Relaxed parabolic problems

Maciej Smołka *

Jagiellonian University, Institute of Computer Science
ul. Nawojki 11 p. 302
30-072 Kraków (Poland)
e-mail: smolka@ii.uj.edu.pl

0 Introduction

The base of this work is the following problem: having a sequence (G_n) of open subsets of a fixed open and bounded $\Omega \subset \mathbb{R}^N$ describe the asymptotic behaviour (as $n \to \infty$) of the sequence of solutions of parabolic equations

$$\begin{cases} u_n' + Au_n = f_n & \text{in } (0,T) \times G_n \\ u_n(0) = u_n^0 & \text{in } G_n \\ u = 0 & \text{on } (0,T) \times (\Omega \setminus G_n). \end{cases} \tag{0.1}$$

In the elliptic case this problem has been thoroughly investigated in many works, starting from [5], through [10], [11], [6], [2], to [8], [9], [12], [13], [3] and others. There is also a paper [4] concerning the hyperbolic case. In both cases, as well as in ours, the limit of (u_n) (if it exists) in general does not satisfy the equation of the type (0.1), but instead, under some assumptions, it appears to be a solution of the equation

$$\begin{cases} u' + Au + \mu u = f \\ u(0) = u^0 \\ u = 0 \text{ on } (0,T) \times \partial\Omega \end{cases}$$

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for a suitable measure μ. Such an equation is called the relaxed parabolic problem.

The present paper contains the basic theory of relaxed parabolic problems. In particular we show the existence and the uniqueness of solutions of such problems, along with some regularity properties. The main goal is to prove the stability of the class of relaxed parabolic problems under the elliptic γ^A-convergence and this is achieved in Theorem 2.9. When this paper was completed, I came across a work by J.-P. Raymond (private communication) containing similar results and also one by R. Toader (PhD. thesis, SISSA, Trieste) concerning hyperbolic equations, but using the same evolution triplet and similar assumptions in the convergence theorem.

In order to achieve our goals, we adapt the techniques contained in the papers mentioned at the beginning from elliptic and hyperbolic cases.

The results of this work are interesting in themselves and have applications e.g. in homogenization, but they are in fact thought to be applied in shape optimization. Therefore this paper can be considered as the first part of a bigger entirety, [16] being the second part.

The plan of the work is the following. In section 1 we recall some notations, definitions and basic properties from the elliptic case. For convenience of the reader we also show there the separability of $V_\mu(\Omega)$. Section 2 contains the main results of the paper (mentioned above) as well as some useful facts concerning the relaxed parabolic problems and related notions (e.g. a characterization of the pivot space $H_\mu(\Omega)$).

1 Preliminaries. Relaxed Dirichlet problems

Let Ω be an open and bounded subset of $\mathbb{R}^N, N \geq 2$. We denote by $H^1(\Omega)$ and $H_0^1(\Omega)$ the usual Sobolev spaces on Ω, and by $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$. On the latter space we consider the norm

$$||v||_{H_0^1(\Omega)} = \left(\int_\Omega |Dv|^2 dx \right)^{1/2}.$$

By $L^p_\mu(\Omega)$, $1 \leq p \leq +\infty$ we denote the usual Lebesgue space with respect to a measure μ on Ω. If μ is the N-dimensional Lebesgue measure Ω, we shall use the standard notation $L^p(\Omega)$. The Lebesgue measure of a set E we denote simply by $|E|$.

For every $E \subset \Omega$ the *(harmonic) capacity* of E in Ω is defined in the following way

$$\operatorname{cap}(E,\Omega) = \inf\Big\{ ||v||^2_{H^1_0(\Omega)} : v \in H^1_0(\Omega),$$
$$v \geq 1 \text{ a.e. in the neighbourhood of } E \Big\}.$$

We say that a property holds *quasi-everywhere* (q.e.) in a set E if it holds everywhere in $E \setminus \mathcal{N}$ with $\operatorname{cap}(\mathcal{N},\Omega) = 0$. A function $v : \Omega \longrightarrow \mathbb{R}$ is said to be *quasi-continuous* if for any $\varepsilon > 0$ there exists $E \subset \Omega$ with $\operatorname{cap}(E,\Omega) < \varepsilon$, such that the restriction of u to $\Omega \setminus E$ is continuous. We say that $G \subset \Omega$ is *quasi-open* if for every $\varepsilon > 0$ there exists an open subset U_ε of Ω such that $\operatorname{cap}(U_\varepsilon, \Omega) < \varepsilon$ and $G \cup U_\varepsilon$ is open.

From [18] we know that for $v \in H^1(\Omega)$ the limit

$$\lim_{r\to 0+} \frac{1}{|B(x,r)|} \int_{B(x,r)} v(y)dy$$

exists and is finite quasi-everywhere in Ω. So if we adopt the following convention concerning pointwise values of v

$$\liminf_{r\to 0+} \frac{1}{|B(x,r)|} \int_{B(x,r)} v(y)dy \leq v(x) \leq \limsup_{r\to 0+} \frac{1}{|B(x,r)|} \int_{B(x,r)} v(y)dy,$$

then we obtain a representative of the equivalence class v defined uniquely up to a set of capacity 0; moreover this representative is quasi-continuous on Ω. Also from [18] we know that if a sequence (v_n) converges to v in $H^1(\Omega)$, then a subsequence of (v_n) converges to v pointwise q.e. in Ω.

For a quasi-open subset G of Ω, we denote by $H^1_0(G)$ the space of all such functions $v \in H^1_0(\Omega)$ that $v = 0$ q.e. in $\Omega \setminus G$, with the Hilbert space structure inherited from $H^1_0(\Omega)$. If G is open, then this definition is equivalent to the standard one provided we use the convention that every $v \in H^1_0(G)$ is extended by 0 outside G in order to obtain an element of $H^1_0(\Omega)$. The following fact may be found in [8] (Lemma 2.1).

Proposition 1.1. *For every quasi-open subset G of Ω there exists an increasing sequence of nonnegative functions of $H^1_0(G)$ converging to 1_G pointwise q.e. in Ω.*

We denote by $\mathcal{M}_0(\Omega)$ the set of (nonnegative) Borel measures μ on Ω such that:

(M1) $\mu(B) = 0$ for every Borel set $B \subset \Omega$ with $\operatorname{cap}(B,\Omega) = 0$;

(M2) for every Borel set $B \subset \Omega$

$$\mu(B) = \inf\{\mu(G) : B \subset G, G \text{ quasi-open}\}.$$

Example 1.2. If $\mu = \varphi\mathcal{L}^N$ for a nonnegative function $\varphi \in L^\infty(\Omega)$, then $\mu \in \mathcal{M}_0(\Omega)$. It can also be shown (compare [18]) that for $N-2 < \alpha \leq N$ the restriction of the α-dimensional Hausdorff measure to a Borel set E with $\mathcal{H}^\alpha(E) < +\infty$ belongs to $\mathcal{M}_0(\Omega)$. In general, any Radon measure which belongs to $H^{-1}(\Omega)$, belongs to $\mathcal{M}_0(\Omega)$ as well.

Example 1.3. Another example of an element of $\mathcal{M}_0(\Omega)$, which will be useful in the sequel is the measure defined for any quasi-closed subset S of Ω by the formula

$$\infty_S(B) = \begin{cases} 0 & \text{if } \operatorname{cap}(B \cap S, \Omega) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

For $\mu \in \mathcal{M}_0(\Omega)$ we consider the space

$$V_\mu(\Omega) = H_0^1(\Omega) \cap L_\mu^2(\Omega).$$

endowed with the scalar product

$$(u, v)_{V_\mu(\Omega)} = \int_\Omega (Du, Dv)\, dx + \int_\Omega uv d\mu.$$

From [2, Proposition 2.1] we know that $V_\mu(\Omega)$ is a Hilbert space.

Example 1.4. If $\mu = \varphi\mathcal{L}^N$ for $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$, then $V_\mu(\Omega) = H_0^1(\Omega)$ with the equivalence of the norms.

Example 1.5. If $G \subset \Omega$ is quasi-open and $\mu = \infty_{\Omega \setminus G}$, then $V_\mu(\Omega) = H_0^1(G)$ and the respective norms are equal.

By $V_\mu'(\Omega)$ we denote the dual of $V_\mu(\Omega)$ and by $\langle \cdot, \cdot \rangle_\mu$ the duality pairing between those spaces (when μ is the Lebesgue measure we shall use the standard notation $\langle \cdot, \cdot \rangle$). We have two natural embeddings

$$\begin{array}{rcl} i_1 : V_\mu(\Omega) & \longrightarrow & H_0^1(\Omega) \\ i_2 : V_\mu(\Omega) & \longrightarrow & L_\mu^2(\Omega). \end{array}$$

In general neither of them is dense (see below), so the transposed 'embeddings'

$$\begin{array}{rrcl} i_1^* : & H^{-1}(\Omega) & \longrightarrow & V_\mu'(\Omega) \\ i_2^* : & L_\mu^2(\Omega) & \longrightarrow & V_\mu'(\Omega) \end{array}$$

may not be injective.

Example 1.6. If $\mu = \infty_\Omega$, then $V_\mu(\Omega) = V'_\mu(\Omega) = \{0\}$, so we have $i_1^*(f) = 0$ for every $f \in H^{-1}(\Omega)$.

Nevertheless we shall write f instead of $i_1^*(f)$ for $f \in H^{-1}(\Omega)$. Due to this convention, for such f we have

$$\langle f, v\rangle_\mu = \langle f, v\rangle \quad \forall v \in V_\mu(\Omega).$$

For $g \in L^2_\mu(\Omega)$ we use the notation $\mu g = i_2^*(g)$, i.e.

$$\langle \mu g, v\rangle_\mu = \int_\Omega g v d\mu \quad \forall v \in V_\mu(\Omega).$$

Let $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ be a linear symmetric elliptic operator of the divergence form

$$Au = -\sum_{i,j=1}^N D_i(a_{ij} D_j u),$$

where (a_{ij}) is a (symmetric) $N \times N$ matrix of functions of $L^\infty(\Omega)$ satisfying, for a positive constant α, the coercivity condition

$$\sum_{i,j=1}^N a_{ij}(x)\xi_j\xi_i \geq \alpha|\xi|^2$$

for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^N$. Let us also introduce the notation

$$C_0 = \max_{i,j=1,\dots,N} ||a_{ij}||_{L^\infty(\Omega)}.$$

Let $\mu \in \mathcal{M}_0(\Omega)$, $f \in H^{-1}(\Omega)$. The following problem is called *relaxed Dirichlet problem*:

$$\begin{cases} Au + \mu u = f \\ u \in V_\mu(\Omega). \end{cases} \tag{1.1}$$

The above problem, due to the convention concerning the mappings i_1^* and i_2^* should be understood as follows: find $u \in V_\mu(\Omega)$ such that

$$\langle Au, v\rangle + \int_\Omega u v d\mu = \langle f, v\rangle$$

for every $v \in V_\mu(\Omega)$. A straightforward application of Lax-Milgram Lemma is the following theorem.

Theorem 1.7. *For every $f \in H^{-1}(\Omega)$ there exists a unique solution of problem* (1.1)*.*

Remark 1.8. If G is an open subset of Ω and $\mu = \infty_{\Omega \setminus G}$, then it is easy to show that $u \in H_0^1(\Omega)$ is the solution of problem (1.1) if and only if $u = 0$ q.e. in $\Omega \setminus G$ and u is a solution of the classical (homogeneous) Dirichlet problem in G

$$\begin{cases} Au = f|_G \\ u \in H_0^1(G). \end{cases}$$

Remark 1.9. It is obvious that Theorem 1.7 remains true if as the right-hand side of equation (1.1) we take $f \in V_\mu'(\Omega)$.

The following relaxed problem is especially important in the sequel

$$\begin{cases} Aw + \mu w = 1 \\ w \in V_\mu(\Omega). \end{cases} \tag{1.2}$$

There is a close connection between a measure μ and the solution of this problem (see [3], [8], [13], [12] for details). Here we recall only a few basic properties of w. First of all from the comparison principle ([8, Proposition 2.4]) we know that $w \in L^\infty(\Omega)$ and $w \geq 0$ q.e. in Ω. Let us define the sets

$$\begin{aligned} A(\mu) &= \{x \in \Omega : w(x) > 0\} \\ S(\mu) &= \Omega \setminus A(\mu) = \{x \in \Omega : w(x) = 0\}. \end{aligned}$$

w can be considered quasi-continuous, so $A(\mu)$ is quasi-open and $S(\mu)$ quasi-closed (both are defined up to null-capacity sets). $A(\mu)$ is called *the regular set of the measure* μ and $S(\mu)$ *the singular set of* μ.

Example 1.10. If G is an open subset of Ω and $\mu = \infty_{\Omega \setminus G}$, then from the strong maximum principle it follows that $A(\mu) = G$.

Proposition 1.11. *If $\mu \in \mathcal{M}_0(\Omega)$, then $\mu(B) = +\infty$ for any Borel subset B of Ω such that* $\operatorname{cap}(B \cap S(\mu), \Omega) > 0$*.*

Proof. See [8, Lemma 3.2]. $\square$

A straightforward consequence of this proposition is the following fact.

Corollary 1.12. $V_\mu(\Omega) \subset H_0^1(A(\mu))$*.*

Moreover it is known that

Lemma 1.13. *$H_0^1(A(\mu))$ is the closure of $V_\mu(\Omega)$ in $H_0^1(\Omega)$.*

Proof. See [2, Lemma 2.5]. □

The following very useful fact has been proved in [13] (Proposition 5.5).

Proposition 1.14. *Let w be the solution of problem* (1.2) *and let* $\beta \geq 1$*. Then the set* $w^\beta C_0^\infty(\Omega)$ *is dense in* $V_\mu(\Omega)$*.*

Corollary 1.15. *The Hilbert space* $V_\mu(\Omega)$ *is separable.*

Proof. First of all note that the space $C^1(\overline{\Omega})$ endowed with the norm

$$||\varphi|| = \sup_\Omega |\varphi| + \sup_\Omega |D\varphi| = ||\varphi||_{W^{1,\infty}(\Omega)}. \tag{1.3}$$

can be identified by means of the isometry

$$\Lambda : C^1(\overline{\Omega}) \ni v \longmapsto (v, Dv) \in C(\overline{\Omega}) \times C(\overline{\Omega}; \mathbb{R}^N),$$

with a subspace of the separable Banach space $C(\overline{\Omega}) \times C(\overline{\Omega}; \mathbb{R}^N)$, therefore it is separable. Furthermore the space $C_0^\infty(\Omega)$ with the norm (1.3) is separable as a subspace of the separable normed space $C^1(\overline{\Omega})$.

Let $\{\varphi_n\}_{n=1}^\infty$ be a countable and dense subset of $C_0^\infty(\Omega)$. Take arbitrary $v \in V_\mu(\Omega)$ and $\varepsilon > 0$. From Proposition 1.14 we know that there exists such $\varphi \in C_0^\infty(\Omega)$ that

$$||v - w\varphi||_{V_\mu(\Omega)} < \varepsilon.$$

Furthermore we can find $n \in \mathbb{N}$ such that

$$||\varphi - \varphi_n|| < \varepsilon.$$

Hence

$$\begin{aligned}
||v - w\varphi_n||_{V_\mu(\Omega)} &\leq ||u - w\varphi||_{V_\mu(\Omega)} + ||w\varphi - w\varphi_n||_{V_\mu(\Omega)} \\
&\leq \varepsilon + ||D(w\varphi) - D(w\varphi_n)||_{L^2(\Omega)} + ||w\varphi - w\varphi_n||_{L^2_\mu(\Omega)} \\
&\leq \varepsilon + ||(\varphi - \varphi_n)Dw||_{L^2(\Omega)} + ||w(D\varphi - D\varphi_n)||_{L^2(\Omega)} \\
&\quad + ||w||_{L^2_\mu(\Omega)} ||\varphi - \varphi_n||_{L^\infty(\Omega)} \\
&\leq \varepsilon + ||Dw||_{L^2(\Omega)} ||\varphi - \varphi_n||_{L^\infty(\Omega)} \\
&\quad + ||w||_{L^2(\Omega)} ||D\varphi - D\varphi_n||_{L^\infty(\Omega)} + ||w||_{L^2_\mu(\Omega)}\varepsilon \\
&\leq \left(1 + ||Dw||_{L^2(\Omega)} + ||w||_{L^2(\Omega)} + ||w||_{L^2_\mu(\Omega)}\right)\varepsilon.
\end{aligned}$$

This means that the set $\{w\varphi_n\}$ is dense in $V_\mu(\Omega)$, which concludes the proof. □

Definition 1.16. Let (μ_n) be a sequence of measures of $\mathcal{M}_0(\Omega)$ and let $\mu \in \mathcal{M}_0(\Omega)$. We say that (μ_n) *γ^A-converges to* μ if for every $f \in H^{-1}(\Omega)$ the sequence of the solutions of problems

$$\begin{cases} Au_n + \mu_n u_n = f \\ u_n \in V_{\mu_n}(\Omega) \end{cases} \tag{1.4}$$

converges weakly in $H_0^1(\Omega)$ to the solution of problem (1.1).

Remark 1.17. It should be noted that γ^A-limit depends on the operator A (for details see e.g. [8, Chapter 6]).

The paper [8] contains the following conditions equivalent to the γ^A-convergence.

Theorem 1.18. *Let $\mu_n, \mu \in \mathcal{M}_0(\Omega)$. Let w,w_n be the solutions of problems, respectively,* (1.2) *and*

$$\begin{cases} Aw_n + \mu_n w_n = 1 \\ w_n \in V_{\mu_n}(\Omega). \end{cases} \tag{1.5}$$

Then the following conditions are equivalent

1. *(μ_n) γ^A-converges to μ;*
2. *(w_n) converges to w weakly in $H_0^1(\Omega)$;*

Proof. See [8, Theorem 4.3]. □

Main properties of the topology of γ^A-convergence are contained in the following propositions.

Proposition 1.19. *Each sequence of measures of $\mathcal{M}_0(\Omega)$ contains a γ^A-convergent subsequence.*

Proof. See [8, Theorem 4.5]. □

Proposition 1.20. *Let λ be a nonnegative Radon measure. For every $\mu \in \mathcal{M}_0(\Omega)$ there exists a sequence (E_n) of compact subsets of Ω such that the corresponding sequence of measures ∞_{E_n} is γ^A-convergent to μ and $\lambda(E_n) = 0$ for $n \in \mathbb{N}$.*

Proof. See [3, Proposition 1.10]. □

The notion of 'γ^A-convergence' was first defined in [11] in terms of Γ-convergence of energy functionals. In our case there is also a connection between these two types of convergence. Let us recall that if X is (for simplicity) a Banach space, then we say that a sequence of functionals $F_n : X \to \overline{\mathbb{R}}$ is *Γ-convergent to* $F : X \to \overline{\mathbb{R}}$ if and only if two following conditions hold:

1. if $x_n \to x$ in X then
$$F(x) \leq \liminf_{n\to\infty} F_n(x_n);$$

2. for every $x \in X$ there exists a sequence (x_n) convergent to x in X such that
$$F(x) = \lim_{n\to\infty} F_n(x_n).$$

More information about the theory of Γ-convergence can be found in [7].

For $\mu \in \mathcal{M}_0(\Omega)$ we define *the energy functional*

$$F_\mu : L^2(\Omega) \longrightarrow \overline{\mathbb{R}}$$

by the formula

$$F_\mu(v) = \begin{cases} \langle Av, v\rangle + \int_\Omega v^2 d\mu & \text{if } v \in H_0^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, analogously to [11], we can show the following fact.

Proposition 1.21. *A sequence (μ_n) of measures of $\mathcal{M}_0(\Omega)$ is γ^A-convergent to μ if and only if the corresponding sequence of energy functionals (F_{μ_n}) is Γ-convergent to F_μ.*

Let us conclude this section with another property of the topology of γ^A-convergence, which can be proved as in [11, Proposition 4.9].

Proposition 1.22. *γ^A-convergence in $\mathcal{M}_0(\Omega)$ is metrizable.*

2 Main results

Let us fix $0 < T < +\infty$ and denote $Q = (0, T) \times \Omega$. The present section contains the generalization of results of the previous one to the case of first order evolution equations. For an arbitrary measure $\mu \in \mathcal{M}_0(\Omega)$ we introduce the following triplet of Hilbert spaces

$$V_\mu(\Omega) \subset H_\mu(\Omega) \subset V'_\mu(\Omega), \tag{2.1}$$

where $H_\mu(\Omega)$ is the closure of $V_\mu(\Omega)$ in the strong topology of $L^2(\Omega)$. Thus it is a separable Hilbert space with the inherited structure, so it can be identified with its dual by means of the same isometry as $L^2(\Omega)$.

Proposition 2.1. *Both embeddings in* (2.1) *are continuous, dense and compact.*

Proof. Due to the identification of $H_\mu(\Omega)$ with its dual it will be sufficient to prove these properties for first embedding. Its density is a straightforward consequence of the definition of $H_\mu(\Omega)$, its continuity follows from Poincaré inequality (see the proof of Corollary 2.4). To prove its compactness let us take a sequence (v_n) bounded in $V_\mu(\Omega)$. But

$$||v_n||^2_{V_\mu(\Omega)} = ||v_n||^2_{H^1_0(\Omega)} + \int_\Omega v_n^2 d\mu \geq ||v_n||^2_{H^1_0(\Omega)},$$

so (v_n) is also bounded in $H^1_0(\Omega)$, therefore it contains a subsequence convergent (strongly) in $L^2(\Omega)$. But since $H_\mu(\Omega)$ is closed, the limit of this subsequence must belong to $H_\mu(\Omega)$. $\square$

The next proposition gives some characterization of functions of $H_\mu(\Omega)$.

Proposition 2.2. *In the situation above*

$$H_\mu(\Omega) = \left\{ v \in L^2(\Omega) : \; v = 0 \;\; a.e. \; in \; S(\mu) \right\}$$

Proof. Define

$$Y = \left\{ v \in L^2(\Omega) : \; v = 0 \;\; a.e. \; in \; S(\mu) \right\}.$$

Y is a closed subspace of $L^2(\Omega)$. Hence the inclusion $V_\mu(\Omega) \subset Y$, which is a simple consequence of Corollary 1.12, implies that

$$H_\mu(\Omega) \subset Y.$$

In order to prove the opposite inclusion let us recall that $H^1_0(A(\mu))$ is the closure of $V_\mu(\Omega)$ in $H^1_0(\Omega)$ (Lemma 1.13), so it suffices to show that every function of Y may be approximated in $L^2(\Omega)$ by elements of $H^1_0(A(\mu))$. Fix then arbitrary $v \in Y$. Thanks to Proposition 1.1 we know that there exists an increasing sequence $z_n \in H^1_0(A(\mu))$ such that

$$0 \leq z_n \leq 1 \quad \text{q.e. in } A(\mu) \tag{2.2}$$

and $z_n \to 1_{A(\mu)}$ pointwise q.e. in $\mathbb{R}^N$. From the dominated convergence theorem it follows that

$$||v - vz_n||_{L^2(\Omega)} \to 0. \tag{2.3}$$

On the other hand it is well known that we can find such $\varphi_n \in C_0^\infty(\Omega)$ that

$$\varphi_n \to v \quad \text{in } L^2(\Omega). \tag{2.4}$$

Note that

$$z_n\varphi_n \in H_0^1(A(\mu)).$$

Moreover from (2.2), (2.3) and (2.4) we have

$$\begin{aligned} ||v - z_n\varphi_n||_{L^2(\Omega)} &\leq ||v - vz_n||_{L^2(\Omega)} + ||vz_n - \varphi_n z_n||_{L^2(\Omega)} \\ &\leq ||v - vz_n||_{L^2(\Omega)} + ||z_n||_{L^\infty(\Omega)}||v - \varphi_n||_{L^2(\Omega)} \\ &\leq ||v - vz_n||_{L^2(\Omega)} + ||v - \varphi_n||_{L^2(\Omega)} \to 0, \end{aligned}$$

which concludes the proof. □

We shall use the notation

$$W_\mu(0,T) = \left\{u \in L^2(0,T;V_\mu(\Omega)) : u' \in L^2(0,T;V_\mu'(\Omega))\right\}$$

where u' is the derivative in the sense of distributions with values in the Banach space $V_\mu'(\Omega)$. If μ is Lebesgue measure, we shall adopt the standard notation $W(0,T)$.

Proposition 2.3. *Let V and H be two Hilbert spaces such that V is dense in H and*

$$||v||_H \leq C_{V,H}||v||_V, \quad \forall v \in V.$$

Then every function $u \in L^2(0,T;V)$ such that $u' \in L^2(0,T;V')$ is a.e. in $(0,T)$ equal to a function

$$\tilde{u} \in C([0,T];H),$$

with

$$||\tilde{u}||^2_{C([0,T];H)} \leq \left(2 + \frac{C^2_{V,H}}{T}\right)\left(||u||^2_{L^2(0,T;V)} + ||u'||^2_{L^2(0,T;V')}\right). \tag{2.5}$$

Proof. The existence of $\tilde{u}$ is proved e.g. in [1, Proposition 3.2]. The following equality, holding for $s,t \in [0,T]$, is also shown in the proof of this proposition

$$||\tilde{u}(t)||^2_H - ||\tilde{u}(s)||^2_H = 2\int_s^t \langle u'(\tau), u(\tau)\rangle_{V'} d\tau. \tag{2.6}$$

Fix $t \in [0,T]$. From the above equality it follows in particular that

$$\begin{aligned}
||\tilde{u}(t)||_H^2 &= ||\tilde{u}(0)||_H^2 + 2\int_0^t \langle u'(\tau), u(\tau)\rangle_{V'} d\tau \\
&\leq ||\tilde{u}(0)||_H^2 + 2\int_0^T |\langle u'(\tau), u(\tau)\rangle_{V'}| d\tau \\
&\leq ||\tilde{u}(0)||_H^2 + 2\int_0^T ||u'(\tau)||_{V'} ||u(\tau)||_V d\tau.
\end{aligned}$$

Hölder inequality implies that

$$\begin{aligned}
||\tilde{u}(t)||_H^2 &\leq ||\tilde{u}(0)||_H^2 + 2||u'||_{L^2(0,T;V')}||u||_{L^2(0,T;V)} \\
&\leq ||\tilde{u}(0)||_H^2 + ||u'||^2_{L^2(0,T;V')} + ||u||^2_{L^2(0,T;V)}.
\end{aligned} \tag{2.7}$$

On the other hand

$$\begin{aligned}
||\tilde{u}(0)||_H^2 &= ||\tilde{u}(t)||_H^2 - 2\int_0^t \langle u'(\tau), u(\tau)\rangle_{V'} d\tau \\
&\leq ||\tilde{u}(t)||_H^2 + 2\int_0^t |\langle u'(\tau), u(\tau)\rangle_{V'}| d\tau \\
&\leq C_{V,H}^2 ||\tilde{u}(t)||_V^2 + 2\int_0^T |\langle u'(\tau), u(\tau)\rangle_{V'}| d\tau
\end{aligned}$$

and, similarly as before, we compute

$$||\tilde{u}(0)||_H^2 \leq C_{V,H}^2 ||\tilde{u}(t)||_V^2 + ||u'||^2_{L^2(0,T;V')} + ||u||^2_{L^2(0,T;V)}.$$

Taking the average over $(0,T)$ we obtain

$$||\tilde{u}(0)||_H^2 \leq \frac{C_{V,H}^2}{T} ||\tilde{u}||^2_{L^2(0,T;V)} + ||u'||^2_{L^2(0,T;V')} + ||u||^2_{L^2(0,T;V)}. \tag{2.8}$$

But

$$\tilde{u}(t) = u(t), \quad \text{a.e. in } (0,T),$$

hence, in particular,

$$||\tilde{u}||_{L^2(0,T;V)} = ||u||_{L^2(0,T;V)},$$

so joining (2.7), (2.8) and this equality we have finally that

$$||\tilde{u}(t)||_H^2 \leq \left(2 + \frac{C_{V,H}^2}{T}\right)\left(||u'||^2_{L^2(0,T;V')} + ||u||^2_{L^2(0,T;V)}\right) \tag{2.9}$$

for every $t \in [0,T]$. To conclude the proof let us recall that

$$||\tilde{u}||_{C([0,T];H)} = \sup_{0\leq t\leq T} ||\tilde{u}(t)||_H,$$

therefore (2.5) is a simple consequence of (2.9). $\square$

Of course $\tilde{u}$ is the representative of the class u, thus in the sequel we shall identify u with this representative. In our situation due to Proposition 2.3 we have the following corollary.

Corollary 2.4. *The space $W_\mu(0,T)$ is continuously embedded in $C([0,T];H_\mu(\Omega))$ and*

$$||u||_{C([0,T];H_\mu(\Omega))} = ||u||_{C([0,T];L^2(\Omega))} \leq C(|\Omega|,N,T)||u||_{W_\mu(0,T)}. \tag{2.10}$$

Proof. Take arbitrary $v \in V_\mu(\Omega)$. Then we have

$$||v||_{H_\mu(\Omega)} = ||v||_{L^2(\Omega)} \leq \overline{C}(|\Omega|,N)||v||_{H_0^1(\Omega)} \leq \overline{C}(|\Omega|,N)||v||_{V_\mu(\Omega)},$$

where $\overline{C}$ is the constant from Poincaré inequality. Therefore, using the notation from Proposition 2.3 we may write

$$C_{V_\mu(\Omega),H_\mu(\Omega)} = \overline{C}(|\Omega|,N)$$

and applying Propositions 2.1 and 2.3 we finish the proof. □

Let

$$A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$$

be an elliptic operator such as in the previous section. We define a new operator

$$A_\mu : V_\mu(\Omega) \longrightarrow V'_\mu(\Omega)$$

by the formula

$$\langle A_\mu u, v\rangle_\mu = \langle Au, v\rangle + \int_\Omega uv d\mu$$

for every $u, v \in V_\mu(\Omega)$. We consider the following first order evolution problem

$$\begin{cases} u'(t) + A_\mu u(t) = f(t) \\ u(0) = u^0 \\ u \in L^2(0,T;V_\mu(\Omega)) \end{cases}$$

for some $f \in L^2(0,T;V'_\mu(\Omega))$ and $u^0 \in H_\mu(\Omega)$, where the first equality is the equality of elements of $V'_\mu(\Omega)$ for almost every $t \in (0,T)$. Due to our convention concerning the 'embedding' i_2^* we may rewrite the above problem in the following way

$$\begin{cases} u' + Au + \mu u = f \\ u(0) = u^0 \\ u \in L^2(0,T;V_\mu(\Omega)) \end{cases} \tag{2.11}$$

From the form of the problem it is clear that we should look for solutions in the class $W_\mu(0,T)$. According to what is written above a function $u \in W_\mu(0,T)$, which in particular is continuous function with values in $H_\mu(\Omega)$ (see Corollary 2.4 together with the preceding remark), is a solution of (2.11) if it satisfies the initial condition (a.e. in Ω) and the equation

$$\langle u'(t), v\rangle_\mu + \langle Au(t), v\rangle + \int_\Omega u(t) v d\mu = \langle f(t), v\rangle_\mu$$

for each $v \in V_\mu(\Omega)$ almost everywhere in $(0,T)$ or, equivalently, the equation

$$\int_0^T \langle u'(t), v\psi(t)\rangle_\mu dt + \int_0^T \langle Au(t), v\psi(t)\rangle dt + \int_Q u(t,x)v(x)\psi(t) d\mu(x) dt \\ = \int_0^T \langle f(t), v\psi(t)\rangle_\mu dt$$

for every $v \in V_\mu(\Omega)$ and $\psi \in C_0^\infty((0,T))$.

From the theory of abstract parabolic problems we can derive the following theorem, which assures the existence and the uniqueness of the solution of (2.11).

Theorem 2.5. *Let $f \in L^2(0,T;V_\mu'(\Omega))$ and $u^0 \in H_\mu(\Omega)$. Then the problem (2.11) admits a unique solution, which belongs to $W_\mu(0,T)$. Moreover, the mapping*

$$L^2(0,T;V_\mu'(\Omega)) \times H_\mu(\Omega) \ni (f, u^0) \longmapsto u \in W_\mu(0,T) \tag{2.12}$$

is a topological isomorphism.

Proof. See e.g. [14, Vol. I, Chapter 3, Theorem 1.1]. □

Remark. In fact we can prove the existence and the uniqueness of the solution for (2.11) in much more general situation, e.g. when $A(t)$ is nonlinear time-dependent operator satisfying the following conditions:

(A1) for every $v_1, v_2 \in H_0^1(\Omega)$ the function $\langle A(\cdot)v_1, v_2\rangle$ is Lebesgue measurable on $(0,T)$;

(A2) for a.e. $t \in (0,T)$ operator $A(t)$ is monotone and demicontinuous;

(A3) there exists a positive constant C such that

$$||A(t)v||_{H^{-1}(\Omega)} \le C\left(1 + ||v||_{H_0^1(\Omega)}\right)$$

for $v \in H_0^1(\Omega)$;

(A4) there exist such constants $\alpha > 0$, $\omega \in \mathbb{R}$ that

$$\langle A(t)v, v\rangle + \omega ||v||^2_{L^2(\Omega)} \geq \alpha ||v||^2_{H^1_0(\Omega)}$$

for an arbitrary $v \in H^1_0(\Omega)$.

In this case we simply need to apply, e.g. [1, Theorem 4.5].

The special form of the operator A enables us to increase the regularity of the solution of (2.11), which will be important in the sequel.

Proposition 2.6. *Let $u^0 \in V_\mu(\Omega)$, $f \in L^2(0, T; H_\mu(\Omega))$ and let u be the solution of (2.11). Then $u' \in L^2(0, T; H_\mu(\Omega))$ and $u \in C([0, T]; V_\mu(\Omega))$. Moreover the following estimates hold:*

$$||u||^2_{L^2(0,T;V_\mu(\Omega))} + ||u'||^2_{L^2(Q)} \leq C_1 \left(||u^0||^2_{V_\mu(\Omega)} + ||f||^2_{L^2(Q)}\right)$$

and

$$||u||^2_{C([0,T];V_\mu(\Omega))} \leq C_2 \left(||u^0||^2_{V_\mu(\Omega)} + ||f||^2_{L^2(Q)}\right),$$

where the constants C_1 and C_2 depend only upon α, C_0, T, N and $|\Omega|$.

Proof. From [1, Corollary 4.3 and Remark 4.4] we know that in our situation $u' \in L^2(0, T; H_\mu(\Omega))$ and

$$||u'||^2_{L^2(0,T;H_\mu(\Omega))} = ||u'||^2_{L^2(Q)} \leq C' \left(||u^0||^2_{V_\mu(\Omega)} + ||f||^2_{L^2(Q)}\right). \tag{2.13}$$

From (2.11), using the inequality (2.6) and the coercivity of A we obtain

$$||u||^2_{L^2(0,T;V_\mu(\Omega))} \leq \frac{1}{\tilde{\alpha}^2} \left(||u^0||^2_{H_\mu(\Omega)} + ||f||^2_{L^2(0,T;V'_\mu(\Omega))}\right)$$

where $\tilde{\alpha} = \min\{\alpha, 1\}$, and, furthermore,

$$||u||^2_{L^2(0,T;V_\mu(\Omega))} \leq C'' \left(||u^0||^2_{V_\mu(\Omega)} + ||f||^2_{L^2(0,T;H_\mu(\Omega))}\right). \tag{2.14}$$

Joining (2.13) and (2.14) we obtain the first inequality from the thesis. From (2.13) and (2.11) it follows that $A_\mu u \in L^2(0, T; H_\mu(\Omega))$ and

$$||A_\mu u||_{L^2(0,T;H_\mu(\Omega))} \leq ||u'||_{L^2(0,T;H_\mu(\Omega))} + ||f||_{L^2(0,T;H_\mu(\Omega))}.$$

It means that $u \in L^2(0, T; D(A_\mu))$, where

$$D(A_\mu) = \{v \in V_\mu(\Omega) : Av + \mu v \in H_\mu(\Omega)\}$$

is the domain of the operator A_μ. Therefore the remaining part of our thesis is a consequence of Theorem 3.1 and Proposition 2.1 from Chapter 1 of [14, Vol. I]. □

We call the problem (2.11) *the relaxed parabolic problem*, analogously to relaxed Dirichlet problems considered in the previous section. Like there, this terminology is justified by the fact that problems (2.11) can be considered as a generalization of classical parabolic problems of the type

$$\begin{cases} u' + Au = f & \text{in } (0,T) \times G \\ u(0) = u^0 & \text{in } G \\ u = 0 & \text{on } (0,T) \times \partial G \end{cases} \tag{2.15}$$

where G is an open subset of Ω (see remark below), while with some natural assumptions the limit of a sequence of the solutions of problems (2.15) on varying domains is the solution of (2.11) (see Theorem 2.9) and, conversely, the solution of (2.11) can be approximated by the solutions of (2.15).

Remark 2.7. Let G be an open subset of Ω. If μ is the measure from Example 1.3, then u is the solution of (2.15) if and only if $\tilde{u}$ is the solution of (2.11), where $\tilde{u}(t)$ is the extension of $u(t)$ by 0 outside G.

Remark 2.8. We can exchange the right-hand side of (2.11) for an arbitrary $\bar{f} \in L^2(Q)$ such that

$$f(t)|_{A(\mu)} = \bar{f}(t)|_{A(\mu)}$$

(i.e. $f(t) = 1_{A(\mu)} \bar{f}(t)$) with no influence on the solution. It is so because both functions give the same element in $L^2(0,T; V'_\mu(\Omega))$ (this is a consequence of Proposition 1.13 and the density of $H^1_0(A(\mu))$ in $H_\mu(\Omega)$).

We are now able to state the main result of this paper, showing the stability of the class of relaxed parabolic problems under the γ^A-convergence. We consider the following sequence of problems

$$\begin{cases} u'_n + Au_n + \mu_n u_n = f_n \\ u_n(0) = u^0_n \\ u_n \in L^2(0,T; V_{\mu_n}(\Omega)). \end{cases} \tag{2.16}$$

Theorem 2.9. *Let u, u_n be the solutions of problems* (2.11) *and* (2.16)*, respectively, for some $\mu, \mu_n \in \mathcal{M}_0(\Omega)$, $f, f_n \in L^2(Q)$, $u^0 \in V_\mu(\Omega)$ and $u^0_n \in V_{\mu_n}(\Omega)$. Take $M > 0$ and assume that*

1. $\mu_n \xrightarrow{\gamma^A} \mu$,

2. $f_n \longrightarrow f$ *weakly in* $L^2(Q)$,

3. $u^0_n \longrightarrow u^0$ *weakly in* $H^1_0(\Omega)$ *and*

4. $||u_n^0||_{L^2_{\mu_n}(\Omega)} \le M$, $\forall n \in \mathbb{N}$.

Then

$$u_n \longrightarrow u \quad \textit{weakly in } W(0,T).$$

Remark 2.10. If $\mu_n = \infty_{\Omega\setminus G_n}$ for some open $G_n \subset \Omega$, then the condition $u_n^0 \in L^2_{\mu_n}(\Omega)$ means simply that

$$u_n^0 = 0 \quad \text{q.e. in } \Omega \setminus G_n,$$

hence, in particular,

$$\int_\Omega |u_n^0|^2 d\mu_n = 0,$$

therefore in this case assumption 4 holds automatically.

Proof of Theorem 2.9. From Proposition 2.6 it follows that

$$\begin{array}{rcl} u_n & \in & L^2(0,T;V_{\mu_n}(\Omega)) \subset L^2(0,T;H_0^1(\Omega)) \\ u_n' & \in & L^2(0,T;H_{\mu_n}(\Omega)) \subset L^2(Q) \subset L^2(0,T;H^{-1}(\Omega)), \end{array}$$

hence $u_n \in W(0,T)$. The same proposition gives us also the following estimates

$$\begin{array}{rcl} ||u_n||^2_{W(0,T)} & = & ||u_n||^2_{L^2(0,T;H_0^1(\Omega))} + ||u_n'||^2_{L^2(0,T;H^{-1}(\Omega))} \\ & \le & ||u_n||^2_{L^2(0,T;V_\mu(\Omega))} + \overline{C}||u_n'||^2_{L^2(Q)} \\ & \le & \overline{C}_1 \left(||u_n^0||^2_{V_{\mu_n}(\Omega)} + ||f_n||^2_{L^2(Q)} \right) \\ & = & \overline{C}_1 \left(||u_n^0||^2_{H_0^1(\Omega)} + ||u_n^0||^2_{L^2_{\mu_n}(\Omega)} + ||f_n||^2_{L^2(Q)} \right) \end{array}$$

and

$$||u_n||^2_{C([0,T];V_{\mu_n}(\Omega))} \le C_2 \left(||u_n^0||^2_{H_0^1(\Omega)} + ||u_n^0||^2_{L^2_{\mu_n}(\Omega)} + ||f_n||^2_{L^2(Q)} \right).$$

Therefore assumptions 2–4 imply

$$\begin{array}{c} ||u_n||_{W(0,T)} \le const \\ ||u_n'||_{L^2(Q)} \le const \\ ||u_n||_{C([0,T];H_0^1(\Omega))} \le ||u_n||_{C([0,T];V_{\mu_n}(\Omega))} \le const. \end{array} \tag{2.17}$$

The last inequality means in particular that

$$||u_n(t)||_{H_0^1(\Omega)} \le const$$

for every $t \in [0,T]$ as well as

$$||u_n||_{L^\infty(0,T;H_0^1(\Omega))} \leq const.$$

The above estimates imply that there exists a sequence of (u_n) (still denoted by the same symbol) convergent weakly in $W(0,T)$ to a function u. We may also assume (perhaps passing to a further subsequence) that

$$u_n' \longrightarrow u' \quad \text{weakly in } L^2(Q) \tag{2.18}$$

and

$$u_n \longrightarrow u \quad \text{weakly * in } L^\infty(0,T;H_0^1(\Omega)) \tag{2.19}$$

Moreover, due to (2.18), (2.19) and [15, Corollary 4] we may assume that

$$u_n \to u \quad \text{strongly (uniformly) in } C([0,T];L^2(\Omega)). \tag{2.20}$$

This implies in particular that for every $t \in [0,T]$

$$u_n(t) \longrightarrow u(t) \quad \text{weakly in } H_0^1(\Omega). \tag{2.21}$$

Furthermore, assumption 1 is equivalent to the condition

$$\Gamma - \lim_{n\to\infty} F_{\mu_n} = F_\mu$$

(see Proposition 1.21), hence from (2.21) it follows that

$$F_\mu(u(t)) \leq \liminf_{n\to\infty} F_{\mu_n}(u_n(t)). \tag{2.22}$$

But since

$$\tilde{\alpha}||v||^2_{V_\mu(\Omega)} \leq F_\mu(v) = \langle Av, v\rangle + \int_\Omega v^2 d\mu \leq \tilde{C}_0||v||^2_{V_\mu(\Omega)},$$

where the constants $\tilde{\alpha}$ and $\tilde{C}_0$ depend only on α and C_0, so (2.22) and the last inequality from (2.17) imply that

$$u \in L^\infty(0,T;V_\mu(\Omega)).$$

Let us consider the problem (2.16) in the integral form with the test function $\psi(t)\varphi(x)w_n(x)$, where $\psi \in C_0^\infty((0,T))$, $\varphi \in C_0^\infty(\Omega)$ and $w_n \in V_{\mu_n}(\Omega)$ is the solution of the equation $Aw_n + \mu_n w_n = 1$:

$$\int_Q u_n'\psi\varphi w_n dtdx + \int_Q \sum_{i,j=1}^N a_{ij}D_j u_n D_i(\psi\varphi w_n)dtdx + \int_Q u_n\psi\varphi w_n dtd\mu_n = \int_Q f_n\psi\varphi w_n dtdx. \tag{2.23}$$

Since $\mu_n \xrightarrow{\gamma^A} \mu$, from Theorem 1.18 it follows that (w_n) converges weakly in $H_0^1(\Omega)$, hence also strongly in $L^2(\Omega)$, to the solution of relaxed Dirichlet problem

$$\begin{cases} Aw + \mu w = 1 \\ w \in V_\mu(\Omega). \end{cases}$$

Therefore, from (2.18) and assumption 2 we have

$$\begin{aligned} \int_Q u_n' \psi \varphi w_n dtdx &\longrightarrow \int_Q u' \psi \varphi w dtdx, \\ \int_Q f_n \psi \varphi w_n dtdx &\longrightarrow \int_Q f \psi \varphi w dtdx. \end{aligned}$$

Because ψ does not depend on x, the following equalities hold:

$$\begin{aligned} D_j u_n D_i(\psi\varphi w_n) = \psi D_j u_n D_i(\varphi w_n) &= \psi\varphi D_j u_n D_i w_n + \psi w_n D_j u_n D_i \varphi \\ &= D_j(\psi\varphi u_n) D_i w_n - \psi u_n D_j \varphi D_i w_n + \psi w_n D_j u_n D_i \varphi. \end{aligned}$$

On the other hand, using Fubini Theorem, we obtain

$$\int_Q \sum_{i,j=1}^N a_{ij} D_j(\psi\varphi u_n) D_i w_n dtdx + \int_Q \psi\varphi u_n w_n dtd\mu_n = \int_Q \psi\varphi u_n dtdx.$$

Hence

$$\begin{aligned} \int_Q \sum_{i,j=1}^N a_{ij} D_j u_n D_i(\psi\varphi w_n) dtdx + \int_Q u_n \psi\varphi w_n dtd\mu_n = \int_Q \psi\varphi u_n dtdx \\ + \int_Q \sum_{i,j=1}^N a_{ij} \psi w_n D_j u_n D_i \varphi dtdx - \int_Q \sum_{i,j=1}^N a_{ij} \psi u_n D_j \varphi D_i w_n dtdx. \quad (2.24) \end{aligned}$$

From the weak convergence of (u_n) in $W(0,T)$ and from (2.20)) it follows that

$$\begin{aligned} u_n &\longrightarrow u \quad \text{strongly in } L^2(Q) \text{ and} \\ D_i u_n &\longrightarrow D_i u \quad \text{weakly in } L^2(Q), \end{aligned}$$

so the right-hand side of (2.24) converges to

$$\int_Q \psi\varphi u dtdx + \int_Q \sum_{i,j=1}^N a_{ij} \psi w D_j u D_i \varphi dtdx - \int_Q \sum_{i,j=1}^N a_{ij} \psi u D_j \varphi D_i w dtdx.$$

Gathering together the above results and using the inequality analogous to (2.24) for w we draw a conclusion that the sequence of equations (2.23) is convergent sidewise to

$$\int_Q u'\psi\varphi w dt dx + \int_Q \sum_{i,j=1}^N a_{ij} D_j u D_i(\psi\varphi w) dt dx + \int_Q u\psi\varphi w dt d\mu$$
$$= \int_Q f\psi\varphi w dt dx.$$

ψ is arbitrary, so we may write

$$\langle u'(t), w\varphi\rangle_\mu + \langle Au(t), w\varphi\rangle + \int_\Omega u(t) w\varphi d\mu = \int_\Omega f(t) w\varphi dx,$$

but this, due to the density of $wC_0^\infty(\Omega)$ in $V_\mu(\Omega)$ (Proposition 1.14) means that

$$u' + Au + \mu u = f.$$

Furthermore, note that (2.21) implies in particular that

$$u_n(0) \longrightarrow u(0) \quad \text{weakly in } H_0^1(\Omega),$$

hence from assumption 3 it follows that

$$u(0) = u^0.$$

To conclude the proof let us note that from the uniqueness of the solution of (2.11) it follows that in fact the whole sequence (u_n) converges to u. □

Remark 2.11. From the proof of Theorem 2.9 we can obtain in fact some additional convergence conditions:

$$\begin{aligned}
u_n &\longrightarrow u \quad \text{strongly in } L^2(Q),\\
u_n &\longrightarrow u \quad \text{strongly in } C([0,T]; L^2(\Omega)),\\
u_n &\longrightarrow u \quad \text{weakly * in } L^\infty(0,T; H_0^1(\Omega)),\\
u_n' &\longrightarrow u' \quad \text{weakly in } L^2(Q),\\
u_n(t) &\longrightarrow u(t) \quad \text{weakly in } H_0^1(\Omega) \text{ for every } t \in [0,T].
\end{aligned}$$

Moreover for every $t \in [0,T]$ we have

$$F_\mu(u(t)) \le \liminf_{n\to\infty} F_{\mu_n}(u_n(t)).$$

As a conclusion, let us state the following corollary, which is a straightforward consequence of Proposition 1.20 and Theorem 2.9.

Corollary 2.12. *For every solution of the relaxed problem* (2.11) *there exists a sequence of solutions of problems* (2.15) *convergent to it weakly in* $W(0,T)$.

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