\, ds\\ &\ge -\frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\, ds - c_0^2\int_0^t \left(|\Omega| + \|u_\varepsilon(s)\|_H^2\right) ds\\ &\ge -\frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\, ds \;-\; c_0^2\, T\, |\Omega|\\ &\quad - c_0^2 \int_0^t \left(2T\int_0^s \|u_\varepsilon'(\tau)\|_H^2\, d\tau + 2\,\|\psi_0\|_H^2\right) ds\\ &\ge -\frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\, ds - 2Tc_0^2 \int_0^t\!\!\int_0^s \|u_\varepsilon'(\tau)\|_H^2\, d\tau\, ds\\ &\quad - Tc_0^2\left(|\Omega| + 2\,\|\psi_0\|_V^2\right).\end{aligned}$$

Finally, for all $t \in (0,T)$ we have

$$\begin{aligned}\int_0^t \langle f(s), u_\varepsilon'(s)\rangle_{V'\times V}\, ds &\le \int_0^t (f(s), u_\varepsilon'(s))_H\, ds\\ &\le \int_0^t \|f(s)\|_H\, \|u_\varepsilon'(s)\|_H\, ds\\ &\le \frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\, ds + \frac{1}{2}\int_0^t \|f(s)\|_H^2\, ds\\ &\le \frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\, ds + \frac{1}{2}\,\|f\|_{\mathcal{H}}^2\,.\end{aligned}$$

Putting all the above estimates into (3.4), for all $t \in (0,T)$ we obtain

$$\begin{aligned} &\frac{1}{2}\,\|u_\varepsilon'(t)\|_H^2 + \frac{\beta_B}{2}\,\|u_\varepsilon(t)\|_V^2 + \varepsilon\beta_B \int_0^t \|u_\varepsilon'(s)\|_V^2\,ds \\ \le\ & c_1 + c_2\,\|\psi_0\|_V^2 + \frac{1}{2}\,\|\psi_1\|_H^2 + \frac{1}{2}\,\|f\|_{\mathcal{H}}^2 + \int_0^t \|u_\varepsilon'(s)\|_H^2\,ds \\ &+ 2\,Tc_0^2 \int_0^t\!\!\int_0^s \|u_\varepsilon'(\tau)\|_H^2\,d\tau\,ds\,, \end{aligned}$$

with $c_1 \stackrel{df}{=} Tc_0^2\,|\Omega|$, $c_2 \stackrel{df}{=} 2\,Tc_0^2 + \frac{\alpha_B}{2}$. Thus we have

$$\begin{aligned} &\frac{1}{2}\,\|u_\varepsilon'(t)\|_H^2 + \frac{\beta_B}{2}\,\|u_\varepsilon(t)\|_V^2 + \varepsilon\beta_B \int_0^t \|u_\varepsilon'(s)\|_V^2 \\ \le\ & c_3\big(1 + \|\psi_0\|_V^2 + \|\psi_1\|_H^2 + \|f\|_{\mathcal{H}}^2\big) + c_4 \int_0^t \|u_\varepsilon'(s)\|_H^2\,ds \\ &+ c_4 \int_0^t\!\!\int_0^s \|u_\varepsilon'(\tau)\|_H^2\,d\tau\,ds\,, \end{aligned} \tag{3.5}$$

where $c_3 \stackrel{df}{=} \max\big\{\frac{1}{2}, c_1, c_2\big\}$ and $c_4 \stackrel{df}{=} \max\{1, 2\,Tc_0^2\}$. Now, using the generalization of the Gronwall–Bellman inequality (see Pachpatte [20, Theorem 1, p. 758]), for all $t \in (0,T)$, we obtain

$$\|u_\varepsilon'(t)\|_H^2 \ \le\ c_5\big(1 + \|\psi_0\|_V^2 + \|\psi_1\|_H^2 + \|f\|_{\mathcal{H}}^2\big)\,, \tag{3.6}$$

where $c_5 \stackrel{df}{=} 2c_3\,\big(1 + 2\,Tc_4 e^{T(2c_4+1)}\big)$, so

$$\|u_\varepsilon'(t)\|_H \ \le\ c_6\,(1 + \|\psi_0\|_V + \|\psi_1\|_H + \|f\|_{\mathcal{H}})\,, \tag{3.7}$$

where $c_6 \stackrel{df}{=} \sqrt{c_5}$. Applying (3.6) to (3.5), we obtain

$$\|u_\varepsilon(t)\|_V^2 \ \le\ c_7\,\big(1 + \|\psi_0\|_V^2 + \|\psi_1\|_H^2 + \|f\|_{\mathcal{H}}^2\big) \tag{3.8}$$

and

$$\varepsilon\,\|u_\varepsilon'\|_{\mathcal{V}}^2 \ \le\ c_7\,\big(1 + \|\psi_0\|_V^2 + \|\psi_1\|_H^2 + \|f\|_{\mathcal{H}}^2\big)\,, \tag{3.9}$$

where $c_7 \stackrel{df}{=} \frac{2}{\beta_B}\,\big(c_3 + Tc_4c_5\big(\frac{T}{2}+1\big)\big)$, hence

$$\|u_\varepsilon(t)\|_V \ \le\ c_8\,(1 + \|\psi_0\|_V + \|\psi_1\|_H + \|f\|_{\mathcal{H}}) \tag{3.10}$$

and

$$\sqrt{\varepsilon}\,\|u_\varepsilon'\|_{\mathcal{V}} \ \le\ c_8\,(1 + \|\psi_0\|_V + \|\psi_1\|_H + \|f\|_{\mathcal{H}})\,, \tag{3.11}$$

where $c_8 \stackrel{df}{=} \sqrt{c_7}$.

Using Lemma 3.3, continuity of the embedding $\mathcal{V} \subset \mathcal{H}$ and estimate (3.8), for any $\varepsilon > 0$, we have

$$\begin{aligned}
\|\chi_\varepsilon\|^2_{\mathcal{H}} &\leq 2\,\bar{c}^{\,2}\big(1 + \|u_\varepsilon\|^2_{\mathcal{H}}\big) \\
&\leq 2\bar{c}^{\,2}\big(1 + \|u_\varepsilon\|^2_{\mathcal{V}}\big) \\
&\leq 2\bar{c}^{\,2} + 2\bar{c}^{\,2}\int_0^T \|u_\varepsilon(t)\|^2_V\,dt \\
&\leq c_9\big(1 + \|\psi_0\|^2_V + \|\psi_1\|^2_H + \|f\|^2_{\mathcal{H}}\big)\,,
\end{aligned}$$

where $c_9 \overset{df}{=} 4\bar{c}^{\,2}Tc_7$.

Finally using the equation in (HVI_ε), hypothesis H(B)(i), continuity of the embedding $H \subset V'$, inequalities (3.8) and (3.9) and the last inequality, for all $\varepsilon \in (0,1)$ we can estimate $\|u''_\varepsilon\|_{\mathcal{V}'}$, as follows

$$\begin{aligned}
\|u''_\varepsilon\|^2_{\mathcal{V}'} &= \int_0^T \|u''_\varepsilon(t)\|^2_{V'}\,dt \\
&\leq 4\varepsilon^2\int_0^T \|B(u'_\varepsilon(t))\|^2_{V'}\,dt + 4\int_0^T \|B(u_\varepsilon(t))\|^2_{V'}\,dt \\
&\quad + 4\int_0^T \|\chi_\varepsilon(t)\|^2_{V'}\,dt + 4\int_0^T \|f(t)\|^2_{V'}\,dt \\
&\leq 4\varepsilon^2\alpha_B^2\int_0^T \|u'_\varepsilon(t)\|^2_V\,dt + 4\alpha_B^2\int_0^T \|u_\varepsilon(t)\|^2_V\,dt \\
&\quad + 4\int_0^T \|\chi_\varepsilon(t)\|^2_H\,dt + 4\int_0^T \|f(t)\|^2_H\,dt \\
&\leq 4\varepsilon\alpha_B^2\,\|u'_\varepsilon\|^2_{\mathcal{V}} + 4\alpha_B^2\int_0^T \|u_\varepsilon(t)\|^2_V\,dt + 4\,\|\chi_\varepsilon\|^2_{\mathcal{H}} + 4\,\|f\|^2_{\mathcal{H}} \\
&\leq c_{10}\big(1 + \|\psi_0\|^2_V + \|\psi_1\|^2_H + \|f\|^2_{\mathcal{H}}\big)\,,
\end{aligned}$$

where $c_{10} \overset{df}{=} 4\big(\alpha_B^2 c_7(T+1) + c_9 + 1\big)$, so

$$\|u''_\varepsilon\|_{\mathcal{V}'} \leq c_{11}(1 + \|\psi_0\|_V + \|\psi_1\|_H + \|f\|_{\mathcal{H}})\,, \tag{3.12}$$

with $c_{11} = \sqrt{c_{10}}$.

Finally, from (3.7), (3.10), (3.11) and (3.12), we obtain (3.2), with $\bar{\bar{c}} \overset{df}{=} c_6 + 2c_8 + c_{11}$. □

Now we are in position to prove our main result.

P r o o f of Theorem 3.1. From Lemma 3.4, it follows that for any $\varepsilon \in (0,1)$, we have

$$\max_{t\in[0,T]}\,(\|u_\varepsilon(t)\|_V + \|u'_\varepsilon(t)\|_H) + \|u''_\varepsilon\|_{\mathcal{V}'} \leq c_{12}\,,$$

with some constant $c_{12} > 0$ not depending on $\varepsilon \in (0,1)$. Thus, we can choose a sequence $\{\varepsilon_n\}_{n\geq 1} \subset (0,1)$, such that $\varepsilon_n \searrow 0$ and

$$u_{\varepsilon_n} \longrightarrow u \quad \text{weakly}^* \ \text{in} \ L^\infty(0,T;V)\,, \tag{3.13}$$

$$u'_{\varepsilon_n} \longrightarrow \overline{u} \quad \text{weakly}^* \ \text{in} \ L^\infty(0,T;H)\,, \tag{3.14}$$

$$u''_{\varepsilon_n} \longrightarrow \overline{\overline{u}} \quad \text{weakly} \ \text{in} \ \mathcal{V}'\,. \tag{3.15}$$

But in fact $\overline{u} = u'$ and $\overline{\overline{u}} = u''$.

It is easy to see that (HVI_ε) is equivalent to the following problem:

Find $u_\varepsilon \in C([0,T];V)$ with $u'_\varepsilon \in \mathcal{W}$ and $\chi_\varepsilon \in \mathcal{H}$, such that

$$(HVI'_\varepsilon) \qquad \begin{cases} u''_\varepsilon + \varepsilon \widehat{B} u'_\varepsilon + \widehat{B} u_\varepsilon + \chi_\varepsilon \ = \ f \quad \text{in} \quad \mathcal{V}' , \\ u_\varepsilon(0) \ = \ \psi_0 , \quad u'_\varepsilon(0) \ = \ \psi_1 \quad \text{in} \quad \Omega , \\ \chi_\varepsilon(t,x) \in \partial j(u_\varepsilon(t,x)) \quad \text{for a. a.} \quad (t,x) \in (0,T) \times \Omega , \end{cases}$$

where $\widehat{B} : \mathcal{V} \to \mathcal{V}'$ is the Nemytski operator corresponding to the operator B. Our aim now is to "pass to the limit" in (HVI'_ε).

First, as $\widehat{B}$ is a linear and bounded operator, from (3.13) we obtain that

$$(3.16) \qquad \widehat{B} u_{\varepsilon_n} \longrightarrow \widehat{B} u \quad \text{weakly in} \quad \mathcal{V}' .$$

Next, from Lemma 3.4, we see that the sequence $\{\sqrt{\varepsilon_n}\, u'_{\varepsilon_n}\}_{n \geq 1}$ remains bounded in $\mathcal{V}$, hence

$$\varepsilon_n u'_{\varepsilon_n} \longrightarrow 0 \quad \text{in} \quad \mathcal{V} .$$

But using hypothesis H(B)(i), we have that $\|\varepsilon_n \widehat{B} u'_{\varepsilon_n}\|_{\mathcal{V}'} \leq \alpha_B \|\varepsilon_n u'_{\varepsilon_n}\|_{\mathcal{V}}$, thus in fact

$$(3.17) \qquad \varepsilon_n \widehat{B} u'_{\varepsilon_n} \longrightarrow 0 \quad \text{in} \quad \mathcal{V}' .$$

As $\{u_{\varepsilon_n}\}_{n \geq 1} \subset \mathcal{W}$ (see (3.13) and (3.14)) and the embedding $\mathcal{W} \subset \mathcal{H}$ is compact we obtain

$$u_{\varepsilon_n} \longrightarrow u \quad \text{in} \quad \mathcal{H} ,$$

and, in particular, possibly passing to a subsequence,

$$(3.18) \qquad u_{\varepsilon_n}(t,x) \longrightarrow u(t,x) \quad \text{for a. a.} \quad (t,x) \in (0,T) \times \Omega .$$

Using convergence (3.13), Lemma 3.3 and extracting a new subsequence if necessary, we obtain

$$(3.19) \qquad \chi_{\varepsilon_n} \longrightarrow \chi \quad \text{weakly in} \quad \mathcal{H} ,$$

with some $\chi \in \mathcal{H}$, hence also

$$(3.20) \qquad \chi_{\varepsilon_n} \longrightarrow \chi \quad \text{weakly in} \quad L^1((0,T) \times \Omega) .$$

Now, because of (3.14), (3.16), (3.17) and (3.19), we can "pass to the limit" in the equation in (HVI'_ε) and obtain

$$(3.21) \qquad u'' + \widehat{B} u + \chi \ = \ f \quad \text{in} \quad \mathcal{V}' .$$

Since for all $n \geq 1$, we have that $\chi_{\varepsilon_n}(t,x) \in \partial j(u_{\varepsilon_n}(t,x))$ for almost all $(t,x) \in (0,T) \times \Omega$, thus, using convergences (3.18) and (3.20) and applying Theorem 7.2.2, p. 273 of Aubin and Frankowska [2] (recall that ∂j is a lower semicontinuous multifunction with convex and closed values), we get

$$(3.22) \qquad \chi(t,x) \in \partial j(u(t,x)) \quad \text{for a. a.} \quad (t,x) \in (0,T) \times \Omega .$$

Finally, from (3.13) and (3.14), we have that $u_{\varepsilon_n} \to u$ weakly in $H^1(0,T;H)$, hence also weakly in $C([0,T];H)$. Analogously from (3.14) and (3.14), we have that $u'_{\varepsilon_n} \to u'$ weakly in $H^1(0,T;V')$, hence also weakly in $C([0,T];V')$. In particular, we have that

$$(3.23)\qquad \begin{aligned} u_{\varepsilon_n}(0) &\longrightarrow u(0) \quad \text{weakly in} \quad H\,, \\ u'_{\varepsilon_n}(0) &\longrightarrow u'(0) \quad \text{weakly in} \quad V'\,. \end{aligned}$$

To end our proof it remains to show that

$$(3.24)\qquad u \in C([0,T];V) \cap C^1([0,T];H)\,.$$

For this purpose let us recall the definition of the following function space introduced in the book of J. L. Lions and E. Magenes [15]

$$C_s([0,T];X) \overset{df}{=} \{u \in L^\infty(0,T;X) : \langle u^*, u(\,\cdot\,)\rangle_{X'\times X} \text{ is continuous for all } u^* \in X'\}\,.$$

Of course one has

$$C([0,T];X) \subset C_s([0,T];X)\,.$$

Moreover, if X and Y are two Banach spaces, X being reflexive, with the dense embedding $X \subset Y$, from [15, Lemma 8.1, p. 297] we know that

$$(3.25)\qquad C_s([0,T];Y) \cap L^\infty(0,T;X) \;=\; C_s([0,T];X)\,.$$

In our case, due to (3.13) – (3.14) we have that

$$\begin{aligned} u &\in C([0,T];H) \cap L^\infty(0,T;V)\,, \\ u' &\in C([0,T];V') \cap L^\infty(0,T;H)\,, \end{aligned}$$

hence, from (3.25)

$$(3.26)\qquad u \in C_s([0,T];V)\,,$$

$$(3.27)\qquad u' \in C_s([0,T];H)\,.$$

Next, using the same argument as in the proof of Lemma 3.4 (see (3.4) and the sequel), for any $t \in [0,T]$ we can prove the following energy equality

$$\begin{aligned} \|u'(t)\|_H^2 + \langle Bu(t), u(t)\rangle_{V'\times V} &= \|\psi_1\|_H^2 + \langle B\psi_0, \psi_0\rangle_{V'\times V} \\ &\quad + 2\int_0^t \big\langle f(s) - \chi(s), u'(s)\big\rangle_{V'\times V}\, ds\,. \end{aligned}$$

This shows that the function

$$E \;:\; [0,T] \ni t \;\longmapsto\; \|u'(t)\|_H^2 + \langle Bu(t), u(t)\rangle_{V'\times V} \in \mathbb{R}$$

is continuous.

Take $t_n, t \in [0,T]$ such that $t_n \to t$ and put

$$\begin{aligned} \delta_n &= \|u'(t_n) - u'(t)\|_H^2 + \langle Bu(t_n) - Bu(t), u(t_n) - u(t)\rangle_{V'\times V} \\ &= E(t_n) + E(t) - 2\langle Bu(t), u(t_n)\rangle_{V'\times V} - 2\,(u'(t_n), u'(t))_H\,. \end{aligned}$$

Thanks to (3.26), (3.27) and the continuity of E we have that

$$\delta_n \;\longrightarrow\; 2E(t) - 2\langle Bu(t), u(t)\rangle_{V'\times V} - 2\,\|u'(t)\|_H^2 \;=\; 0\,,$$

which, together with the inequality

$$\delta_n \;\geq\; \|u'(t_n) - u'(t)\|_H^2 + \beta_B\,\|u(t_n) - u(t)\|_V^2\,,$$

gives us (3.24).

Now, from (3.21), (3.22), (3.23) and (3.24) we obtain that u is a solution of the following problem:

Find $u \in C([0,T];V) \cap C^1([0,T];H)$ with $u'' \in \mathcal{V}'$ and $\chi \in \mathcal{H}$, such that

$$(HVI') \qquad \begin{cases} u'' + \widehat{B}u + \chi \;=\; f \quad \text{in} \quad \mathcal{V}'\,, \\ u(0) \;=\; \psi_0\,, \quad u'(0) \;=\; \psi_1 \quad \text{in} \quad \Omega\,, \\ \chi(t,x) \in \partial j(u(t,x)) \quad \text{for a. a.} \quad (t,x) \in (0,T)\times\Omega\,, \end{cases}$$

and in particular u is a solution of (HVI). □

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