Numerical Methods

Lecture 5.

Numerical integration

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Outline

- Trapezoidal rule
- Multi-segment trapezoidal rule
- Richardson extrapolation
- Romberg's method
- Simpson's rule
- Gaussian method

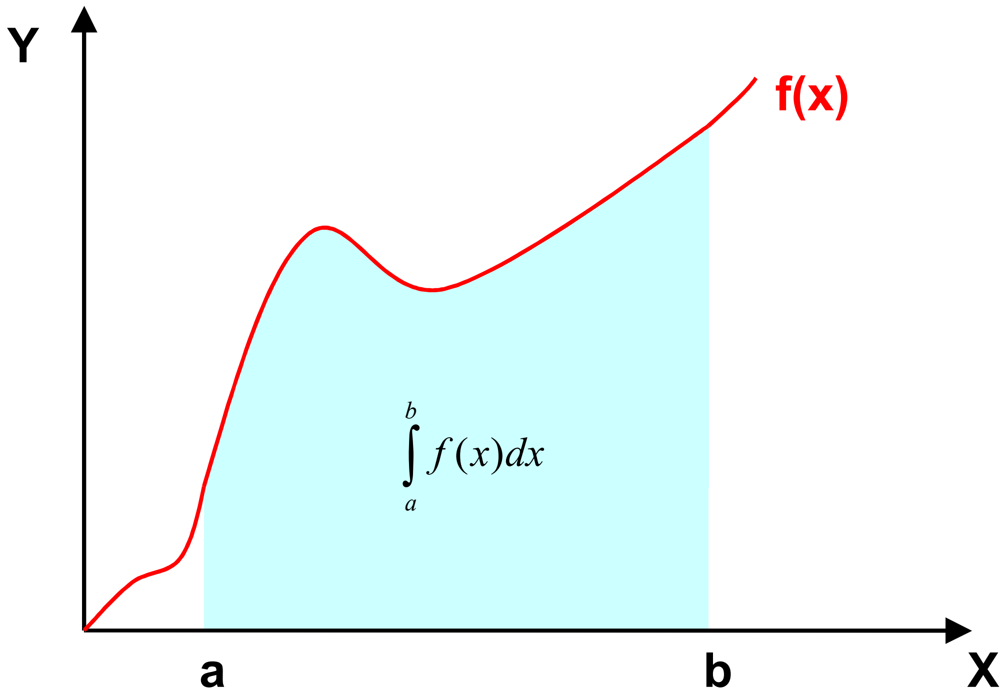
Numerical integration - idea

$$I = \int_a^b f(x)dx$$

The integral can be approximated by the sum of the

$$S = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$x_i \le c_i \le x_{i+1}$$



Newton–Cotes methods

Newton – Cotes integration belongs to a class of methods with fixed nodes: function f(x) is interpolated by a polynomial (e.g. Lagrange polynomial)

$$f(x) \approx f_n(x)$$

where:

$$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n$$

Then, the integral of f(x) can be approximated as an integral of the interpolated function $f_n(x)$

$$I = \int_a^b f(x)dx \approx \int_a^b f_n(x)dx$$

Trapezoidal rule

The trapezoidal rule assumes: n = 1, thus:

$$f_1(x) = a_0 + a_1 x$$

$$I = \int_a^b f(x)dx \approx \int_a^b f_1(x)dx = \int_a^b (a_0 + a_1 x)dx$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2}$$

We are looking for a_0 and a_1 ?

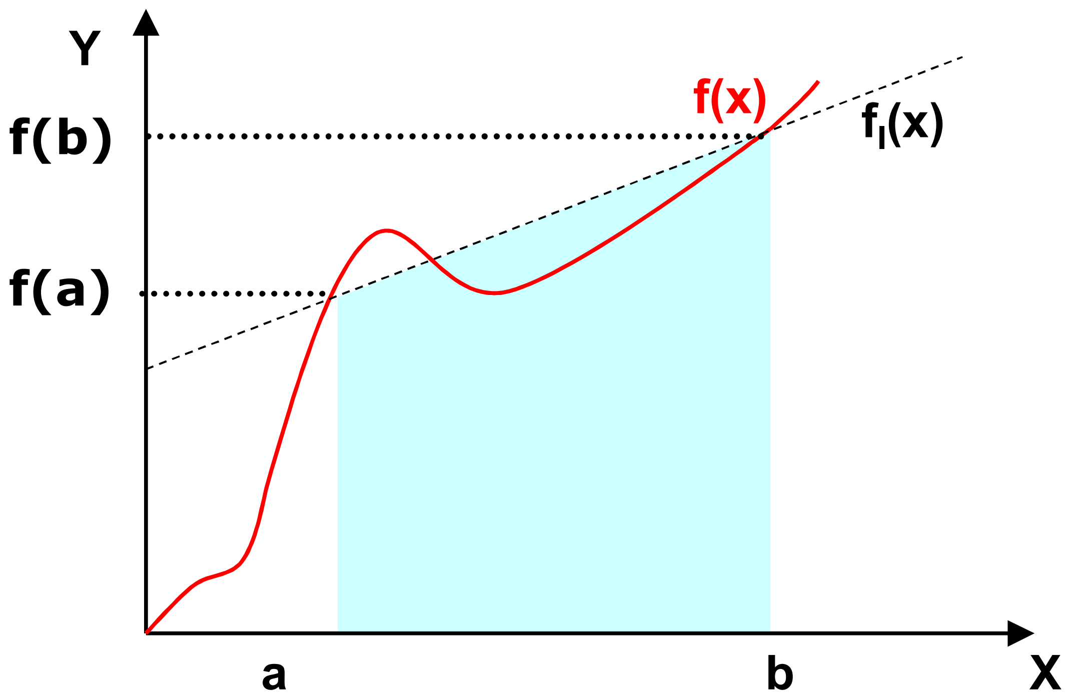
Now if one chooses, (a,f(a)) i (b,f(b)) as the two points to approximate f(x) by a straight line from a to b, then:

$$\begin{cases} f(a) = f_1(a) = a_0 + a_1 a \quad \Longrightarrow \quad a_0 = \dfrac{f(a)b - f(b)a}{b-a} \\ f(b) = f_1(b) = a_0 + a_1 b \quad \Longrightarrow \quad a_1 = \dfrac{f(b) - f(a)}{b-a} \end{cases}$$

Trapezoidal rule

$$\int_a^b f(x)dx = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

$$\int_a^b f(x)dx \approx area\,of\,trapezoid$$



Trapezoidal rule

Example 1:

The vertical distance covered by a rocket from $t_1=8$ s to $t_2=30$ s is given by:

$$v(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t$$

a) Use the single segment trapezoidal rule to find the distance covered from $t_1=8$ s to $t_2=30$ s
b) Find the true relative error.

Trapezoidal rule

a)

$$I \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right] \qquad a=8\ s \qquad b=30\ s$$

$$f(t)=2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t$$

$$f(8)=2000\ln\left[\frac{140000}{140000-2100(8)}\right]-9.8(8) \ =177.27\ m/s$$

$$f(30)=2000\ln\left[\frac{140000}{140000-2100(30)}\right]-9.8(30) \ =901.67\ m/s$$

$$I=(30-8)\left[\frac{177.27+901.67}{2}\right] \ =11868\ m$$

Trapezoidal rule

b) The true value

$$\Delta x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt = 11061\, m$$

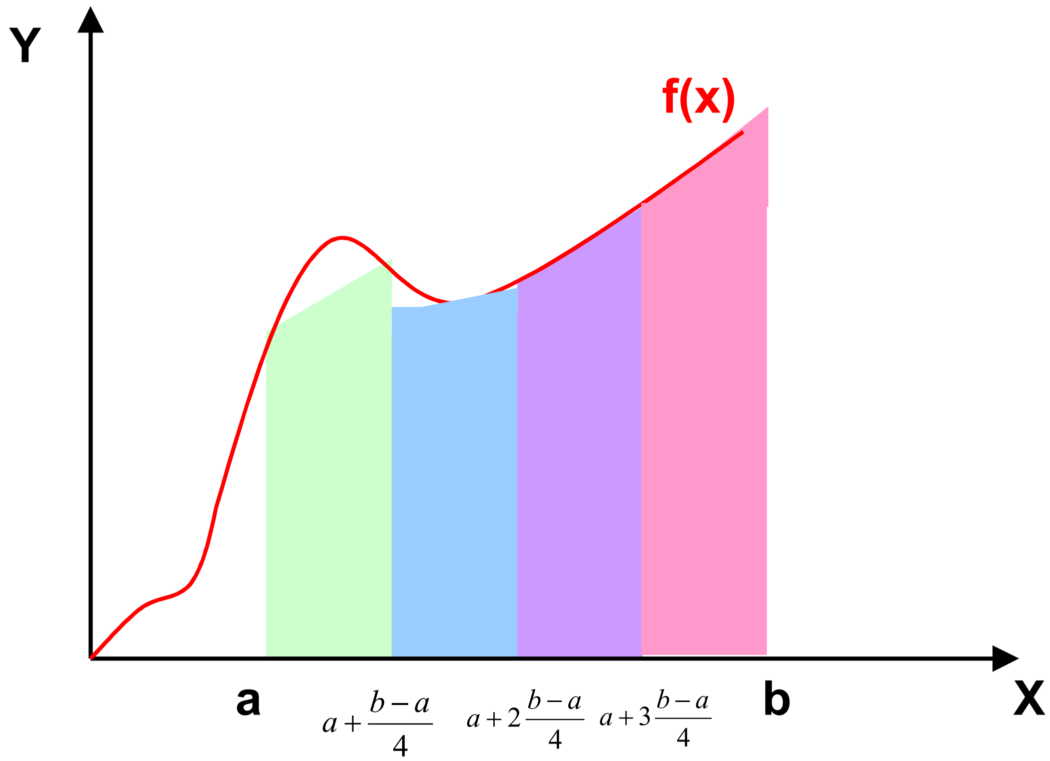
The relative error:

$$\left| \in_t \right| = \left| \frac{11061 \quad - 11868}{11061} \right| \times 100 \quad = 7.2959\,\%$$

Multi-segment trapezoidal rule

The true error using a single segment trapezoidal rule was large. We can divide the interval into n subsections (of equal length - h) and apply the trapezoidal rule over each segment:

$$h = \frac{b-a}{n}$$



for n=4

$$I = \int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \int_{a+2h}^{a+3h} f(x)dx + \int_{a+3h}^{a+4h} f(x)dx$$

Multi-segment trapezoidal rule

$$\int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^{b} f(x)dx$$

$$= h\left[\frac{f(a)+f(a+h)}{2}\right] + h\left[\frac{f(a+h)+f(a+2h)}{2}\right] + \ldots$$

$$\ldots + [b-(a+(n-1)h]\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a)+2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\}+f(b)\right]$$

Multi-segment trapezoidal rule

Example 2:

The vertical distance covered by a rocket from t_1=8 s to t_2=30 s seconds is given by:

$$v(t) = 2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

a) Use the complex segment trapezoidal rule to find the distance covered from t_1=8 s to t_2=30 s for n = 2

b) Find the true relative error.

Multi-segment trapezoidal rule

a)

$$I = \frac{b-a}{2n}\left[f(a) + 2\left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right]$$

n = 2 a = 8 s b = 30 s

$$h = \frac{b-a}{n} = \frac{30-8}{2} = 11s$$

$$I = \frac{30-8}{2(2)}\left[f(8) + 2\left\{ \sum_{i=1}^{2-1} f(a+ih) \right\} + f(30) \right]$$

$$= \frac{22}{4}\left[f(8) + 2f(19) + f(30) \right] = \frac{22}{4}\left[177.27 + 2(484.75) + 901.67 \right]$$

$$= 11266\ m$$

Multi-segment trapezoidal rule

b) the true value:

$$x = \int_{8}^{30} \left(2000 ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt \quad = 11061\, m$$

The relative error:

$$\left| \in_t \right| = \left| \frac{11061 - 11266}{11061} \right| \times 100 \ = 1.8534\%$$

Multi-segment trapezoidal rule

n	Δx	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

Trapezoidal rule errors

The relative error for a simple trapezoidal rule

$$E_t = \frac{(b-a)^3}{12} f''(\zeta), \quad a < \zeta < b$$

where ζ is a point in: $[a,b]$

The relative error in the complex trapezoidal rule is a sum of errors for each segment. The realative error of one segment is given by:

$$E_1 = \frac{[(a+h)-a]^3}{12} f''(\zeta_1), \quad a < \zeta_1 < a+h$$

$$= \frac{h^3}{12} f''(\zeta_1)$$

Trapezoidal rule errors

By analogy:

$$E_i = \frac{\left[(a+ih)-(a+(i-1)h)\right]^3}{12} f''(\zeta_i), \quad a+(i-1)h < \zeta_i < a+ih$$

$$= \frac{h^3}{12} f''(\zeta_i)$$

for n:

$$E_n = \frac{\left[b-\{a+(n-1)h\}\right]^3}{12} f''(\zeta_n), \quad a+(n-1)h < \zeta_n < b$$

$$= \frac{h^3}{12} f''(\zeta_n)$$

Trapezoidal rule errors

The total error in the complex trapezoidal rule is a sum of the errors of single segments:

$$E_t = \sum_{i=1}^{n} E_i = \frac{h^3}{12} \sum_{i=1}^{n} f''(\zeta_i) = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

Formula: $$\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

is an approximate average value of the second derivative in the range of $a < x < b$

$$E_t \propto \alpha \frac{1}{n^2}$$

Trapezoidal rule errors

Table of results for integral $\int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$

as a function of the number of segments – n. When n increases twice the error decreases four times!

n	**Value**	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
2	11266	-205	1.854	5.343
4	11113	-51.5	0.4655	0.3594
8	11074	-12.9	0.1165	0.03560
16	11065	-3.22	0.02913	0.00401

Romberg integration method

Romberg method is an extension of the trapezoidal method and gives a better approximation of the integral by reducing the error (true error)

Richardson extrapolation

The true error obtained when using the multiple segment trapezoidal rule (complex) with n segments to approximate an integral is given by:

$$E_t \cong \frac{C}{n^2}$$

where: C is an approximate constant of proportionality

Since:

$$E_t = TV - I_n$$

True value

approximate value using n-segments

Richardson extrapolation

It can be shown, that:

$$\frac{C}{(2n)^2} \cong TV - I_{2n}$$

If the number of segments is doubled from n to 2n in the trapezoidal rule:

$$\begin{cases} \dfrac{C}{(n)^2} \cong TV - I_n \\[2ex] \dfrac{C}{(2n)^2} \cong TV - I_{2n} \end{cases}$$

We get:

$$\boxed{TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}}$$

Richardson extrapolation

Example 3:

The vertical distance covered by a rocket from t_1=8 s to t_2=30 s seconds is given by:

$$v(t) = 2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

a) Use Richardson extrapolation rule to find the distance covered for n = 2
b) Find the absolute relative true error

Table of results from a complex trapezoid rules for n = 8 segments

n	Δx	E_t	$\left\|\in_t\right\|\%$	$\left\|\in_a\right\|\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

Richardson extrapolation

a)

$$I_2 = 11266m \qquad I_4 = 11113m$$

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$ for n=2

$$TV \cong I_4 + \frac{I_4 - I_2}{3} = 11113 + \frac{11113 - 11266}{3}$$

$$= 11062m$$

Richardson extrapolation

b)

The true value:

$$x = \int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt \quad = 11061\ m$$

The error:

$$E_t = 11061 - 11062 \qquad = -1\ m$$

Richardson extrapolation

c) The relative error:

$$\left|\in_t\right| = \left|\frac{11061 - 11062}{11061}\right| \times 100 = 0.00904\%$$

Comparison of the results with the trapezoidal rule

n	**Δx** (m) **Trapezoidal rule**	$\left\|\in_t\right\|\%$ **Trapezoidal rule**	**Δx** (m) **Richardson extrapolation**	$\left\|\in_t\right\|\%$ **Richardson extrapolation**
1	11868	7.296	--	--
2	11266	1.854	11065	0.03616
4	11113	0.4655	11062	0.009041
8	11074	0.1165	11061	0.0000

Romberg's method

Romberg’s method uses the same pattern as Richardson extrapolation. However, Romberg used a recursive algorithm for the extrapolation as follows:

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

which can be expressed as:

$$\left(I_{2n}\right)_R = I_{2n} + \frac{I_{2n} - I_n}{3} \quad = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$

The true value TV is replaced by result of Richardson extrapolation

$$\left(I_{2n}\right)_R$$

Note also that the sign $\cong$ is replaced by the sign $=$

Romberg's method

Estimated true value is given by: $TV \cong \left(I_{2n}\right)_R + Ch^4$

where: Ch^4 is the value of the approximation error

Determine another integral value with further halving the step size (doubling the number of segments):

$$\left(I_{4n}\right)_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

Estimated true value is given by:

$$TV \cong \left(I_{4n}\right)_R + \frac{\left(I_{4n}\right)_R - \left(I_{2n}\right)_R}{15}$$

$$= \left(I_{4n}\right)_R + \frac{\left(I_{4n}\right)_R - \left(I_{2n}\right)_R}{4^{3-1} - 1}$$

Romberg's method

A general expression for Romberg integration can be written as:

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index k represents the order of extrapolation

k=1 represents the values obtained from the regular trapezoidal rule

k=2 represents the values obtained using the true error estimate as $O(h^2)$

The value of an integral with a *j+1* index is more accurate than the value of the integral with a *j* index

Romberg's method

Example 4:

The vertical distance covered by a rocket from t_1=8 s to t_2=30 s seconds is given by:

$$v(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

a) Use Romberg's method to find the distance covered. Use the n = 1, 2, 4, and 8-segment

b) Find the absolute and relative true error

Table of results from a complex trapezoid rules for n = 8 segments

n	Δx	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
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8	11074	-12.9	0.1165	0.03560

Romberg's method

From the table, the needed values from the original the trapezoidal rule are:

$$I_{1,1} = 11868 \qquad I_{1,2} = 11266$$

$$I_{1,3} = 11113 \qquad I_{1,4} = 11074$$

To get the k=2 order extrapolation values:

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$

$$= 11266 + \frac{11266 - 11868}{3}$$

Romberg's method

Then

$$I_{2,2} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3}$$

$$= 11113 + \frac{11113 - 11266}{3}$$

$$= 11062$$

$$I_{2,3} = I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3}$$

$$= 11074 + \frac{11074 - 11113}{3}$$

$$= 11061$$

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

Romberg's method

For the k=3 extrapolation values,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

$$= 11062 + \frac{11062 - 11065}{15}$$

$$= 11062$$

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$

$$= 11061 + \frac{11061 - 11062}{15}$$

$$= 11061$$

For k-=4

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63} = 11061 + \frac{11061 - 11062}{63} = 11061m$$

Romberg's method

		K=2	*K=3*	*K=4*
1-segment	11868			
		11065		
2-segment	11262		11062	
		11062		11061
4-segment	11113		11061	
		11061		
8-segment	11074			

Improved estimates of the value of an integral using Romberg integration

Simpson's rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's rule is an extension of trapezoidal rule where the integrand is approximated by a second order polynomial.

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

where:

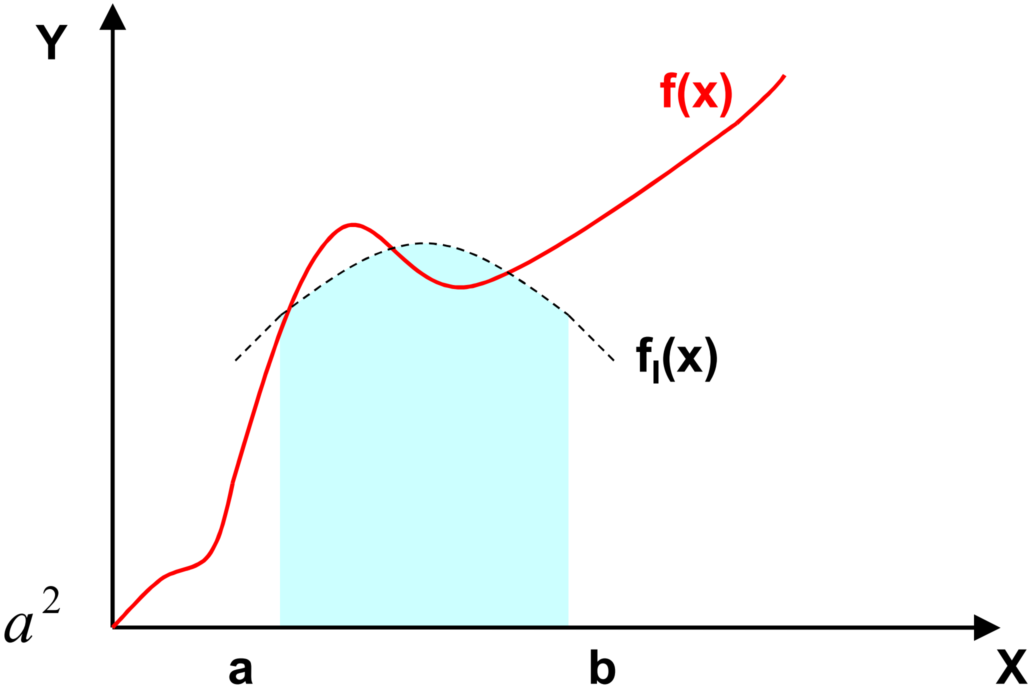
$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

Simpson's rule

$$(a, f(a)),$$

$$\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right),$$

$$(b, f(b))$$



$$\begin{cases} f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2 \\ f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2 \\ f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2 \end{cases}$$

Simpson's rule

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Simpson's rule

Since:

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} \right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Simpson's rule

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$h = \frac{b-a}{2}$$

It follows that:

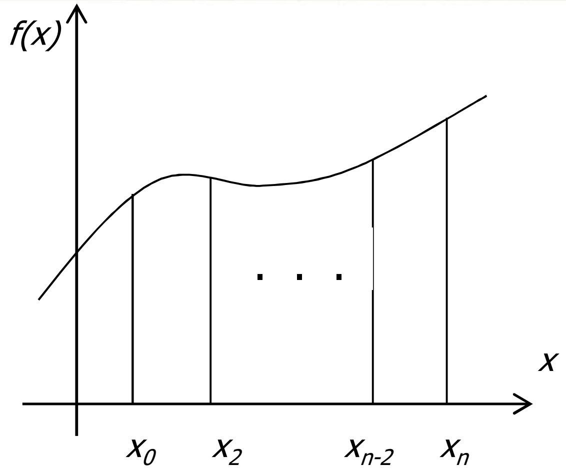
$$\int_a^b f_2(x)dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

it is called Simpson's 1/3 rule

Complex Simpson's rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots..$$

$$\ldots. + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$



$$\int_a^b f(x)dx = (x_2 - x_0)\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + \ldots$$

$$+ (x_4 - x_2)\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + \ldots$$

$$\ldots + (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6}\right] + \ldots$$

$$+ (x_n - x_{n-2})\left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right]$$

$$x_i - x_{i-2} = 2h$$

$$i = 2, 4, \ldots, n$$

Simpson's rule

$$\int_a^b f(x)dx = 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Simpson's rule

$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0)+4\{f(x_1)+f(x_3)+\ldots+f(x_{n-1})\}+\ldots]$$

$$\ldots+2\{f(x_2)+f(x_4)+\ldots+f(x_{n-2})\}+f(x_n)\}]$$

$$= \frac{h}{3}\left[f(x_0)+4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i)+f(x_n)\right]$$

$$= \frac{b-a}{3n}\left[f(x_0)+4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i)+f(x_n)\right]$$

Simpson's rule errors

Approximate values, for mentioned earlier example, using Simpson's rule with multiple segments

n	Approximate values	E_t	$\lvert \epsilon_t \rvert$
2	11065.72	4.38	0.0396%
4	11061.64	0.30	0.0027%
6	11061.40	0.06	0.0005%
8	11061.35	0.01	0.0001%
10	11061.34	0.00	0.0000%

Simpson's rule errors

Error for one segment $$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

Error for the multi-segment

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1) \quad = -\frac{h^5}{90} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2) \quad = -\frac{h^5}{90} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$

$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i) = -\frac{h^5}{90} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$

Simpson's rule errors

True error

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i = -\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) = -\frac{(b-a)^5}{90n^5}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^4}\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

$$E_t = -\frac{(b-a)^5}{90n^4}\overline{f}^{(4)}$$

the average value of the derivative

Gauss-Quadrature Method

Integral Gaussian method is given by:

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

constant coefficients

Points x_1 and x_2, which define the value of the integrand are not fixed (as before in the interval <a,b>), but there are *a priori* distributed freely between <a,b>.

Gauss-Quadrature Method

There are four unknowns x_1, x_2, c_1, c_2. These are found by assuming that the formula gives exact results for integrating a general third order polynomial:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

Gauss-Quadrature Method

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

Hence:

$$\int_a^b f(x)dx = c_1\left(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3\right)$$

$$= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2\left(c_1x_1^2 + c_2x_2^2\right) + a_3\left(c_1x_1^3 + c_2x_2^3\right)$$

The formula would then give:

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1x + a_2x^2 + a_3x^3\right)dx$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

Gauss-Quadrature Method

$$a_0(c_1 + c_2) + a_1(c_1 x_1 + c_2 x_2) + a_2\left(c_1 {x_1}^2 + c_2 {x_2}^2\right) + a_3\left(c_1 {x_1}^3 + c_2 {x_2}^3\right)$$

$$= a_0(b - a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

$$\Downarrow$$

$$b - a = c_1 + c_2$$

$$\frac{b^2 - a^2}{2} = c_1 x_1 + c_2 x_2$$

$$\frac{b^3 - a^3}{3} = c_1 {x_1}^2 + c_2 {x_2}^2$$

$$\frac{b^4 - a^4}{4} = c_1 {x_1}^3 + c_2 {x_2}^3$$

Gauss-Quadrature Method

This gives us four equations as follows:

$$\begin{cases} b-a = c_1 + c_2 \\ \dfrac{b^2-a^2}{2} = c_1x_1 + c_2x_2 \\ \dfrac{b^3-a^3}{3} = c_1x_1{}^2 + c_2x_2{}^2 \\ \dfrac{b^4-a^4}{4} = c_1x_1{}^3 + c_2x_2{}^3 \end{cases}$$

we can find that the above four simultaneous nonlinear equations have only one acceptable solution

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2} \qquad x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$c_1 = \frac{b-a}{2} \qquad c_2 = \frac{b-a}{2}$$

Gauss-Quadrature Method

Hence:

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$= \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)$$

General n-point rules would approximate the integral:

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots\ldots + c_n f(x_n)$$

Gauss-Quadrature Method

The coefficients and arguments given for n-point Gauss quadrature rule are given for integrals of the form in the <-1,1> integral:

$$\int_{-1}^{1} g(x)dx \cong \sum_{i=1}^{n} c_i g(x_i)$$

n	coefficients	Function arguments
2	c_1 = 1.000000000 c_2 = 1.000000000	x_1 = -0.577350269 x_2 = 0.577350269
3	c_1 = 0.555555556 c_2 = 0.888888889 c_3 = 0.555555556	x_1 = -0.774596669 x_2 = 0.000000000 x_3 = 0.774596669
4	c_1 = 0.347854845 c_2 = 0.652145155 c_3 = 0.652145155 c_4 = 0.347854845	x_1 = -0.861136312 x_2 = -0.339981044 x_3 = 0.339981044 x_4 = 0.861136312

Gauss-Quadrature Method

n	coefficients	Function arguments
5	c_1 = 0.236926885 c_2 = 0.478628670 c_3 = 0.568888889 c_4 = 0.478628670 c_5 = 0.236926885	x_1 = -0.906179846 x_2 = -0.538469310 x_3 = 0.000000000 x_4 = 0.538469310 x_5 = 0.906179846
6	c_1 = 0.171324492 c_2 = 0.360761573 c_3 = 0.467913935 c_4 = 0.467913935 c_5 = 0.360761573 c_6 = 0.171324492	x_1 = -0.932469514 x_2 = -0.661209386 x_3 = -0.2386191860 x_4 = 0.2386191860 x_5 = 0.661209386 x_6 = 0.932469514

Gauss-Quadrature Method

So if the table is given for: $\int_{-1}^{1} g(x)dx$ integrals, how does one solve

$$\int_{a}^{b} f(x)dx \quad ?$$

The answer lies in that any integral with limits of $[a, b]$

can be converted into an integral with limits: $[-1, 1]$

Let, $x = mt + c$

if $x = a,$ $t = -1$

ir $x = b,$ $t = 1$

It follows that:

$$\begin{cases} m = \dfrac{b-a}{2} \\ c = \dfrac{b+a}{2} \end{cases}$$

Gauss-Quadrature Method

Hence: $$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

$$dx = \frac{b-a}{2}dt$$

Substituting our values of x and dx into the integral gives us:

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\frac{b-a}{2}dt$$

Gauss-Quadrature Method

Example 5:

Use two-point Gauss quadrature rule to approximate the distance covered by a rocket from t = 8 s to t = 30 s as given by:

$$v(t) = 2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

a) Use the Gauss-Quadrature Method to find the distance covered from t_1=8 s to t_2=30 s
b) Find the true error.

Gauss-Quadrature Method

First, change the limits of integration from from [8,30] to [-1,1]

$$\int_{8}^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right)dx$$

$$= 11\int_{-1}^{1} f(11x+19)dx$$

The weighting factors and function argument values are (n=2):

$$c_1 = 1.000000000$$

$$x_1 = -0.577350269$$

$$c_2 = 1.000000000$$

$$x_2 = 0.577350269$$

Gauss-Quadrature Method

The formula is:

$$11\int_{-1}^{1} f(11x+19)dx \approx 11c_1 f(11x_1+19) + 11c_2 f(11x_2+19)$$

$$= 11f(11(-0.5773503)+19) + 11f(11(0.5773503)+19)$$

$$= 11f(12.64915) + 11f(25.35085)$$

$$= 11(296.8317) + 11(708.4811)$$

$$= 11058.44\ m$$

Gauss-Quadrature Method

Since:

$$f(12.64915) = 2000\,ln\left[\frac{140000}{140000 - 2100(12.64915)}\right] - 9.8(12.64915)$$

$$= 296.8317$$

$$f(25.35085) = 2000\,ln\left[\frac{140000}{140000 - 2100(25.35085)}\right] - 9.8(25.35085)$$

$$= 708.4811$$

b) The absolute relative true error: E_t

$$E_t = 11061.34 - 11058.44$$

$$= 2.9000 \ m$$

c) The relative error: $\left|\in_t\right|$

$$\left|\in_t\right| = \left|\frac{11061.34 - 11058.44}{11061.34}\right| \times 100\%$$

$$= 0.0262\%$$