1 Notes on the computation of $h'(\theta)$ in the c++ code

In the computer assisted proof, the parameter $\theta = G$. All our considerations here are in the space extended by the parameter. The parameter is placed on the first coordinate both in these notes as well as in the c++ code.

1.1 Local coordinates

Let $q = (G, X, Y, Z)$ be the original coordinates of the Lorenz-84 model, extended to include the parameter on the first coordinate.

We consider two local coordinates:

$$q = q_0 + C_i p_i, \qquad \text{for } i = 1, 2.$$

We choose

$$C_i = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{C}_i \end{pmatrix} \qquad \text{for } i = 1, 2,$$

where $\tilde{C}_i$ align X, Y, Z coordinates with local dynamics.

We now discuss these local coordinates.

In the coordinates p_1 we parameterise the one dimensional unstable manifold as

$$W^u_G = \left\{ (G, x, w^u_1(G, x), w^u_2(G, x)) : x \in \overline{B}_u(R) \right\}.$$

This means that in original coordinates of the system, the manifold is given as

$$q_0 + C_1 W^u_G.$$

When we consider

$$p^u_G := q_0 + C_1 \begin{pmatrix} G \\ R \\ w^u_1(G, R) \\ w^u_2(G, R) \end{pmatrix},$$

we obtain

$$\frac{d}{dG} p^u_G = C_1 \begin{pmatrix} 1 \\ 0 \\ \frac{\partial w^u_1}{\partial G} \\ \frac{\partial w^u_2}{\partial G} \end{pmatrix} \in C_1 \begin{pmatrix} 1 \\ 0 \\ [-1/M, 1/M] \\ [-1/M, 1/M] \end{pmatrix}.$$

In the coordinates p_2, we parameterise the two dimensional stable manifold as

$$W^s_G = \left\{ (G, x_1, x_2, w^s(G, x_1, x_2)) : (x_1, x_2) \in \overline{B}_s(R) \right\}. \tag{1}$$

In the original coordinates, the stable manifold is

$$q_0 + C_2 W^s_G.$$

For any G, x we have the bound

$$Dw^s = \left[\frac{\partial w^s}{\partial G} \frac{\partial w^s}{\partial x_1} \frac{\partial w^s}{\partial x_2} \right] \in [[-1/M, 1/M]\ [L, L]\ [L, L]]\,.$$

Above is a 1×3 matrix.

1.2 Computation of h and h'

We compute the h in the coordinates p_2:

$$h\left(G\right) = \pi_4 C_2^{-1}\left(\Phi_T\left(p_G^u\right) - q_0\right) - w^s\left(G, \pi_{x_1,x_2}(C_2^{-1}\left(\Phi_T\left(p_G^u\right) - q_0\right))\right),$$

where Φ_t is the flow in original coordinates, extended to include the parameter G. The π_4 is the projection ont the last, fourth coordinate. We project there, since this is how W^s is parameterised in coordinates p_2 (1).

Remark 1 *In the paper, the coordinates are such that the projection π_4 in the code corresponds to π_x in the paper.*

The projection π_{x_1,x_2} is the projection onto the second and third coordinate of p_2.

We can now compute h' as follows

$$\begin{aligned} h'\left(G\right) &= \pi_4 C_2^{-1} D\Phi_T\left(p_G^u\right) \frac{d}{dG} p_G^u - Dw^s \frac{d}{dG}\left(G, \pi_{x_1,x_2}(C_2^{-1}\left(\Phi_T\left(p_G^u\right) - q_0\right))\right) \\ &= \pi_4 C_2^{-1} D\Phi_T\left(p_G^u\right) \frac{d}{dG} p_G^u - Dw^s\, v, \end{aligned} \tag{2}$$

where

$$v = \begin{pmatrix} 1 \\ \pi_{x_1 x_2} C_2^{-1} D\Phi_T \frac{d}{dG} p_G^u \end{pmatrix}.$$

The (2) is the formula that is used in the `DerivativeOf_h()` function in the code for the computer assisted proof.