Asymptotic behaviour of optimal solutions of control problems governed by inclusions

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1 Introduction

In this paper we consider control problems of the form

$$(CP_h) \quad \min \{ J_h(u,y) : A_h(y) \in B_h(u), (u,y) \in U \times Y \},$$

where some topological spaces U and Y are, respectively, the space of controls and the space of states. We search for an explicit form of the limit problem (CP_∞) for the sequence (CP_h), i.e. such control problem that if (u_h, y_h) is a solution of (CP_h) and

$$(u_h, y_h) \to (u_\infty, y_\infty),$$

then (u_∞, v_∞) is a solution of (CP_∞). Under the assumption of G-convergence of operators A_h we are able to find an explicit form of (CP_∞) in two different cases.

1. When (B_h) converges 'K-continuously' to B, i.e.

   $$u_h \to u \Rightarrow B_h(u_h) \xrightarrow{K} B(u)$$

   where 'K' means the sequential convergence in the sense of Kuratowski;

2. B_h satisfy a weaker convergence condition, but they have the local form

   $$B_h(u) = b_h(x, u(x))$$

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for some multivalued functions b_h and J_h are integral functionals

$$J_h(u,y) = \int_\Omega f_h(x, y(x), u(x))dx.$$

In the first case the limit problem has the same form as (CP_h), in the latter, in general, it is not true — the state inclusion may disappear and the minimization may extend to the whole space $U \times Y$, but then some additional terms are inserted into the cost functional.

In this paper we generalize the results of G.Buttazzo and L.Freddi contained in [BF] to the case of multivalued input operators B_h. Namely, they considered control problems with the states of the system described by equations, i.e.

$$\min\{J_h(u,y) : A_h(y) = B_h(u), (u,y) \in U \times Y\}$$

where B_h were single-valued. Our cases 1 and 2 correspond, respectively, to the case of the continuous convergence of B_h (see [BF], pp.406–408) and the case of a weak convergence of nonlinear, local input operators (see [BF], section 5). On the other hand, the results of section 6 formulated for abstract operators A_h can be considered as a generalization of the results obtained by G.Buttazzo and E.Cavazzuti in [BC] concerning ordinary differential inclusions

$$y' \in a_h(t,y) + B_h(t,y)b_h(t,u)$$

where only b_h were multimappings.

The main tool used in the paper is the 'auxiliary variable' method due to G.Buttazzo (see [B]).

2 Γ-convergence, G-convergence and Kuratowski convergence

This section contains definitions and basic properties of sequential Γ-limits of functionals, Kuratowski limits of sets as well as a definition of the G-convergence of abstract operators (see [BM], [BF], [M]).

Let X be a topological space and let $F_h : X \to \overline{\boldsymbol{R}}$ be a sequence of functionals. Let us denote by $Z(+)$ and $Z(-)$, respectively, 'sup' and 'inf' operators. For $x \in X$ we define

$$\begin{aligned} \Gamma(X^\alpha) \limsup_{h\to\infty} F_h(x) &= Z(\alpha)_{x_h \to x} \limsup_{h\to\infty} F_h(x_h), \\ \Gamma(X^\alpha) \liminf_{h\to\infty} F_h(x) &= Z(\alpha)_{x_h \to x} \liminf_{h\to\infty} F_h(x_h), \end{aligned}$$

where α stands for $+$ or $-$. If these limits are equal, then their common value is denoted by

$$\Gamma(X^\alpha) \lim_{h\to\infty} F_h(x).$$

If some Γ-limit does not depend on α, we shall drop the sign, e.g. if

$$\Gamma(X^+) \limsup_{h\to\infty} F_h = \Gamma(X^-) \limsup_{h\to\infty} F_h,$$

then the common value is denoted by

$$\Gamma(X) \limsup_{h\to\infty} F_h.$$

If one of the following conditions is satisfied:

(G1) X satisfies the first axiom of countability;

(G2) X is a Banach space endowed with its weak topology, X^* is separable and $F_h \geq \Psi$ for $h \in \boldsymbol{N}$ where $\Psi : X \to \overline{\boldsymbol{R}}$ is such that

$$\lim_{||x||\to+\infty} \Psi(x) = +\infty; \tag{1}$$

then $\Gamma(X^-)$-limits defined above coincide with topological Γ-limits introduced by E.De Giorgi in [GF] (see [M], propositions 8.1 and 8.10). The $\Gamma(X)$-convergence is nothing else than the (sequential) continuous convergence, i.e. such that

$$x_h \to x \Rightarrow F_h(x_h) \to F(x).$$

If (F_h) is a constant sequence, i.e. $F_h = F$, $\forall\ h \in \boldsymbol{N}$ and (G1) or (G2) is true, then ([M], remark 4.5)

$$\Gamma(X^-) \liminf_{h\to\infty} F_h = \Gamma(X^-) \limsup_{h\to\infty} F_h = \mathrm{sc}^- F,$$

where $\mathrm{sc}^- F$ denotes *the lower semicontinuous envelope for* F, i.e. the greatest l.s.c. function bounded from above by F. Moreover, we have

Proposition 2.1 ([M], proposition 6.11) *If one of the conditions (G1), (G2) holds, then*

$$\begin{aligned} \Gamma(X^-)\liminf_{h\to\infty} F_h &= \Gamma(X^-)\liminf_{h\to\infty}\ \mathrm{sc}^- F_h, \\ \Gamma(X^-)\limsup_{h\to\infty} F_h &= \Gamma(X^-)\limsup_{h\to\infty}\ \mathrm{sc}^- F_h. \end{aligned}$$

In particular, (F_h) *is* $\Gamma(X^-)$*-convergent to* F *iff* $(\mathrm{sc}^- F_h)$ $\Gamma(X^-)$*-converges to* F*.*

The next proposition shows the relation between the Γ-convergence and the pointwise one.

Proposition 2.2 ([M], proposition 5.9) *Assume that* X *satisfies the first axiom of countability and* (F_h) *is equi l.s.c. at* $x \in X$*. Then*

$$\begin{aligned} \Gamma(X^-)\liminf_{h\to\infty} F_h(x) &= \liminf_{h\to\infty} F_h(x), \\ \Gamma(X^-)\limsup_{h\to\infty} F_h(x) &= \limsup_{h\to\infty} F_h(x). \end{aligned}$$

In particular, if (F_h) *is equi l.s.c. on* X*, then* (F_h) *is* $\Gamma(X^-)$*-convergent to* F *iff it converges to* F *pointwise.*

In the optimal control theory it is useful to introduce the notion of *double* Γ*-limits.* Let X and Y be topological spaces and let

$$F_h : X \times Y \to \overline{\boldsymbol{R}}$$

be a sequence of functionals. For $x \in X$, $y \in Y$ we define

$$\begin{aligned} \Gamma(X^\alpha, Y^\beta)\limsup_{h\to\infty} F_h(x,y) &= Z(\alpha)_{x_h\to x} Z(\beta)_{y_h\to y} \limsup_{h\to\infty} F_h(x_h, y_h), \\ \Gamma(X^\alpha, Y^\beta)\liminf_{h\to\infty} F_h(x,y) &= Z(\alpha)_{x_h\to x} Z(\beta)_{y_h\to y} \liminf_{h\to\infty} F_h(x_h, y_h), \end{aligned}$$

where α and β stand for $+$ or $-$. We shall use similar notation conventions as in the case of simple Γ-limits. Moreover, an expression like

$$\Gamma(X^\alpha) F_h(x,y)$$

means that the limit is taken with respect to the first variable, while the other is fixed.

The following proposition shows the role of the Γ-convergence in the optimal control theory.

Proposition 2.3 ([BM]) *Let U and Y be topological spaces and let $F_h : U \times Y \to \overline{\boldsymbol{R}}$ be a sequence of functionals. Let (u_h, y_h) be a sequence of minimal points for F_h or, more generally, such sequence that*

$$\lim_{h\to\infty} F_h(u_h, y_h) = \lim_{h\to\infty} \left[\inf_{U\times Y} F_h \right].$$

Assume that (u_h, y_h) converges in $U \times Y$ to some (u_∞, y_∞) and that for every $(u, y) \in U \times Y$ the limit

$$F(u, y) = \Gamma(U^-, Y^-) \lim_{h\to\infty} F_h(u, y).$$

exists. Then

1. $$\lim_{h\to\infty} \left[\inf_{U\times Y} F_h \right] = \min_{U\times Y} F;$$

2. *(u_∞, y_∞) is a minimal point for F.*

Γ-limits are not, in general, linear operators. The following proposition allows us to compute Γ-limit of a sequence of sums of functions.

Proposition 2.4 ([BM]) *Let $F_h, G_h : X \times Y \to \overline{\boldsymbol{R}}$ be two sequences of functions. Assume that the limits*

$$\begin{aligned} \Gamma(X^-, Y) \lim_{h\to\infty} F_h(x, y) &= a, \\ \Gamma(X, Y^-) \lim_{h\to\infty} G_h(x, y) &= b, \end{aligned}$$

exist. Then

$$\Gamma(X^-, Y^-) \lim_{h\to\infty} (F_h + G_h)(x, y) = a + b.$$

Let (E_h) be a sequence of subsets of X. We define *(sequential) upper and lower limits of (E_h) in the sense of Kuratowski* by:

$$\begin{aligned} K-\limsup_{h\to\infty} E_h &= \left\{ x \in X \mid \exists\ (h_n), h_n < h_{n+1}, x_n \in E_{h_n} : \lim_{n\to\infty} x_n = x \right\}, \\ K-\liminf_{h\to\infty} E_h &= \left\{ x \in X \mid \exists\ x_h \in E_h : \lim_{h\to\infty} x_h = x \right\}, \end{aligned}$$

If these limits are equal, then their common value is called *the Kuratowski limit of* (E_h) and denoted by

$$K-\lim_{h\to\infty} E_h.$$

Let E be a subset of X. By χ_E we shall denote *the indicator function of* E, i.e.

$$\chi_E(x) = \begin{cases} 0 & \text{when} \quad x \in E, \\ +\infty & \text{when} \quad x \in X \setminus E. \end{cases}$$

Let Y and V be topological spaces. We say that a sequence of operators

$$A_h : Y \to V$$

G-converges to $A : Y \to V$ if

$$\Gamma(V, Y^-) \lim_{h\to\infty} \chi_{\{A_h(y)=v\}} = \chi_{\{A(y)=v\}},$$

i.e. if the following conditions are satisfied:

1. if $y_h \to y$ in Y, $v_h \to v$ in V and $A_h(y_h) = v_h$ for infinitely many $h \in \boldsymbol{N}$, then $A(y) = v$;

2. if $A(y) = v$ and $v_h \to v$ in V, then there exists a sequence (y_h) convergent to y in Y and such that $A_h(y_h) = v_h$ for sufficiently large h.

Let us denote by $S_h(v)$ the set of solutions of the equation $A_h(y) = v$, and by $S(v)$ — the set of solutions of the equation $A(y) = v$.

Proposition 2.5 ([DM], proposition 4.4) *The above conditions are equivalent to the following:*

$$v_h \to v \Rightarrow S_h(v_h) \xrightarrow{K} S(v)$$

where 'K' means Kuratowski convergence with respect to the topology of Y*.*

Remark ([BM]). If for Y we substitute $H_0^1(\Omega)$ with the weak topology, for V — $H^{-1}(\Omega)$ with the norm topology and if operators A_h are linear and uniformly elliptic, then the given definition will coincide with the classical one introduced by S.Spagnolo ([S]).

3 Fenchel conjugate functions

Let X be a Banach space and let X^* be its dual. Let

$$F : X \to \overline{\boldsymbol{R}}$$

be *a proper function*, i.e. such that is not everywhere equal to $+\infty$ and anywhere to $-\infty$. We define a function

$$F^* : X^* \to \overline{\boldsymbol{R}},$$

which we shall call *the Fenchel conjugate to* F, by the formula

$$F^*(x^*) = \sup_{x \in X} \left\{ \langle x^*, x \rangle - F(x) \right\},$$

where

$$\langle \cdot, \cdot \rangle : X^* \times X \to \boldsymbol{R}$$

denotes the duality pairing between X^* and X. F^* is always convex and l.s.c.; of course we can say the same about the bi-conjugate

$$F^{**}(x) = \sup_{x^* \in X^*} \left\{ \langle x^*, x \rangle - F^*(x^*) \right\}.$$

Moreover, F^{**} is *the convex and lower semicontinuous envelope for* F, so if F is proper, convex and l.s.c., then $F^{**} = F$.

For integral functionals we have the following representation theorem.

Theorem 3.1 ([ET]) *Let Ω be an open and bounded subset of $\boldsymbol{R}^d$ and let*

$$f : \Omega \times \boldsymbol{R}^n \to \boldsymbol{R}$$

be a Borel function bounded from below. Define for $u \in L^p(\Omega; \boldsymbol{R}^n)$, $p \in (1, +\infty)$

$$F(u) = \int_\Omega f(x, u(x)) dx.$$

Assume that

1. *$f(x, \cdot)$ is l.s.c. for almost every $x \in \Omega$;*
2. *there exists $\overline{u} \in L^\infty(\Omega; \boldsymbol{R}^n)$ such that $F(\overline{u}) < +\infty$.*

Then

$$\begin{aligned} F^*(u^*) &= \int_\Omega f^*(x, u^*(x))dx, \\ F^{**}(u) &= \int_\Omega f^{**}(x, u(x))dx, \end{aligned}$$

where $u \in L^p(\Omega; \boldsymbol{R}^n)$, $u^* \in L^{p'}(\Omega; \boldsymbol{R}^n)$, $1/p + 1/p' = 1$, *and* f^*, f^{**} *are taken with respect to the second variable.*

The next theorem relates Γ-convergence of convex functions with Γ-convergence of their conjugates.

Theorem 3.2 ([A], theorem 3.2.4) *Let* X *be a separable Banach space and let* F, F_h *be proper, convex and l.s.c. functionals on* X*. Assume that*

1.
$$\Gamma(X^-) \lim_{h\to\infty} F_h = F;$$

2.
$$\limsup_{h\to\infty} F_h^*(x_h^*) < +\infty \Rightarrow \sup_h ||x_h^*|| < +\infty.$$

Then
$$\Gamma(w^* X^{*-}) \lim_{h\to\infty} F_h^* = F^*,$$

where $w^* X^*$ *denotes the dual to* X *endowed with its weak* topology.*

4 Abstract model

Let U and Y be two topological spaces called, respectively *the space of controls* and *the space of states.* Let

$$J_h : U \times Y \to \overline{\boldsymbol{R}}$$

be a sequence of *cost functionals*. In the sequel we will always assume that they are uniformly bounded from below. We consider the following sequence of abstract optimal control problems

$$\min \{J_h(u, y) : A_h(y) \in B_h(u),\ (u, y) \in U \times Y\}, \tag{2}$$

where

$$A_h \ : \ Y \to V,$$
$$B_h \ : \ U \to \mathcal{P}(V),$$

map Y and U into another topological space V.

Obviously, the problem (2) is equivalent to the following

$$\min \left\{ J_h(u,y) + \chi_{\{A_h(y)\in B_h(u)\}}(u,y) : (u,y) \in U \times Y \right\}$$

with the minimization over the whole Cartesian product $U \times Y$.

In this paper the method of *auxiliary variable* (see e.g. [BC],[B]) is used. It is based on the following proposition.

Proposition 4.1 ([BC], proposition 2.3) *Let U, V and Y be topological spaces. Let $F_h : U \times Y \to \overline{\boldsymbol{R}}$ be a sequence of functions uniformly bounded from below and $\Xi_h : U \times Y \to \mathcal{P}(V)$ — a sequence of multimappings. Put*

$$\Phi_h(u,v,y) = \begin{cases} F_h(u,y) & \text{when} \quad v \in \Xi_h(u,y), \\ +\infty & \text{otherwise} \end{cases}$$

and assume that

1. *for any convergent sequence (u_h, y_h) such that $\{F_h(u_h,y_h)\}$ is bounded there exists a sequence $v_h \in \Xi_h(u_h,y_h)$ which is relatively compact in V;*

2. *for every $(u,v,y) \in U \times V \times Y$ the limit*

$$\Gamma((U\times V)^-, Y) \lim_{h\to\infty} \Phi_h(u,v,y)$$

exists.

Then

$$\Gamma(U^-,Y^-) \lim_{h\to\infty} F_h(u,y) = \inf\left\{ \Gamma((U\times V)^-, Y^-) \lim_{h\to\infty} \Phi_h(u,v,y) : v \in V \right\}.$$

Put

$$G_h(u,v,y) = J_h(u,y) + \chi_{\{v\in B_h(u)\}}(u,v). \tag{3}$$

In our abstract model the form of the limit problem is given by the following theorem, corresponding to the Theorem 3.3 of [BF].

Theorem 4.2 *Assume that*

1. *A_h is G-convergent to A;*
2. *if $u_h \to u$, $y_h \to y$, $A_h(y_h) \in B_h(u_h)$ and $\{J_h(u_h, y_h)\}$ is bounded, then $A_h(y_h)$ is relatively compact in V;*
3. *the limit*

$$G(u, v, y) = \Gamma((U \times V)^-, Y^-) \lim_{h\to\infty} G_h(u, v, y)$$

exists.

Then

$$\Gamma(U^-, Y^-) \lim_{h\to\infty} \left(J_h + \chi_{\{A_h(y)\in B_h(u)\}}\right)(u, y) = G(u, A(y), y).$$

Proof. Put

$$\begin{aligned} F_h(u, y) &= J_h(u, y) + \chi_{\{A_h(y)\in B_h(u)\}}, \\ \Xi_h(u, y) &= \{A_h(y)\} \cap B_h(u) \end{aligned}$$

and let Φ_h be the same as in the previous proposition. Notice that

$$\begin{aligned} \Phi_h(u, v, y) &= J_h(u, y) + \chi_{\{v\in B_h(u)\}}(u, v) + \chi_{\{A_h(y)=v\}}(v, y) \\ &= G_h(u, v, y) + \chi_{\{A_h(y)=v\}}(v, y). \end{aligned}$$

The assumption 2 makes it possible to use the proposition 4.1. Applying it as well as the proposition 2.4 and the definition of the G-convergence we obtain

$$\begin{aligned} &\Gamma(U^-, Y^-) \lim_{h\to\infty} \left(J_h + \chi_{\{A_h(y)\in B_h(u)\}}\right)(u, y) \\ = &\inf_{v\in V} \left\{\Gamma((U \times V)^-, Y^-) \lim_{h\to\infty} \left(G_h + \chi_{\{A_h(y)=v\}}\right)(u, v, y)\right\} \\ = &\inf_{v\in V} \left\{\Gamma((U \times V)^-, Y) \lim_{h\to\infty} G_h(u, v, y) + \Gamma(V, Y^-) \lim_{h\to\infty} \chi_{\{A_h(y)=v\}}(v, y)\right\} \\ = &\inf_{v\in V} \left\{G(u, v, y) + \chi_{\{A(y)=v\}}(v, y)\right\} \\ = &G(u, A(y), y). \qquad \square \end{aligned}$$

Due to this theorem our task is now to find the explicit expression for G and we shall concentrate on it in the next sections.

5 The case of 'K-continuous' convergence of B_h

Let us first assume that operators $B_h : U \to \mathcal{P}(V)$ converge to $B : U \to \mathcal{P}(V)$ in the following manner:

$$u_h \to u \Rightarrow B_h(u_h) \stackrel{K}{\to} B(u), \tag{4}$$

where 'K' denotes Kuratowski convergence with respect to the topology of V.
Remark The above convergence condition is equivalent to the following (see proposition 2.5):

$$\Gamma(U, V^-) \lim_{h\to\infty} \chi_{\{v\in B_h(u)\}} = \chi_{\{v\in B(u)\}}.$$

Let us take the hypotheses:

(K1) there exists a function $\Psi : U \to \boldsymbol{R}$ such that if (u_h) is convergent, then $(\Psi(u_h))$ is bounded, and a function $\omega : Y \times Y \to \boldsymbol{R}$ such that for any $y \in Y$

$$\lim_{z\to y} \omega(y, z) = \lim_{z\to y} \omega(z, y) = 0$$

and for $u \in U$, $y, z \in Y$, $h \in \boldsymbol{N}$

$$|J_h(u, y) - J_h(u, z)| \le \Psi(u)\omega(y, z);$$

(K2) if $u_h \to u$, $y_h \to y$, $v_h \in B_h(u_h)$ and $\{J_h(u_h, y_h)\}$ is bounded, then $\{v_h\}$ is relatively compact.

Proposition 5.1 *Let G_h be defined by (3). Assume (4), (K1) and that the following limit*

$$\Gamma(U^-) \lim_{h\to\infty} J_h.$$

exists. Then

$$\Gamma((U \times V)^-, Y) \lim_{h\to\infty} G_h(u, v, y) = \Gamma(U^-) \lim_{h\to\infty} J_h(u, y) + \chi_{\{v\in B(u)\}}(u, v).$$

Proof. From (4) we know that if $v \in B(u)$, then there exist such $v_h \in B_h(u_h)$ that $v_h \to v$, and if, on the other hand, $v \notin B(u)$ with $v_h \to v$, then $v_h \notin B_h(u_h)$ for sufficiently large h. Let us take $u \in U$, $y \in Y$. If $v \in B(u)$, then from (4) we obtain

$$\begin{aligned} & \Gamma((U \times V)^-, Y^-) \liminf_{h\to\infty} G_h(u,v,y) \\ = & \inf_{u_h \to u} \inf_{v_h \to v} \inf_{y_h \to y} \liminf_{h\to\infty} G_h(u_h, v_h, y_h) \\ = & \inf_{u_h \to u} \inf_{y_h \to y} \liminf_{h\to\infty} J_h(u_h, y_h) \\ \geq & \inf_{u_h \to u} \inf_{y_h \to y} \liminf_{h\to\infty} [J_h(u_h, y) - \Psi(u_h)\omega(y_h, y)] \\ = & \inf_{u_h \to u} \liminf_{h\to\infty} J_h(u_h, y) \\ = & \Gamma(U^-) \liminf_{h\to\infty} J_h(u,y); \end{aligned}$$

on the other hand

$$\begin{aligned} & \Gamma((U \times V)^-, Y^+) \limsup_{h\to\infty} G_h(u,v,y) \\ = & \inf_{u_h \to u} \inf_{v_h \to v} \sup_{y_h \to y} \limsup_{h\to\infty} G_h(u_h, v_h, y_h) \\ = & \inf_{u_h \to u} \sup_{y_h \to y} \limsup_{h\to\infty} J_h(u_h, y_h) \\ \leq & \inf_{u_h \to u} \sup_{y_h \to y} \limsup_{h\to\infty} [J_h(u_h, y) + \Psi(u_h)\omega(y_h, y)] \\ = & \inf_{u_h \to u} \limsup_{h\to\infty} J_h(u_h, y) \\ = & \Gamma(U^-) \limsup_{h\to\infty} J_h(u,y). \end{aligned}$$

If $v \notin B(u)$, then both inequalities become trivial. Joining these facts we finish the proof. □

Theorem 5.2 *Assume (4), (K1), (K2) and*

1. $A_h \xrightarrow{G} A$,
2. $\Gamma(U^-) \lim_{h\to\infty} J_h = J$.

Then

$$\Gamma(U^-, Y^-) \lim_{h\to\infty} \left(J_h + \chi_{\{A_h(y) \in B_h(u)\}}\right) = J + \chi_{\{A(y)\in B(u)\}}.$$

Proof. (K2) makes sure that the assumption 2 of the theorem 4.2 is satisfied. Applying that theorem together with the proposition 5.1 we complete the proof. □

Remark Under the assumptions of the theorem the limit problem has the form

$$\min\left\{J(u,y) : A(y) \in B(u), (u,y) \in U \times Y\right\}.$$

Remark The hypothesis (K2) holds e.g. when U and V are reflexive Banach spaces endowed with their weak topologies, and for some constants $a, b > 0$, $c \geq 0$ we have

$$J_h(u,y) \geq a||u||_U + b \sup_{v \in B_h(u)} ||v||_V - c.$$

Remark The theorem 5.2 is a generalization of the theorem 3.6 of [BF] to the case of state inclusions (see also the introduction).

6 The local case

In this section we shall deal with the case when B_h are local operators between Lebesgue spaces and J_h are integral functionals. Namely, let Ω be an open and bounded subset of $\boldsymbol{R}^d$. Let $p, q \in (1, +\infty)$ and let p', q' be their adjoint exponents. We take $U = L^p(\Omega; \boldsymbol{R}^m)$, $V = L^q(\Omega; \boldsymbol{R}^k)$, both endowed with their weak topologies, but in the dual spaces U^* and V^* we choose the strong topologies. As Y we may take any space of measurable functions from Ω into $\boldsymbol{R}^n$ which can be embedded in $L^s(\Omega; \boldsymbol{R}^n)$ (with the norm topology) for some $s \in [1, +\infty]$. Let

$$B_h : U \to \mathcal{P}(V)$$

be defined by:

$$B_h(u) = \{v \in V \mid v(x) \in b_h(x, u(x)) \text{ } a.e. \text{ } in \text{ } \Omega\},$$

where

$$b_h : \Omega \times \boldsymbol{R}^m \to \mathcal{P}(\boldsymbol{R}^k)$$

are some multivalued Borel mappings, i.e. such that

$$\text{Gr } b_h = \left\{(x, u, v) \in \Omega \times \boldsymbol{R}^m \times \boldsymbol{R}^k \mid v \in b_h(x, u)\right\}$$

are Borel sets. Cost functionals are of the integral type

$$J_h(u,y) = \int_\Omega f_h(x, y(x), u(x))dx$$

for some Borel functions

$$f_h : \Omega \times \boldsymbol{R}^n \times \boldsymbol{R}^m \to (-\infty, +\infty].$$

In the sequel we shall take the following hypotheses:

(L1) b_h have non empty values and the sets

$$\text{Gr } b_h(x,\cdot) = \big\{(u,v) \in \boldsymbol{R}^m \times \boldsymbol{R}^d \mid v \in b_h(x,u)\big\}$$

are closed;

(L2) $f_h(x,y,\cdot)$ is l.s.c. on $\boldsymbol{R}^m$ for almost every $x \in \Omega$ and every $y \in \boldsymbol{R}^n$;

(L3) there exist $a, b > 0$ and $c \in L^\infty(\Omega)$ such that for every $(x,y,u) \in \Omega \times \boldsymbol{R}^n \times \boldsymbol{R}^m$, $v \in \boldsymbol{R}^k$ and $h \in \boldsymbol{N}$

$$f_h(x,y,u) + \chi_{\{v \in b_h(x,u)\}}(x,u,v) \geq a|u|^p + b|v|^q - c(x);$$

(L4) there exists a function $\sigma : \Omega \times \boldsymbol{R}^n \times \boldsymbol{R}^n \to [0,+\infty)$ measurable with respect to the first variable, continuous with respect to the others and such that

$$z \xrightarrow{L^s} y \quad \Rightarrow \quad \sigma(\cdot, y(\cdot), z(\cdot)) \xrightarrow{L^1} 0$$

and for every $x \in \Omega$, $u \in \boldsymbol{R}^m$, $y, z \in \boldsymbol{R}^n$, $h \in \boldsymbol{N}$

$$f_h(x,y,u) \leq f_h(x,z,u) + \sigma(x,y,z);$$

(L5) there exist $u_h \in L^\infty(\Omega; \boldsymbol{R}^m)$ and $v_h \in B_h(u_h) \cap L^\infty(\Omega; \boldsymbol{R}^k)$ such that for every $y \in Y$

$$\int_\Omega f_h(x, y(x), u_h(x))dx < +\infty.$$

Let G_h be defined by (3). Notice that

$$G_h(u,v,y) = \int_\Omega g_h(x, y(x), u(x), v(x))dx,$$

where $g_h : \Omega \times \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^k \to \overline{\boldsymbol{R}}$ is given by

$$g_h(x,y,u,v) = f_h(x,y,u) + \chi_{\{v \in b_h(x,u)\}}(x,u,v).$$

Thanks to (L1), (L2) and (L5) we can apply the theorem 3.1 to G_h obtaining

$$\begin{aligned} G_h^*(u^*, v^*, y) &= \int_\Omega g_h^*(x, y(x), u^*(x), v^*(x))dx, \\ G_h^{**}(u,v,y) &= \int_\Omega g_h^{**}(x, y(x), u(x), v(x))dx, \end{aligned}$$

while the conjugates for g_h are taken with respect to last two variables.

Lemma 6.1 *If the limit*

$$\Gamma((U^* \times V^*)^-) \lim_{h\to\infty} G_h^*,$$

exists, then

$$\Gamma((U \times V)^-) \lim_{h\to\infty} G_h^{**} = \left(\Gamma((U^* \times V^*)^-) \lim_{h\to\infty} G_h^*\right)^*.$$

Proof. Notice that thanks to (L3) for $y \in Y$, $u \in U$ and $v \in B_h(u)$ we have

$$J_h(u,y) \geq a||u||^p + b||v||^q - c_0, \tag{5}$$

or, in other words, for $y \in Y$, $u \in U$, $v \in V$ we have

$$G_h(u,v,y) \geq a||u||^p + b||v||^q - c_0, \tag{6}$$

where $c_0 = \min\{0, \text{ess inf } c\}$. Thus

$$\begin{aligned} & G_h^*(u^*, v^*, y) = \\ = \ & \sup\{\langle u^*, u\rangle + \langle v^*, v\rangle - J_h(u,y) : u \in U, v \in B_h(u)\} \\ \leq \ & \sup\{||u^*||||u|| + ||v^*||||v|| - a||u||^p - b||v||^q + c_0 : u \in U, v \in V\} \qquad (7) \\ \leq \ & \sup\{||u^*||||u|| - a||u||^p : u \in U\} + \sup\{||v^*||||v|| - b||v||^q : v \in V\} + c_0 \\ = \ & \alpha||u^*||^{p'} + \beta||v^*||^{q'} + c_0, \end{aligned}$$

where

$$\alpha = \frac{1}{p'}(ap)^{\frac{1}{1-p}},$$
$$\beta = \frac{1}{q'}(bq)^{\frac{1}{1-q}}.$$

Similarly (7) implies

$$\begin{aligned} & G_h^{**}(u,v,y) = \\ = \; & \sup\{\langle u^*,u\rangle + \langle v^*,v\rangle - G_h^*(u^*,v^*,y) : u^* \in U^*, v^* \in V^*\} \qquad (8) \\ \geq \; & \sup\left\{\langle u^*,u\rangle - \alpha||u^*||^{p'} : u^* \in U^*\right\} + \sup\left\{\langle v^*,v\rangle - \beta||v^*||^{q'} : v^* \in V^*\right\} - c_0 \\ = \; & a||u||^p + b||v||^q - c_0 \end{aligned}$$

Fix $y \in Y$. Put $X = U^* \times V^*$ and

$$F_h(u^*,v^*) = G_h^*(u^*,v^*,y).$$

As Fenchel conjugates F_h are convex and l.s.c., thanks to (7) they are also proper. The assumption 2 of the theorem 3.2 is satisfied due to the inequality (8) — now our thesis is a consequence of this theorem. □

Proposition 6.2 *For any $(u^*,v^*) \in U^* \times V^*$ and $y \in Y$ we have*

$$\Gamma((U^* \times V^*)^-) \lim_{h\to\infty} G_h^*(u^*,v^*,y) = \lim_{h\to\infty} G_h^*(u^*,v^*,y).$$

Proof. Notice first that $U^* \times V^*$, as Banach space endowed with the strong topology satisfies the first axiom of countability. Functionals $G_h^*(\cdot,\cdot,y)$ are convex, l.s.c. and, thanks to (7), locally uniformly bounded. Hence ([ET], corollaries I.2.4, I.2.5, remark I.2.1) they are locally equi Lipschitz. Now from the proposition 2.2 we obtain the equality between the Γ-limit and the pointwise one. □

Proposition 6.3 *For every $(u,v) \in U \times V$ and $y \in Y$ we have*

$$\Gamma((U \times V)^-, Y) \lim_{h\to\infty} G_h(u,v,y) = \left(\lim_{h\to\infty} G_h^*\right)^*(u,v,y).$$

Proof. Thanks to (L4) we have

$$J_h(u,y) \le J_h(u,z) + \omega(y,z),$$

while

$$\lim_{z\to y} \omega(y,z) = \lim_{z\to y} \omega(z,y) = 0$$

and, similarly as in the proof of the proposition 5.1, we can show that

$$\Gamma((U\times V)^-, Y) \lim_{h\to\infty} G_h = \Gamma((U\times V)^-) \lim_{h\to\infty} G_h.$$

Because of (L5) from the remark 2.6.5 of [B] we have

$$\mathrm{sc}^-(U\times V) G_h(\cdot,\cdot,y) = G_h^{**}(\cdot,\cdot,y).$$

for any fixed $y \in Y$. The inequality (6) allows us to apply the proposition 2.1 to $G_h(\cdot,\cdot,y)$. Thus we obtain

$$\Gamma((U\times V)^-) \lim_{h\to\infty} G_h = \Gamma((U\times V)^-) \lim_{h\to\infty} G_h^{**}.$$

The thesis is now a consequence of the lemma 6.1 and the proposition 6.2. □

Lemma 6.4 *There exists a function*

$$\omega^* : L^s(\Omega; \boldsymbol{R}^n) \times L^s(\Omega; \boldsymbol{R}^n) \to [0, +\infty)$$

such that for any y

$$\lim_{z\to y} \omega^*(y,z) = 0$$

and for $u^* \in U^*$, $v^* \in V^*$, $y, z \in L^s(\Omega; \boldsymbol{R}^n)$

$$G_h^*(u^*, v^*, y) \le G_h^*(u^*, v^*, z) + \omega^*(y,z).$$

Proof. The definition of the Fenchel conjugate implies that for any $\varepsilon > 0$ there exist such $u_\varepsilon \in U$ and $v_\varepsilon \in B_h(u_\varepsilon)$ that

$$G_h^*(u^*, v^*, y) \le \langle u^*, u_\varepsilon\rangle + \langle v^*, v_\varepsilon\rangle - \int_\Omega f_h(x, y(x), u_\varepsilon(x))dx + \varepsilon. \qquad (9)$$

On the other hand

$$G_h^*(u^*, v^*, z) \geq \langle u^*, u_\varepsilon \rangle + \langle v^*, v_\varepsilon \rangle - \int_\Omega f_h(x, z(x), u_\varepsilon(x))dx. \qquad (10)$$

Joining (9) with (10) we obtain

$$G_h^*(u^*, v^*, y) - G_h^*(u^*, v^*, z) \leq \int_\Omega \left[f_h(x, z(x), u_\varepsilon(x)) - f_h(x, y(x), u_\varepsilon(x)) \right] dx + \varepsilon.$$

Now putting

$$\omega^*(y, z) = \int_\Omega \sigma(x, y(x), z(x))dx,$$

and using (L4) we finish the proof. □

Proposition 6.5 ([BF], proposition 5.2) *Assume that for any* $y \in \boldsymbol{R}^n$, $u^* \in \boldsymbol{R}^m$, $v^* \in \boldsymbol{R}^k$

$$g_h^*(\cdot, y, u^*, v^*) \to \varphi(\cdot, y, u^*, v^*) \qquad (11)$$

weakly in $L^1(\Omega)$. *Then for* $y \in Y$, $u^* \in U^*$ *and* $v^* \in V^*$

$$\lim_{h \to \infty} G_h^*(u^*, v^*, y) = \Phi(u^*, v^*, y),$$

where Φ *is given by*

$$\Phi(u^*, v^*, y) = \int_\Omega \varphi(x, y(x), u^*(x), v^*(x))dx.$$

To sum up, we have proved the following theorem.

Corollary 6.6 *Assume that* (A_h) *G-converges to* A. *If the hypotheses (L1)–(L5) and (11) hold, then the limit problem has the form*

$$\min \left\{ \int_\Omega \varphi^*(x, y(x), u(x), Ay(x))dx : (u, y) \in L^p(\Omega; \boldsymbol{R}^m) \times Y \right\}.$$

Remark This corollary generalizes the results of the section 5 of [BF] (see also the introduction).

7 Examples

1. Let $U = V$ be a reflexive Banach space endowed with the weak topology. Let us take a sequence r_h of positive numbers such that

$$r_h \to r$$

and define

$$B_h(u) = B(u, r_h)$$

(the limit problem will not change if, instead of open balls, we take closed ones). We will show that

$$u_h \xrightarrow{w} u \Rightarrow B(u_h, r_h) \xrightarrow{w-K} \overline{B}(u, r). \tag{12}$$

Namely, take $u_h \to u$ and $v_h \in B(u_h, r_h)$ such that

$$v_{h_k} \to v.$$

Then, of course,

$$v_{h_k} - u_{h_k} \to v - u.$$

The norm is weakly l.s.c. (as it is strongly continuous and convex), hence

$$||v - u|| \leq \liminf_{k\to\infty} ||v_{h_k} - u_{h_k}|| \leq \liminf_{k\to\infty} r_{h_k} = r.$$

It means that $v \in \overline{B}(u, r)$, so

$$K-\limsup_{h\to\infty} B(u_h, r_h) \subset \overline{B}(u, r).$$

Now let $v \in \overline{B}(u, r)$. Put

$$v_h = u_h + \lambda_h(v - u),$$

where

$$\lambda_h = \frac{r_h}{r}(1 - \frac{1}{h}).$$

Then

$$||v_h - u_h|| = \lambda_h ||v - u|| = (1 - \frac{1}{h}) r_h \frac{||v - u||}{r} < r_h \tag{13}$$

and

$$v_h \to u + (v - u) = v. \tag{14}$$

The case $r = 0$ is simpler, because then it is appropriate to take $u_h = v_h$. Joining (13) with (14) we obtain

$$\overline{B}(u, r) \subset K - \liminf_{h\to\infty} B(u_h, r_h),$$

which, together with the obvious inclusion

$$K - \liminf_{h\to\infty} B(u_h, r_h) \subset K - \limsup_{h\to\infty} B(u_h, r_h),$$

implies (12). If $u_h \stackrel{w}{\to} u$, then

$$\sup_h ||u_h|| < +\infty$$

and for $v_h \in B(u_h, r_h)$ we have

$$||v|| - ||u_h|| \leq ||v - u_h|| < r_h,$$

therefore

$$||v|| < ||u_h|| + r_h \leq \sup_h ||u_h|| + \sup_h r_h.$$

In other words, there exists a positive number R such that for all $h \in \boldsymbol{N}$

$$B(u_h, r_h) \subset B(0, R),$$

and this, thanks to the Banach-Alaoglu Theorem, means that the assumption (K2) is satisfied. Now, having applied the theorem 5.2, we see that if (A_h) is G-convergent to A, (J_h) is $\Gamma(U^-)$-convergent to J and (K1) is satisfied, then the limit problem for the sequence

$$\min \{J_h(u, y) : ||A_h(y) - u_h|| < r_h, (u, y) \in U \times Y\}$$

has the form

$$\min \{J(u, y) : ||A(y) - u|| \leq r, (u, y) \in U \times Y\}.$$

2. Let Ω be an open and bounded subset of $\boldsymbol{R}^d$. Let us take $U = V = L^2(\Omega)$, $Y \subset L^2(\Omega)$ and

$$J_h(u,y) = \int_\Omega \left[u^2(x) + |y(x) - y_0(x)|^2\right] dx,$$

where y_0 is a function from $L^2(\Omega)$. Let B_h be given by a multifunction

$$b_h(x,u) = [-\beta_h(x)|u|, \beta_h(x)|u|].$$

Assume that $\beta_h \in L^\infty(\Omega)$ are nonnegative and

$$\begin{array}{rcl} \beta_h & \rightarrow & \beta \\ \beta_h^2 & \rightarrow & \beta^2 \end{array}$$

weakly* in $L^\infty(\Omega)$. Then the assumptions (L1)–(L5) can be easily verified and we have

$$\begin{array}{rcl} g_h^*(x,y,u^*,v^*) & = & \frac{1}{4}\left(|u^*| + \beta_h(x)|v^*|\right)^2 - |y - y_0(x)|^2 \\ \varphi(x,y,u^*,v^*) & = & \frac{1}{4}\left(|u^*| + \beta(x)|v^*|\right)^2 - |y - y_0(x)|^2. \end{array}$$

If $\beta > 0$ almost everywhere, then

$$\varphi^*(x,y,u,v) = \max\{u^2, \frac{v^2}{\beta^2}\} + |y - y_0(x)|^2$$

and, thanks to the theorem 6.6, the limit problem for the sequence

$$\min\left\{\int_\Omega [u^2 + |y - y_0|^2]dx : -\beta_h|u| \le A_h y \le \beta_h|u|\right\}$$

is

$$\min\left\{\int_\Omega \left[\left(u^2 \vee \frac{1}{\beta^2}(Ay)^2\right) + |y - y_0|^2\right] dx : (u,y) \in U \times Y\right\}.$$

If, however, $\beta = 0$, then

$$\varphi^*(x,y,u,v) = u^2 + \chi_{\{v=0\}}(v) + |y - y_0(x)|^2$$

and the limit problem has the form

$$\begin{array}{rl} & \min\left\{\int_\Omega [u^2 + |y - y_0|^2]dx : A(y) = 0\right\} \\ = & \min\left\{||y - y_0||^2 : A(y) = 0, u = 0\right\}. \end{array}$$

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