NUMERICAL METHODS

Lecture 3.

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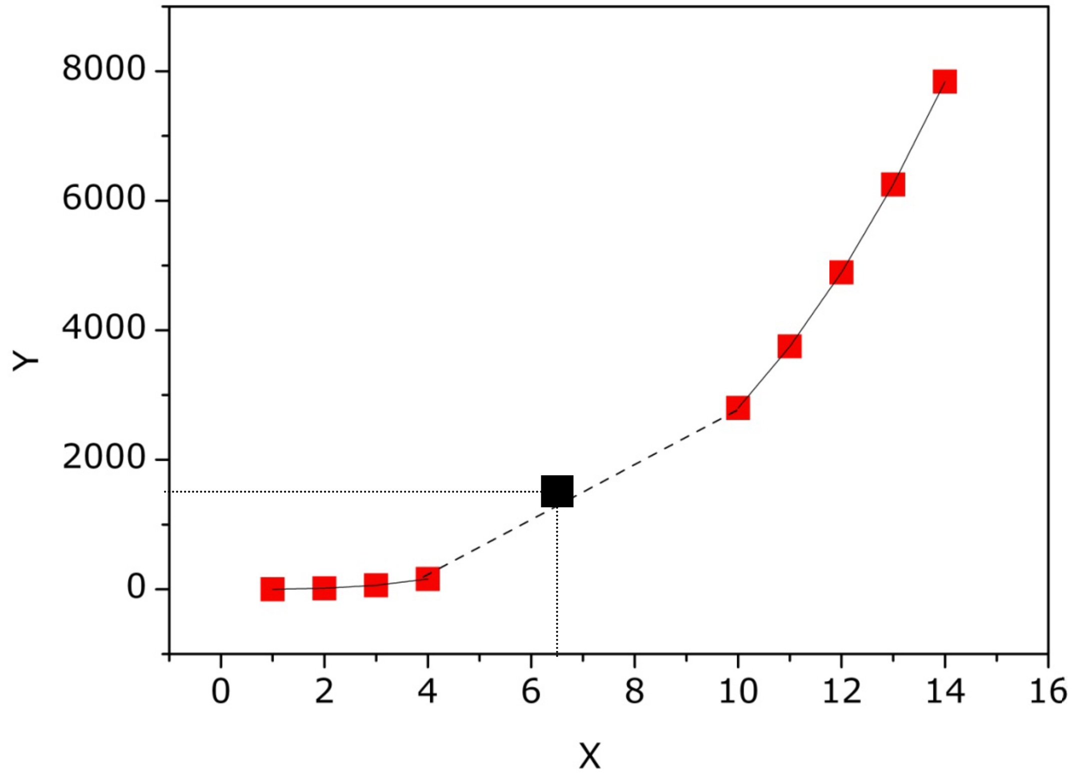
Outline

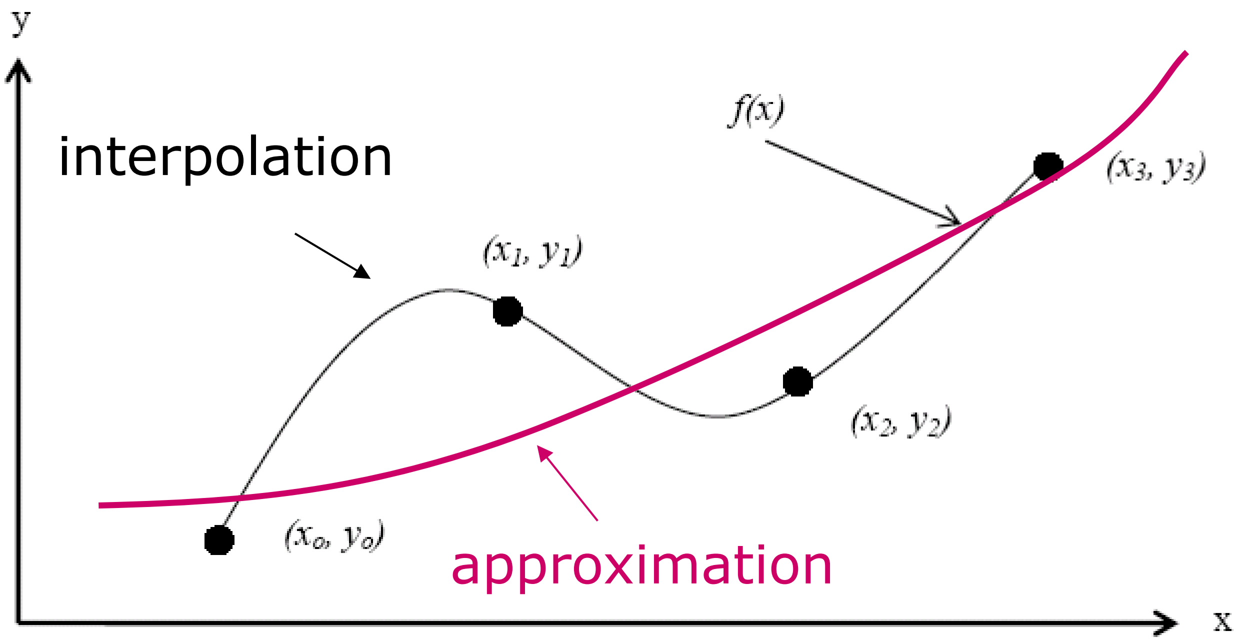
- Approximation
- Polynomial interpolation
- Examples

Principle of interpolation

Consider discrete points (x_0,y_0), (x_1,y_1), ...(x_n,y_n). How does one find the value of *y* at any other value of *x* ?



Approximation vs. interpolation



Approximation

We would like to approximate f(x) by linear combination of functions that belong to a certain, particular class .

$\{x^n\}\ (n = 0, 1, \ldots)$ like in Taylor's series

$\{p_n(x)\}\ (n = 0, 1, \ldots)$ $p_n(x)$ is a polynomial of order n

$\{\sin(nx), \cos(nx)\}\ (n = 0, 1, 2 \ldots)$ trigonometric polynomials

The most important is a polynomial interpolation

Approximation

Linear approximation f(x)

$$f(x) \approx a_0 g_0(x) + a_1 g_1(x) + \ldots + a_m g_m(x)$$

$$\{g_n(x)\} \ (n = 0, 1, \ldots)$$

Constant coefficients: $a_i (i = 0, 1, \ldots, m)$

Linear approximation is widely used, because non-linear approximation is very complex and difficult.

Sometimes rational function approximation is better:

$$f(x) \approx \frac{a_0 g_0(x) + a_1 g_1(x) + \ldots + a_m g_m(x)}{b_0 g_0(x) + b_1 g_1(x) + \ldots + b_k g_k(x)}$$

Approximation

The selection criteria for constant coefficients

$$a_i (i = 0, 1, \ldots, m)$$

There exist three types of approximations of major importance:

- **interpolation**

coefficients are chosen, so that at the points

$$x_i (i = 1, 2, \ldots, p)$$

the approximate function with its first derivatives is equal with f(x) and its derivatives (with the accuracy of rounding errors)

Approximation

The selection criteria for constant coefficients $a_i (i = 0, 1, ..., m)$

- **mean square approximation**

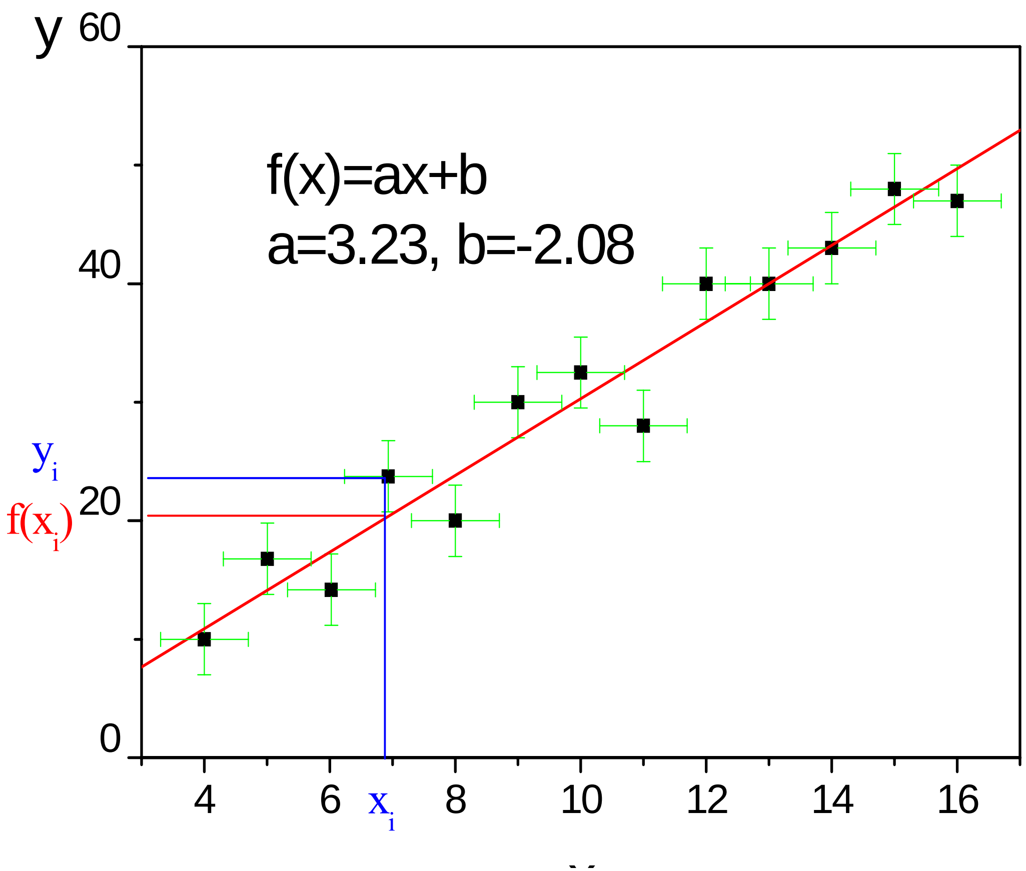
We want to find a minimum of integral of the square of the difference between f(x) and its approximation in interval $<x_1,x_2>$ or weighted sum of squared errors in collection of discrete points within the interval $<x_1,x_2>$

- **uniform approximation**

finding the maximum of the smallest difference between f(x) and its approximation in interval $<x_1,x_2>$

The least squares methods
Linear regression

$$S^2 = \sum_i^n \left[y_i - (ax_i + b)\right]^2 = \min$$



The least squares criterion:

$$\frac{\partial S^2}{\partial a} = 0 \qquad \frac{\partial S^2}{\partial b} = 0$$

Linear equations for *a* and *b*

$$a\sum x_i^2 + b\sum x_i = \sum x_i y_i$$

$$a\sum x_i + bn = \sum y_i$$

Solving the above set of equations gives *a* and *b:*

$$a = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{W}$$

$$b = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{W}$$

where: the main determinant W is given by:

$$W = n\sum x_i^2 - \left(\sum x_i\right)^2$$

Standard deviation for u(a) and u(b):

$$u(a) = \sqrt{\frac{n}{n-2}}\sqrt{\frac{S^2}{W}}$$

$$u(b) = u(a)\sqrt{\frac{\sum x_i^2}{n}}$$

Polynominal approximation

A polynomial is a common choice for an interpolating function becasue polynominals are easy to:

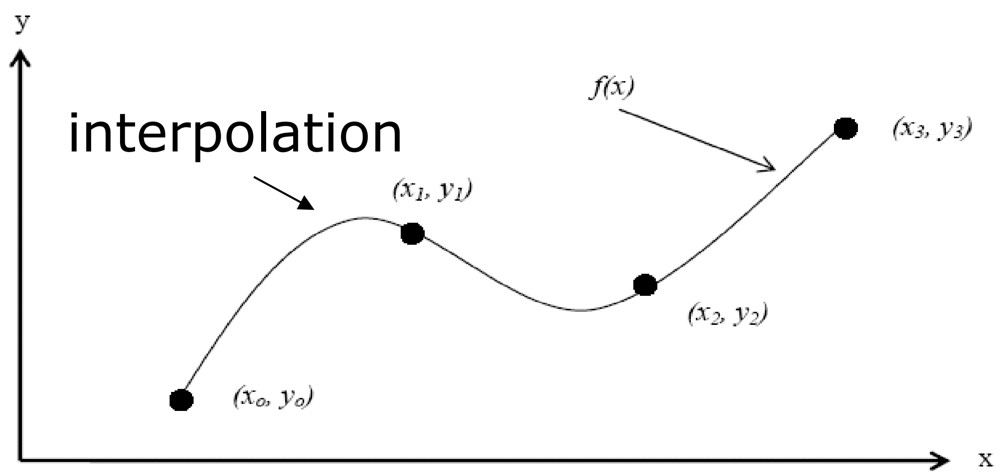
- evaluate
- differentiate
- integrate

Polynominal interpolation

Assumption:

In the [a, b] interval are (n +1) different points $x_0, x_1, ..., x_n$, called interpolation nodes, and the value of a function y = f (x) at these points:

$$f(x_i) = y_i \text{ dla } i = 0, 1, ..., n.$$



Polynomial interpolation

Main goal of interpolation:

Determination of the approximate value of the function at points which are not nodes and estimation of interpolation error

1. We want to find F(x) which approximates function f(x) in the [a,b] interval.
2. The function F(x) in the nodes of interpolation has the same value as a function y = f (x).
3. There is a unique polynomial of degree less than or equal to n passing through $n+1$ given points

Interpolation – direct method

Polynomial interpolation involves finding a polynomial of order *n* that passes through the *n+1* (x_0,y_0), (x_1,y_1), ….(x_n,y_n) points

$$y = a_0 + a_1 x + \ldots\ldots\ldots\ldots\ldots\ldots + a_n x^n.$$

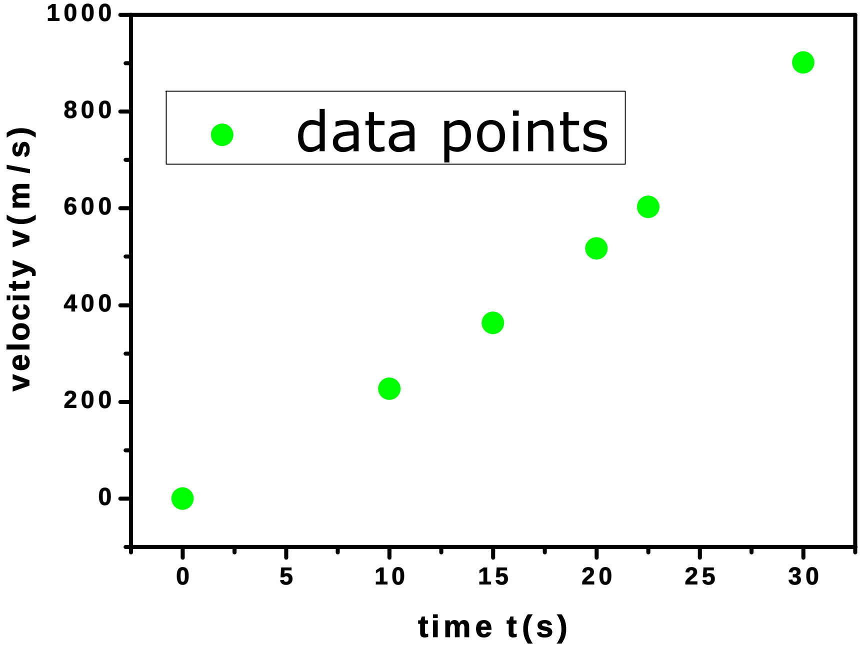
where a_0, a_1, …. a_n are constant coefficients (R)

- Consider *n+1* equations to find *n+1* constant
- Use *x* value in polynomial to find *y* value

Example

Table 1 Velocity *v* as a function of time *t*

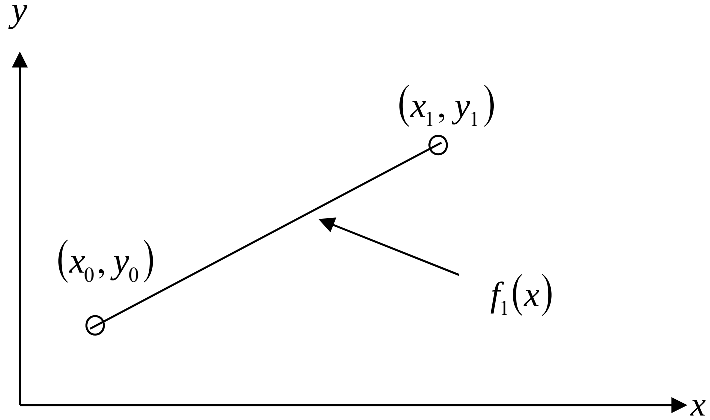
t(s)	v(m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Determine the value of the velocity at t = 16 s using the direct method of interpolation

Linear interpolation

$$v(t) = a_0 + a_1 t$$



$$v(15) = a_0 + a_1(15) = 362.78$$

$$v(20) = a_0 + a_1(20) = 517.35$$

The solution of equations

$$a_0 = -100.93$$

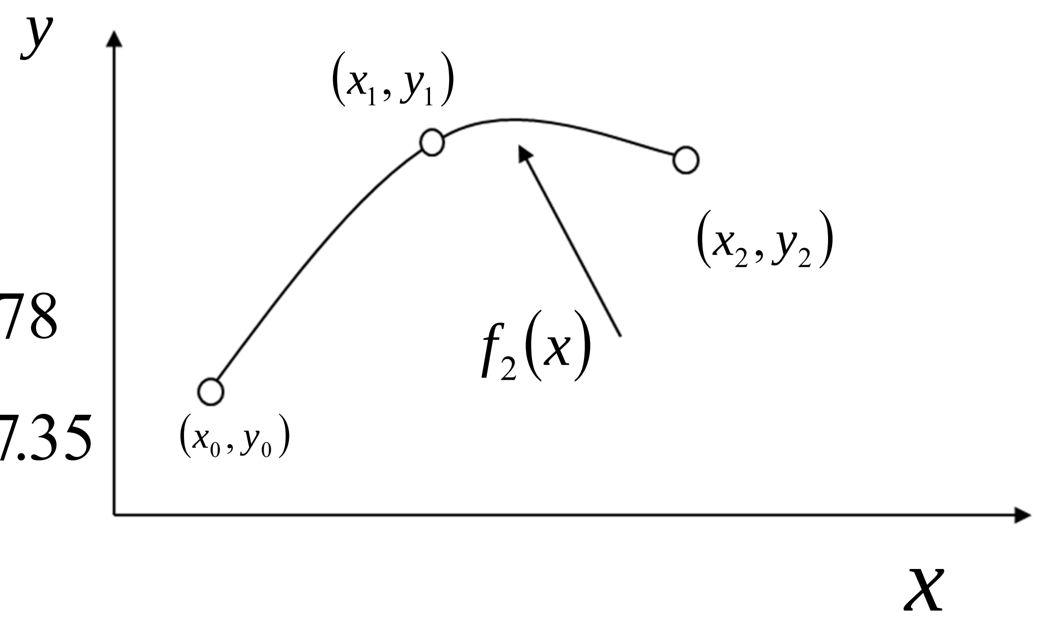
$$a_1 = 30.914$$

So: $$v(t) = -100.93 + 30.914t, \ 15 \le t \le 20$$

$$\mathrm{v}(16) = -100.93 + 30.914(16) = 393.7 \text{ m/s}$$

Quadratic interpolation

$$v(t)=a_0+a_1t+a_2t^2$$



$$v(10)=a_0+a_1(10)+a_2(10)^2=227.04$$

$$v(15)=a_0+a_1(15)+a_2(15)^2=362.78$$

$$v(20)=a_0+a_1(20)+a_2(20)^2=517.35$$

The solution of equations

$$a_0=12.05 \quad a_1=17.733 \quad a_2=0.3766$$

$$v(t)=12.05+17.733t+0.3766t^2,\ 10\le t\le 20$$

$$v(16)=12.05+17.733(16)+0.3766(16)^2 \quad =392.19\ \text{m/s}$$

Quadratic interpolation

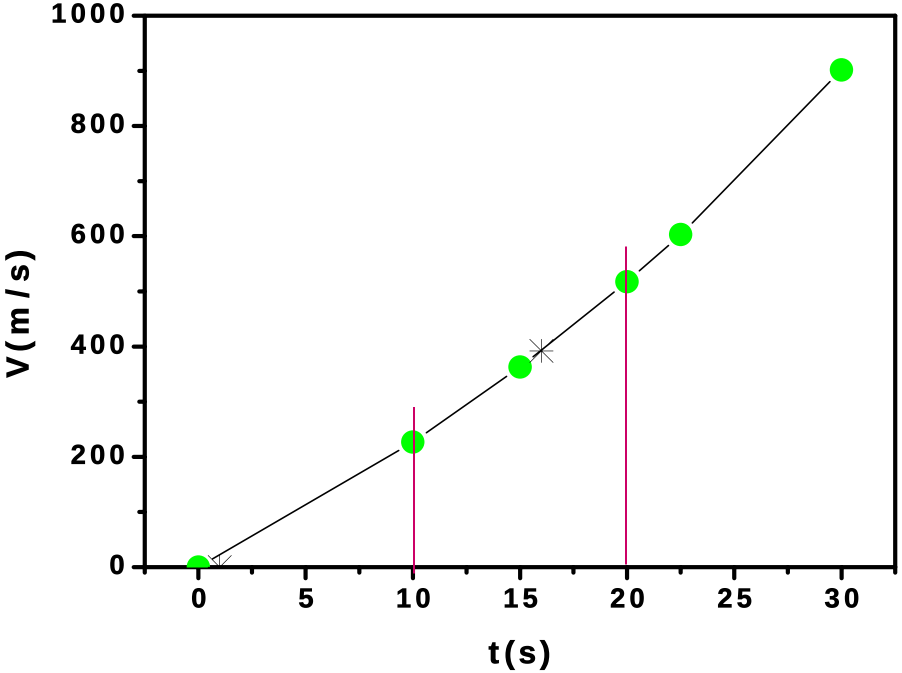
$$v(t) = 12.05 + 17.733t + 0.3766t^2, \ 10 \le t \le 20$$

$$v(16) = 392.19 m/s$$

Relative error:

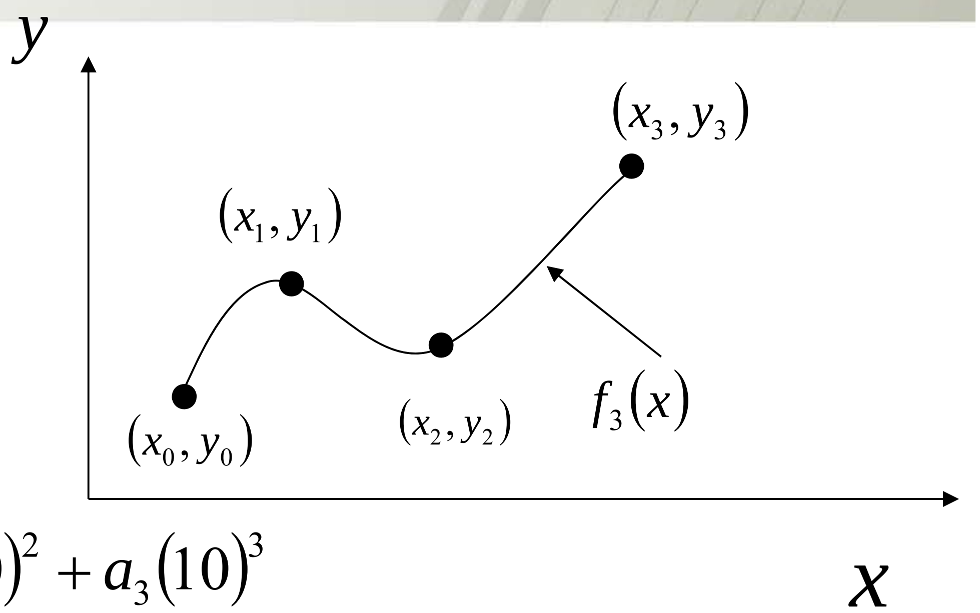
$$|\in_a| = \left| \frac{392.19 - 393.70}{392.19} \right| \times 100$$

$$= 0.38410\%$$



Cubic interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$



$$v(10) = 227.04 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$

$$v(15) = 362.78 = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3$$

$$v(20) = 517.35 = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3$$

$$v(22.5) = 602.97 = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3$$

Cubic interpolation

Homework

The solution of equations:

$$\begin{cases} a_0 + 10a_1 + 100a_2 + 1000a_3 = 227.04 \\ a_0 + 15a_1 + 225a_2 + 3375a_3 = 362.78 \\ a_0 + 20a_1 + 400a_2 + 8000a_3 = 517.35 \\ a_0 + 22.5a_1 + 506.25a_2 + 11390.625a_3 = 602.97 \end{cases}$$

Calculate and draw v(t)

Cubic interpolation - solution

$$a_0 = -4.2540$$

$$a_1 = 21.266$$

$$a_2 = 0.13204$$

$$a_3 = 0.0054347$$

$$10 \le t \le 22.5$$

$$v(t) = -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3,$$

$$v(16) = 392.06\ m/s$$

Relative error

$$|\in_a| = \left|\frac{392.06 - 392.19}{392.06}\right| \times 100$$

$$= 0.033269\ \%$$

Comparison

Polynomianl order	1	2	3
$v(t = 16)$ m/s	393.7	392.19	392.06
Relative error	----------	0.38410 %	0.033269 %

Displacement

from t=11s to t=16s

$$v(t) = -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3, 10 \le t \le 22.5$$

$$s(16) - s(11) = \int_{11}^{16} v(t)\,dt$$

$$= \int_{11}^{16} \left(-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3\right)dt$$

$$= \left[-4.2540t + 21.266\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4}\right]_{11}^{16}$$

$$= 1605\ m$$

Acceleration

$$v(t) = -4.2540 + 21.266t + 0.13204^{2} + 0.0054347\,t^{3}, 10 \le t \le 22.5$$

$$a(t) = \frac{d}{dt}v(t)$$

$$= \frac{d}{dt}\left(-4.2540 + 21.266\,t + 0.13204\,t^{2} + 0.0054347\,t^{3}\right)$$

$$= 21.266 + 0.26408\,t + 0.016304\,t^{2}, \quad 10 \le t \le 22.5$$

$$a(16) = 21.266 + 0.26408(16) + 0.016304(16)^{2}$$

$$= 29.665\ m/s^{2}$$

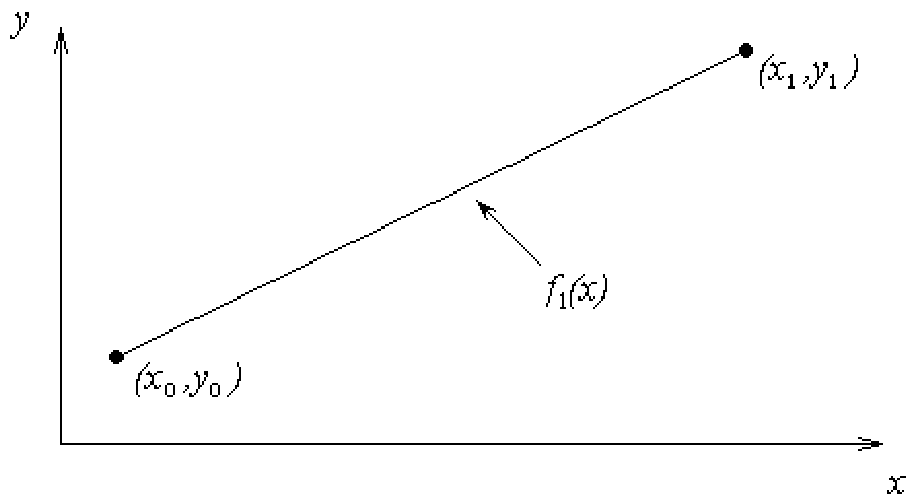
Newton's divided difference interpolation

Linear interpolation: data points $(x_0, y_0),\ (x_1, y_1),$

Let's find:

$$f_1(x) = b_0 + b_1(x - x_0)$$

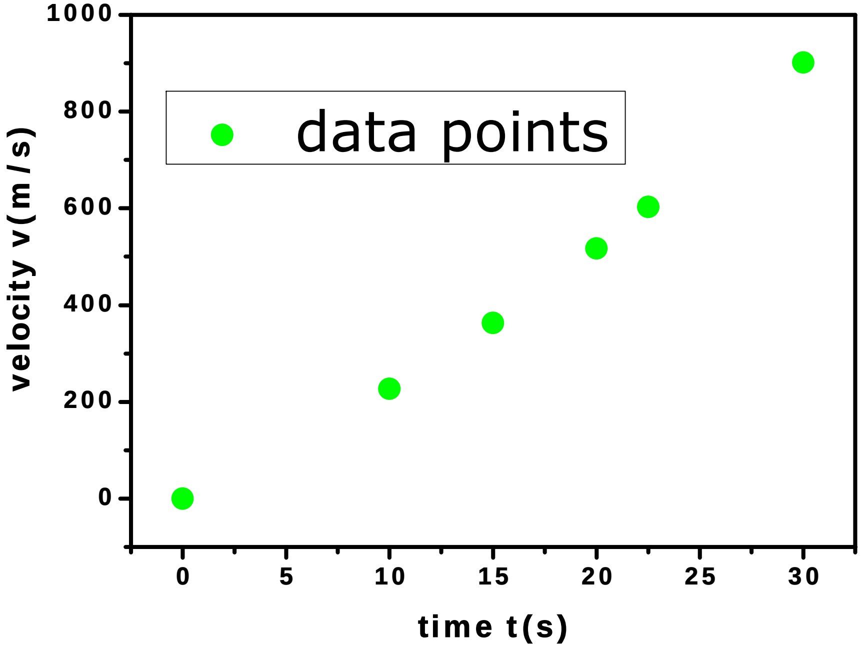
$$\begin{cases} b_0 = f(x_0) \\ b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} \end{cases}$$



Example

Table 1 Velocity v as a time function t

t(s)	v(m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Determine the value of the velocity at time t = 16 s using Newton's method of interpolation

Linear interpolation

$$v(t) = b_0 + b_1(t - t_0)$$

It is known that :

$$t_0 = 15, \;\; v(t_0) = 362.78$$

$$t_1 = 20, \;\; v(t_1) = 517.35$$

We found:

$$b_0 = v(t_0) = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = 30.914$$

So:

$$v(t) = b_0 + b_1(t - t_0) =$$

$$= 362.78 + 30.914(t - 15), \;\; 15 \le t \le 20$$

Homework: please compare obtained result with the direct method of interpolation

Linear interpolation

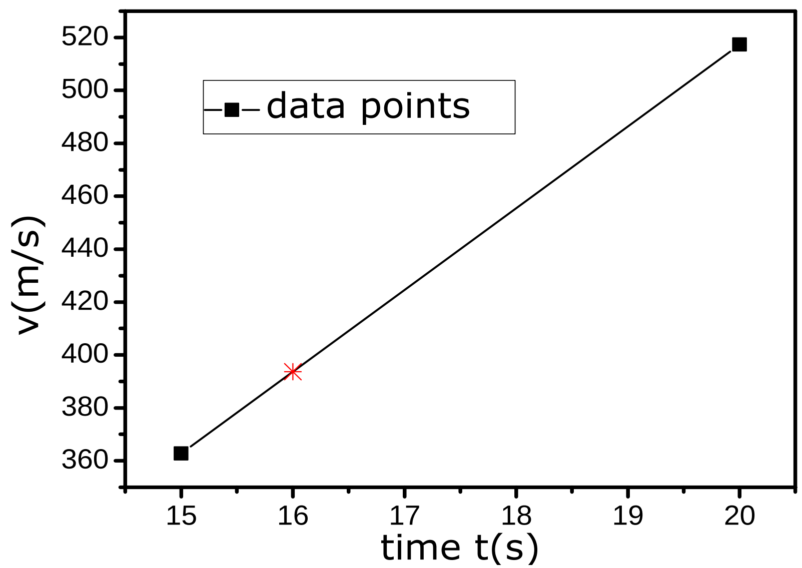
$$v(t) = b_0 + b_1(t - t_0)$$

Velocity at t=16 s is:

$$v(t) = b_0 + b_1(t - t_0) =$$

$$= 362.78 + 30.914(16 - 15)$$

$$= 393.69 \ \ m/s$$

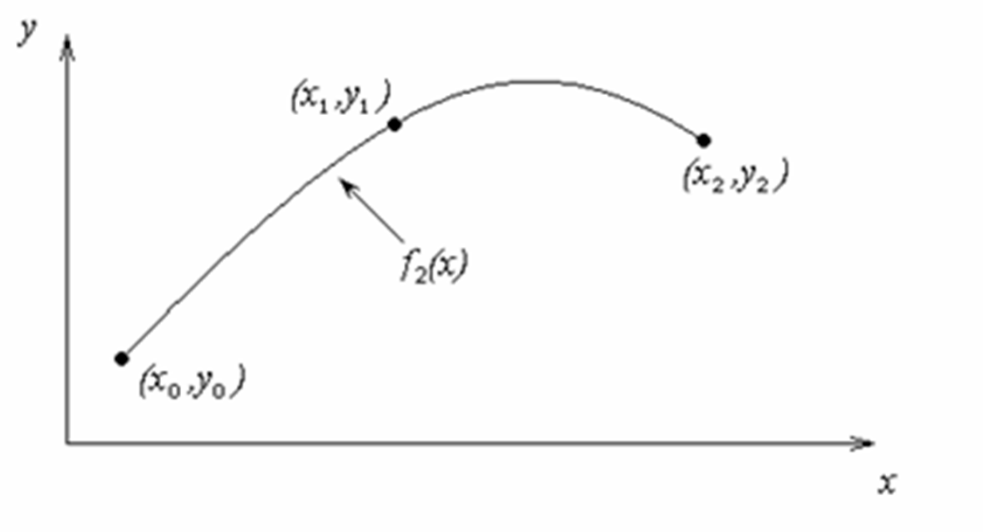


Quadratic interpolation

Consider the points: $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),$

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$\begin{cases} b_0 = f(x_0) \\ b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} \\ b_2 = \dfrac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} \end{cases}$$



Quadratic interpolation

It is known that :

$$t_0 = 10,\ \ v(t_0) = 227.04$$

$$t_1 = 15,\ \ v(t_1) = 362.78$$

$$t_2 = 20,\ \ v(t_2) = 517.35$$

We found:

$$b_0 = v(t_0) = 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10} =$$

$$= 27.148$$

$$b_2 = \frac{\dfrac{v(t_2) - v(t_1)}{t_2 - t_1} - \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{30.914 - 27.148}{10} =$$

$$= 0.37660$$

Quadratic interpolation

So:

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) =$$

$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15),\ \ 10 \le t \le 20$$

for t=16s:

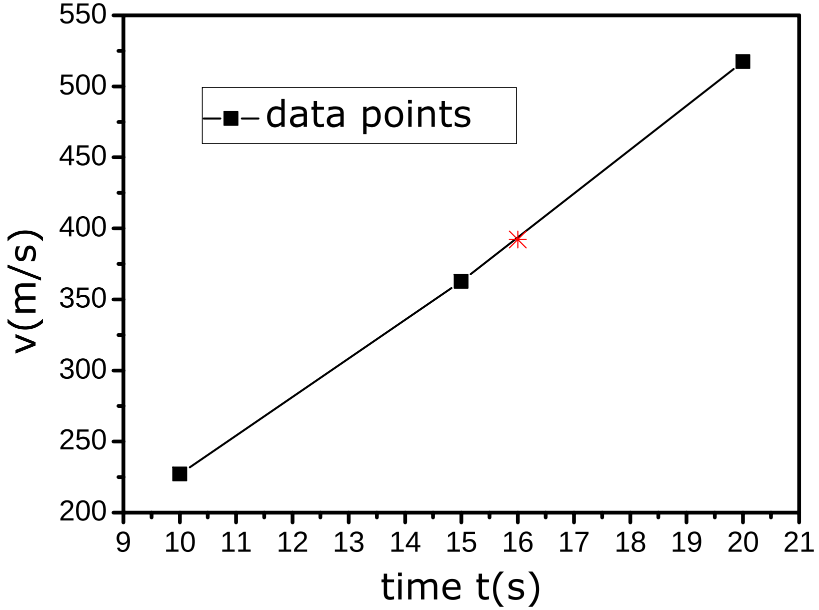
$$v(16) = b_0 + b_1(16 - t_0) + b_2(16 - t_0)(16 - t_1) =$$

$$= 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15)$$

$$= 392.19\ m/s$$

Homework: please compare obtained result with the direct method of interpolation

Quadratic interpolation



The relative error in comparison with the previous interpolation

$$|\in_a| = \left|\frac{392.19-393.69}{392.19}\right| \text{x}100$$

$$= 0.38502\ \%$$

General formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$ ← first order difference quotient

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

second order difference quotient

So:

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General formula

Consider (n+1) points

$$(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$$

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

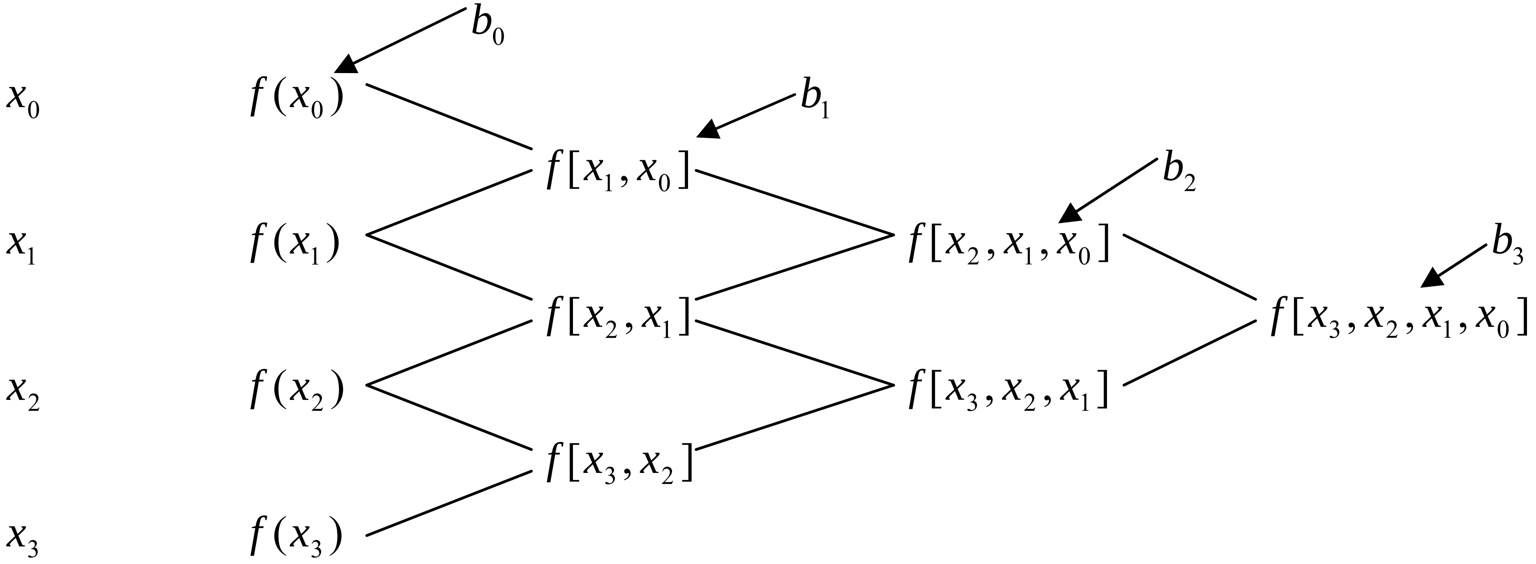
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots., x_0]$$

$$b_n = f[x_n, x_{n-1}, \ldots., x_0]$$

Cubic interpolation

3-rd degree polynomial, with data (x_0, y_0), (x_1, y_1), (x_2, y_2), i (x_3, y_3), is:

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



Cubic interpolation

Homework

Find the equation and calculate the speed v (@ 16s. Use the Newton cubic method of interpolation :

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

Data

$$t_0 = 10,\ v(t_0) = 227.04$$

$$t_1 = 15,\ v(t_1) = 362.78$$

$$t_2 = 20,\ v(t_2) = 517.35$$

$$t_3 = 22.5,\ v(t_3) = 602.97$$

Find the coefficients b_i

Find the distance covered by the rocket from t = 11 s to t = 16 s.
Find the acceleration @ t = 16 s

Solution

$t_0 = 10$	227.04 (b_0)			
		27.148 (b_1)		
$t_1 = 15,$	362.78		0.37660 (b_2)	
		30.914		$5.4347x10^{-3}$ (b_3)
$t_2 = 20,$	517.35		0.44453	
		34.248		
$t_3 = 22.5,$	602.97			

$b_0 = 227.04$; $b_1 = 27.148$; $b_2 = 0.37660$; $b_3 = 5.4347*10^{-3}$

Comparison

Polynomial order	1	2	3
v(t=16) m/s	393.69	392.19	392.06
The relative error of approximation	----------	0.38502 %	0.033427 %

Interpolation at equidistant points

Consider the function $f(x_i)=y_i$ for $i=0,1,\ldots n$ at equidistant points: $x_i = x_0 + ih$

The first Newton's polynomial interpolation is:

$$N_n^I(x) = y_0 + \frac{\Delta y_0}{1!h}(x - x_o) + \frac{\Delta^2 y_0}{2!h^2}(x - x_o)(x - x_1) + \ldots +$$

$$+ \frac{\Delta^n y_0}{n!h^n}(x - x_o)(x - x_1)\ldots\ldots(x - x_{n-1})$$

where $\Delta^k f(x_0)$ is a progressive k-th order difference

$$\Delta f = f(x+h) - f(x)$$

$$\Delta^k f = \Delta(\Delta^{k-1} f)$$

Interpolation at equidistant points

Newton polynomial interpolation is preferred near beginning of the array. Near the end of the array we use

$$N_n^{II}(x) = y_n + \frac{\Delta y_{n-1}}{1!h}(x - x_n) + \frac{\Delta^2 y_{n-2}}{2!h^2}(x - x_n)(x - x_{n-1}) + \ldots +$$

$$+ \frac{\Delta^n y_0}{n!h^n}(x - x_n)(x - x_{n-1}).....(x - x_1)$$

Newton second polynomial interpolation with backward differences

$$\nabla y_i = \Delta y_{i-1} = f(x_i) - f(x_{i-1})$$

$$\nabla^k y_i = \Delta^k y_{i-k}$$

Forward difference

$$\Delta y_i = f(x_i + h) - f(x_i) = y_{i+1} - y_i$$

$$\Delta^2 y_i = \Delta(\Delta y_i) = \Delta y_{i+1} - \Delta y_i$$

Backward difference

$$\nabla y_i = f(x_i) - f(x_i - h) = y_i - y_{i-1}$$

$$\nabla^2 y_i = \nabla(\nabla y_i) = \nabla y_i - \nabla y_{i-1}$$

INTERPOLATION BY LAGRANGE POLYNOMIAL

$$W_n(x) = \sum_{j=0}^{n} f(x_j) \frac{(x-x_0)(x-x_1)\dots(x-x_{j-1})(x-x_{j+1})\dots(x-x_n)}{(x_j-x_0)(x_j-x_1)\dots(x_j-x_{j-1})(x_j-x_{j+1})\dots(x_j-x_n)}$$

General:

$$W_n(x) = \sum_{j=0}^{n} f\left(x_j\right) \frac{\omega_n(x)}{(x-x_j)\left\{\frac{\omega_n(x)}{x-x_j}\right\}\Bigg|_{x=x_j}} = \sum_{j=0}^{n} f\left(x_j\right) \frac{\omega_n(x)}{(x-x_j)\omega'_n(x_j)}$$

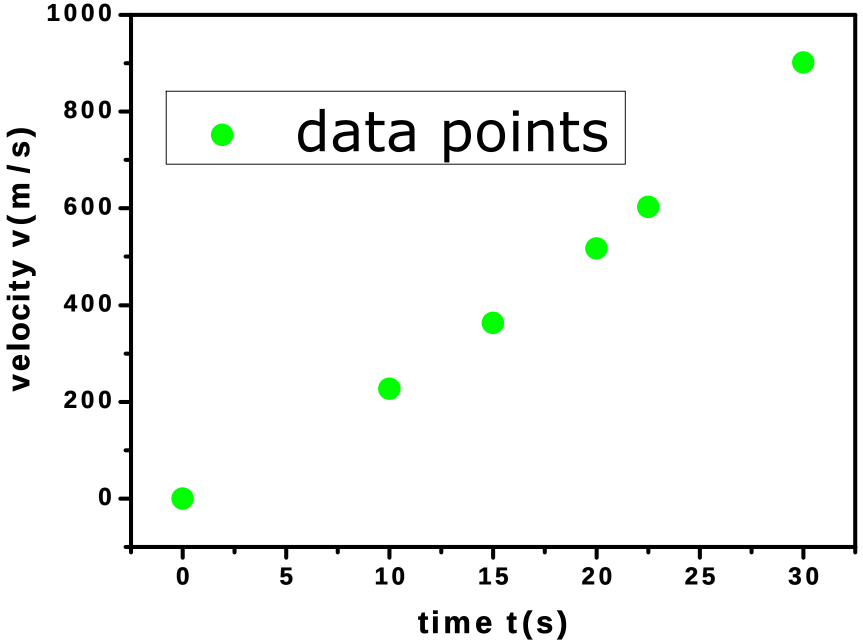
where: $\omega_n(x) = (x-x_0)(x-x_1)\dots(x-x_n)$

$\omega'_n(x_j)$ is the value of the derivative of a polynomial $\omega_n(x)$ at point x_j which is a root of the polynomial

Example

Table 1 Velocity v as a function of time t

t(s)	v(m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Determine the value of the velocity at t = 16 s using the Lagrange polynomial interpolation method

INTERPOLATION BY LAGRANGE POLYNOMIAL

$$v(t)=\sum_{i=0}^{1} L_i(t)v(t_i)=L_0(t)v(t_0)+L_1(t)v(t_1)$$

It is known that:

$$t_0=15,\ \ v(t_0)=362.78$$

$$t_1=20,\ \ v(t_1)=517.35$$

We get:

$$L_0(t)=\prod_{\substack{j=0\\ j\neq 0}}^{1}\frac{t-t_j}{t_0-t_j}=\frac{t-t_1}{t_0-t_1}$$

$$L_1(t)=\prod_{\substack{j=0\\ j\neq 1}}^{1}\frac{t-t_j}{t_1-t_j}=\frac{t-t_0}{t_1-t_0}$$

So:

$$v(t)=\frac{t-t_1}{t_0-t_1}v(t_0)+\frac{t-t_0}{t_1-t_0}v(t_1)=$$

$$=\frac{t-20}{15-20}362.78+\frac{t-15}{20-15}517.35$$

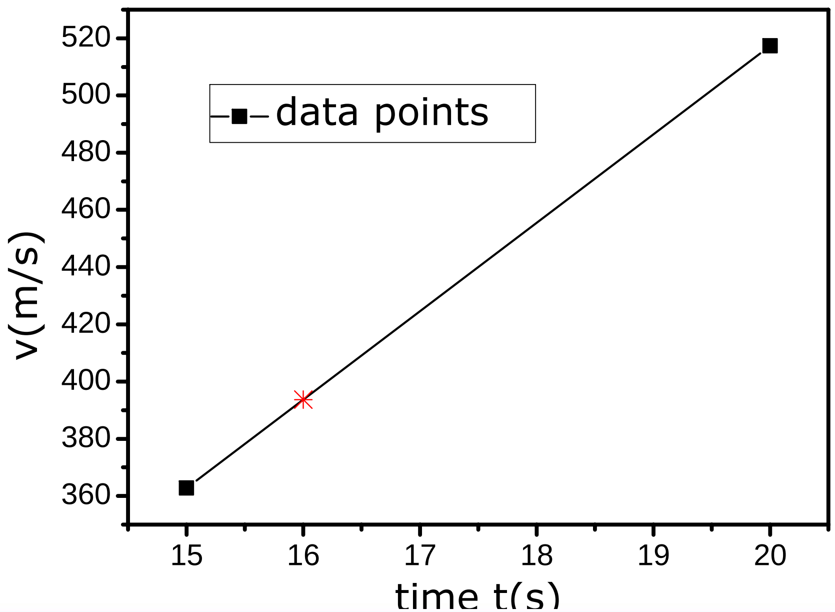
Homework: please compare the obtained result with the direct method of interpolation

INTERPOLATION BY LAGRANGE POLYNOMIAL

$$v(16)=\frac{16-20}{15-20}(362.78)+\frac{16-15}{20-15}(517.35)$$

$$=0.8\ (362.78)+0.2\ (517.35)=$$

$$=393.7\ m/s$$



Quadratic interpolation

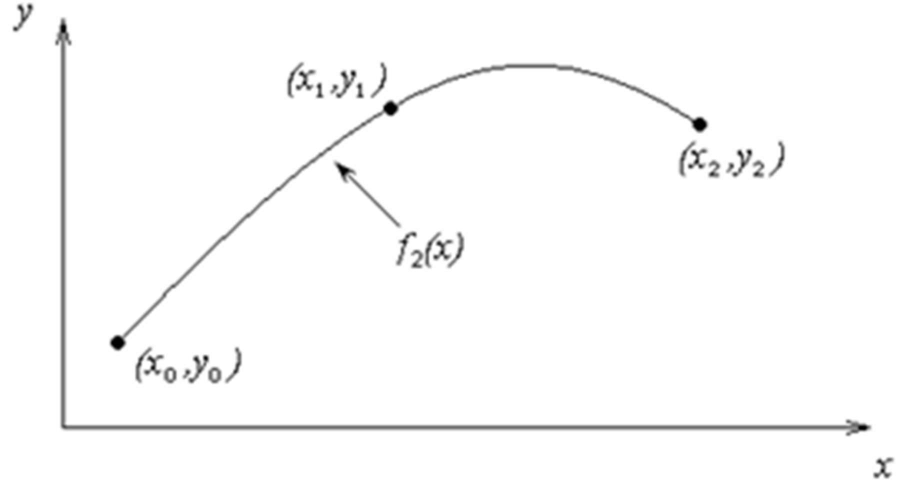
Consider the points: $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2)$

We want to find:

$$v(t) = \sum_{i=0}^{2} L_i(t)v(t_i) =$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$

$$L_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^{2} \frac{t - t_j}{t_i - t_j}$$



Quadratic interpolation

It is known that:

$$t_0 = 10,\ \ v(t_0) = 227.04$$

$$t_1 = 15,\ \ v(t_1) = 362.78$$

$$t_2 = 20,\ \ v(t_2) = 517.35$$

We get:

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{t - t_j}{t_0 - t_j} = \frac{(t - t_1)(t - t_2)}{(t_0 - t_1)(t_0 - t_2)}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{t - t_j}{t_1 - t_j} = \frac{(t - t_0)(t - t_2)}{(t_1 - t_0)(t_1 - t_2)}$$

$$L_2(t) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{t - t_j}{t_2 - t_j} = \frac{(t - t_0)(t - t_1)}{(t_2 - t_0)(t_2 - t_1)}$$

Then:

$$v(t) = \frac{t - t_1}{t_0 - t_1}\frac{t - t_2}{t_0 - t_2} v(t_0) + \frac{t - t_0}{t_1 - t_0}\frac{t - t_2}{t_1 - t_2} v(t_1) + \frac{t - t_0}{t_2 - t_0}\frac{t - t_1}{t_2 - t_1} v(t_2)$$

Quadratic interpolation

for t=16s:

$$v(16)=\frac{(16-15)}{(10-15)}\frac{(16-20)}{(10-20)}(227.04)+\frac{(16-10)}{(15-10)}\frac{(16-20)}{(15-20)}(362.78)$$

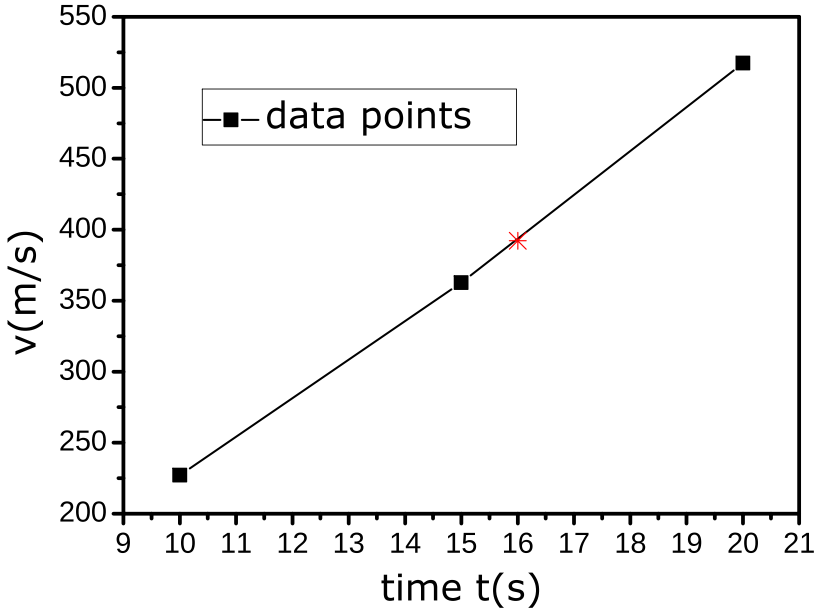
$$+\frac{(16-10)}{(20-10)}\frac{(16-15)}{(20-15)}(517.35)=$$

$$=(-0.08)(227.04)+(0.96)(362.78)+(0.12)(517.35)$$

$$=392.19\ m/s$$

Homework: please compare obtained result with the direct method and the Newton method of interpolation

Quadratic interpolation



The relative error in comparison with the previous interpolation

$$|\in_a| = \left|\frac{392.19 - 393.70}{392.19}\right| \times 100$$

$$= 0.38502\%$$

Cubic interpolation

Homework

Find the equation expressing speed and calculate v (16s) using Lagrange cubic interpolation

Data

$$t_0 = 10, \; v(t_0) = 227.04$$

$$t_1 = 15, \; v(t_1) = 362.78$$

$$t_2 = 20, \; v(t_2) = 517.35$$

$$t_3 = 22.5, \; v(t_3) = 602.97$$

Find the distance covered from t = 11 s to t = 16 s.
Find the acceleration at t = 16 s
Compare the results with those obtained on the basis of the direct interpolation method and Newton method.

Comparison

Polynomial order	1	2	3
v(t=16) m/s	393.69	392.19	392.06
The relative error	----------	0.38502 %	0.033427 %

Interpolation by Lagrange polynomial - example

Suppose we are given points: 0, 1, 3, 6. Find the Lagrange interpolation polynomial

$$f(x) = 2 \cdot \sin\left(\frac{\pi}{6} \cdot x\right)$$

Solution:

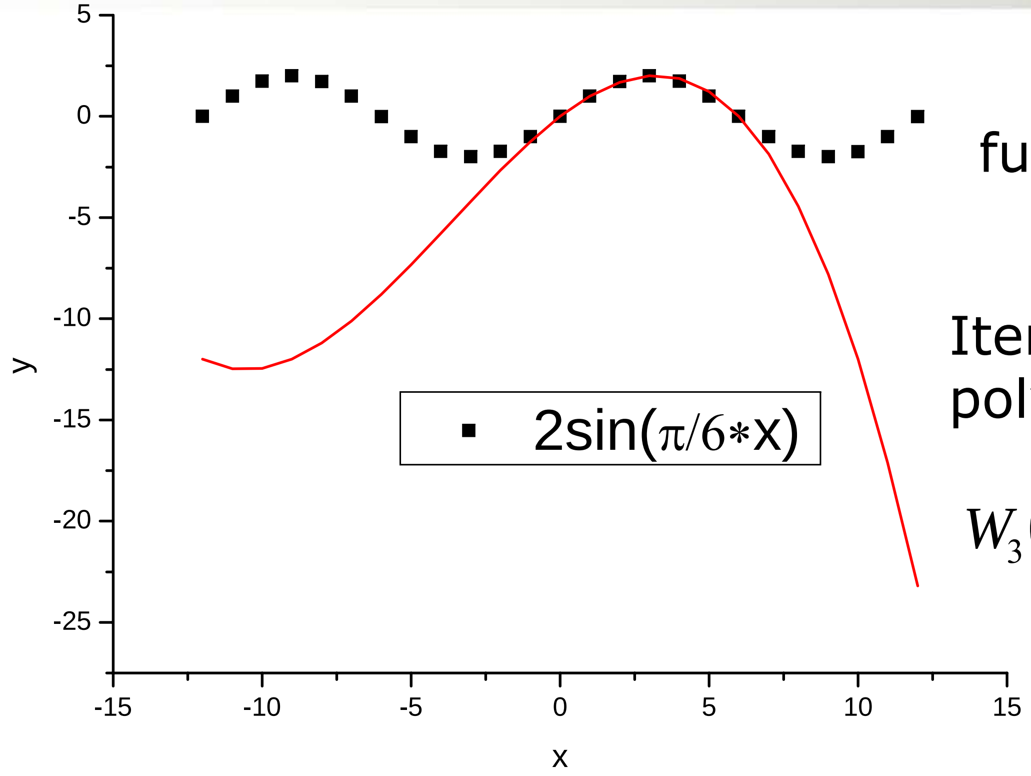
The values of the function f(x) at the interpolation nodes are as follows:

$$y_0 = f(0) = 0, \quad y_1 = f(1) = 1, \quad y_2 = f(3) = 2, \quad y_3 = f(6) = 0.$$

It can be shown that the Lagrange polynomial interpolation is:

$$W_3(x) = -\frac{x^3}{90} - \frac{11x^2}{90} + \frac{17x}{15}$$

Interpolation by Lagrange polynomial - example



function f(x)

Iterpolation polynomial $W_3(x)$

$$W_3(x) = -\frac{x^3}{90} - \frac{11x^2}{90} + \frac{17x}{15}$$

The interpolation polynomial approximates the function f (x) only in the interval [0,6].
The smaller distance between the nodes, the better the approximation.

Spline Method of Interpolation

Motivation

Disadvantages of polynomial interpolation:

Worst results of interpolation when the number of nodes is increased.

Example: $$f(x) = |x|$$

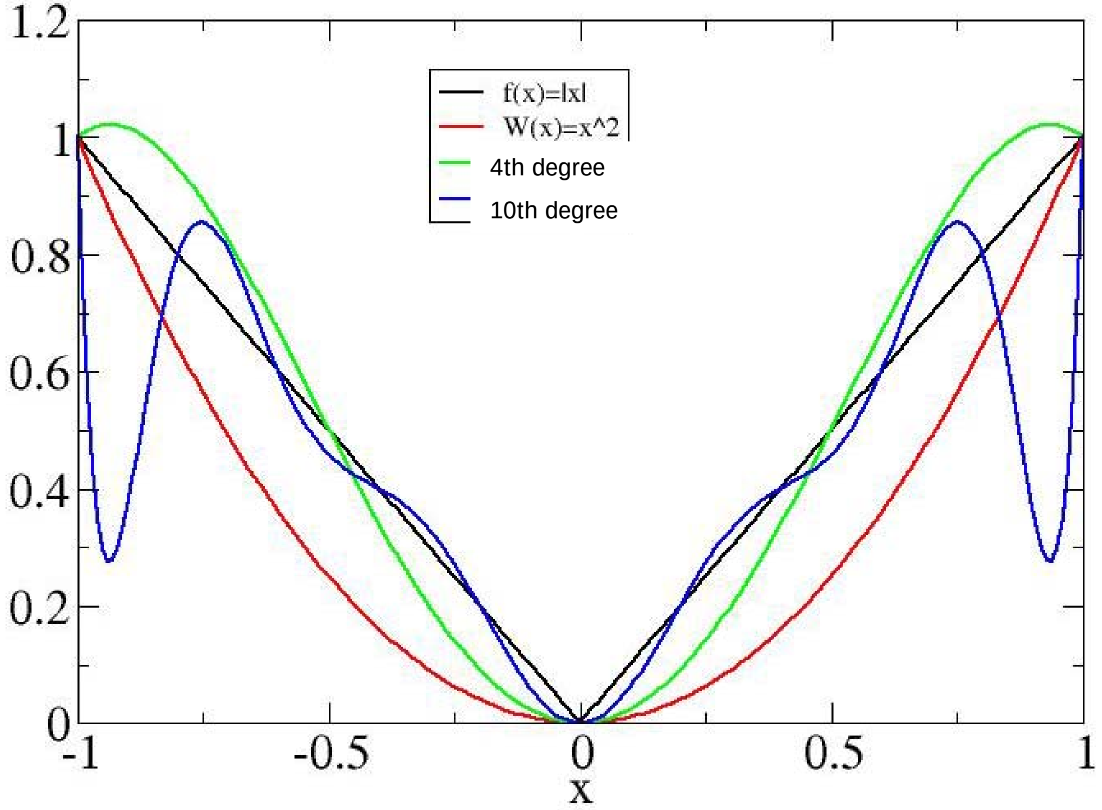
Runge's phenomenon (example of ill- conditioned task):

Polynomial interpolation of high degree at constant node distances leads to a serious deviation from the interpolated function especially at the ends of the interval. Interpolation of the central parts of the range is, however very good and useful.

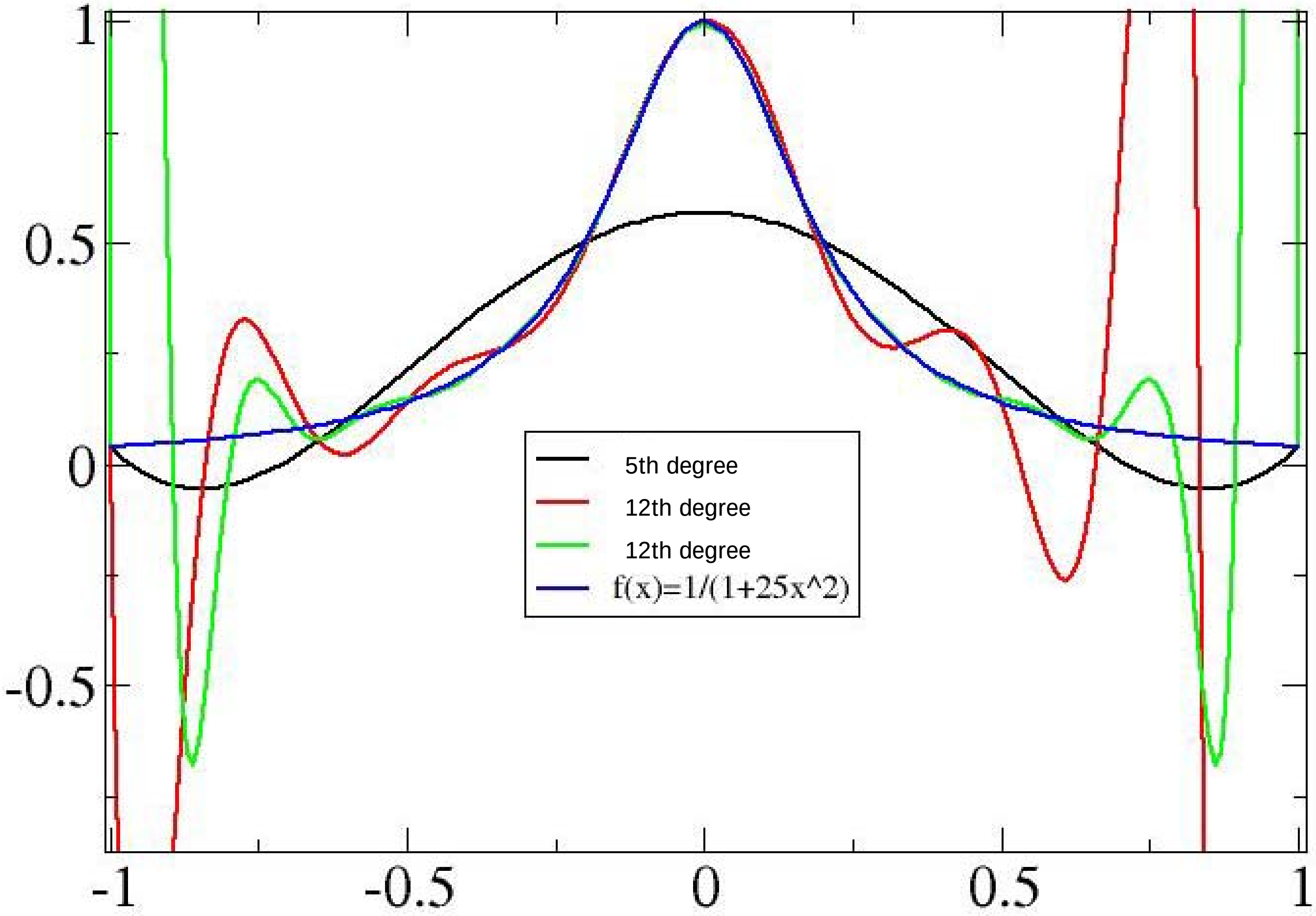
Example: $$f(x) = \frac{1}{1+25x^2}$$

Polynomial interpolation - example

$$f(x) = |x|$$



Runge's phenomenon

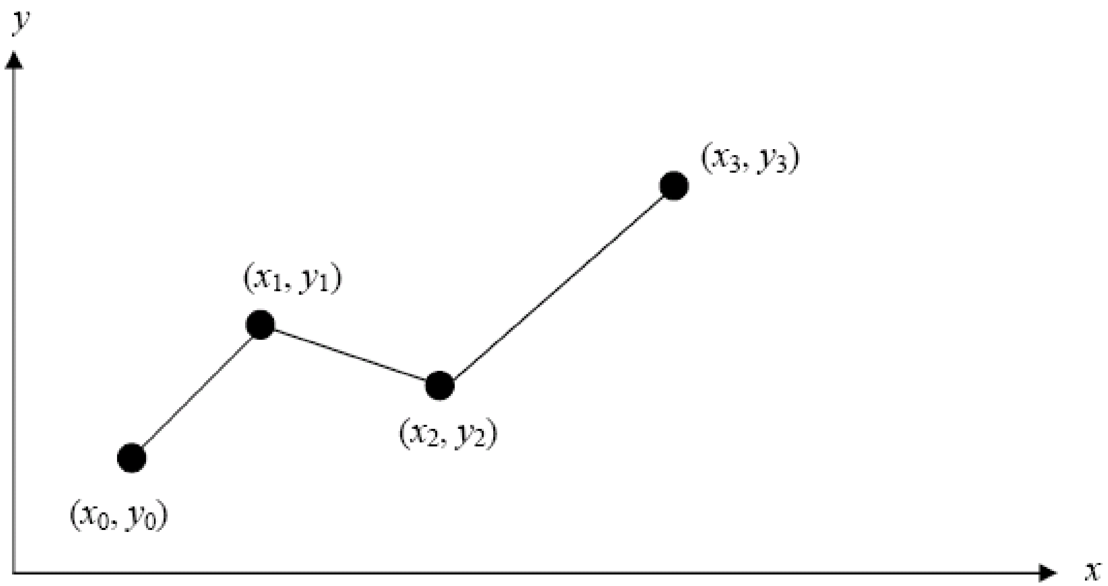


(Linear) Spline Method of Interpolation

With data points:

$$(x_0, y_0), (x_1, y_1), \ldots (x_{n-1}, y_{n-1}), (x_n, y_n)$$

we conduct straight lines between points.



(Linear) Spline Method of Interpolation

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \qquad x_0 \le x \le x_1$$

$$f(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) \qquad x_1 \le x \le x_2$$

.

slope between nodes

.

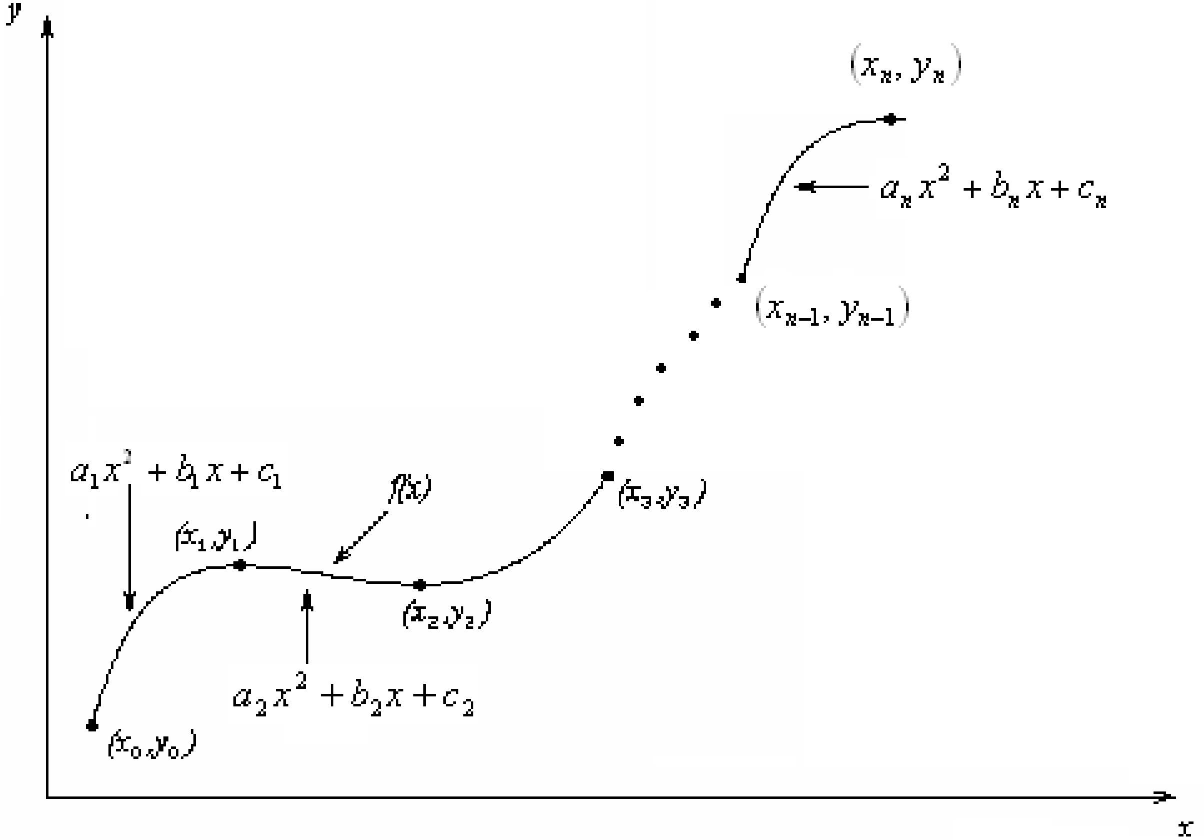
.

$$f(x) = f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}) \qquad x_{n-1} \le x \le x_n$$

(Quadratic) Spline Method of Interpolation

With data points: $(x_0, y_0), (x_1, y_1), \ldots (x_{n-1}, y_{n-1}), (x_n, y_n)$

we write a variety of quadratic functions between each pair of points.



(Quadratic) Spline Method of Interpolation

$$f(x) = a_1 x^2 + b_1 x + c_1 \qquad x_0 \le x \le x_1$$

$$f(x) = a_2 x^2 + b_2 x + c_2 \qquad x_1 \le x \le x_2$$

.

.

.

$$f(x) = a_n x^2 + b_n x + c_n \qquad x_{n-1} \le x \le x_n$$

Find the coefficients a_i, b_i, c_i $\quad i = 1,2,\ldots,n$

We have 3n unknown values so we need 3n equations

(Quadratic) Spline Method of Interpolation

Each parabola passes through two points, so we have 2n equations

$$f(x_0) = a_1 x_0^2 + b_1 x_0 + c_1$$

$$f(x_1) = a_1 x_1^2 + b_1 x_1 + c_1$$

$$\vdots$$

$$f(x_{i-1}) = a_i x_{i-1}^2 + b_i x_{i-1} + c_i$$

$$f(x_i) = a_i x_i^2 + b_i x_i + c_i$$

$$\vdots$$

$$f(x_{n-1}) = a_n x_{n-1}^2 + b_n x_{n-1} + c_n$$

$$f(x_n) = a_n x_n^2 + b_n x_n + c_n$$

(Quadratic) Spline Method of Interpolation

Additional conditions imposing continuity of the first derivatives in n-1 internal nodes can be obtained :

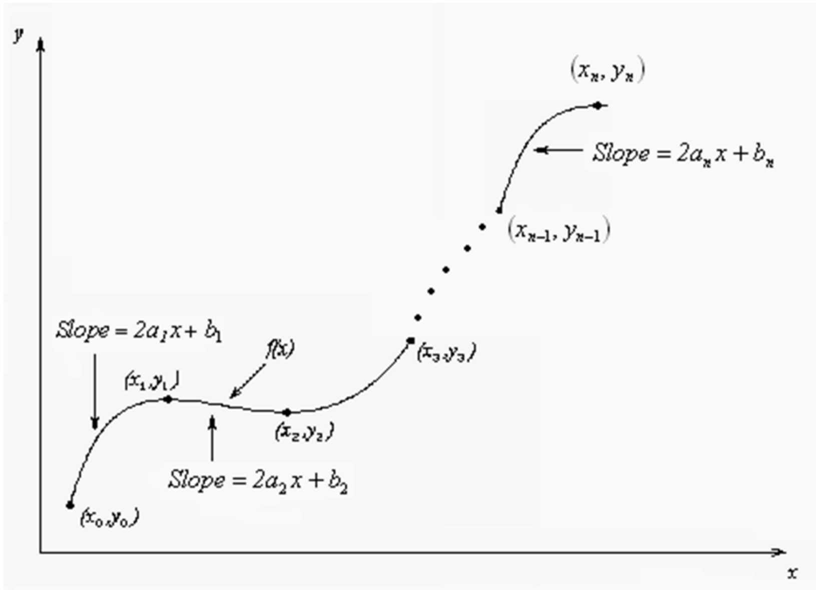
for $f(x) \begin{cases} a_1 x^2 + b_1 x + c_1 \\ a_2 x^2 + b_2 x + c_2 \end{cases} \Rightarrow f'(x) \begin{cases} 2a_1 x + b_1 \\ 2a_2 x + b_2 \end{cases}$

so:

$$2a_1 x_1 + b_1 = 2a_2 x_1 + b_2$$

.
.
.

$$2a_{n-1} x_{n-1} + b_{n-1} = 2a_n x_{n-1} + b_n$$



(Quadratic) Spline Method of Interpolation

As a results, we get n-1 equations:

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

$$\vdots$$

$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

$$\vdots$$

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$

The total number of equations is 2n+(n-1)=3n-1

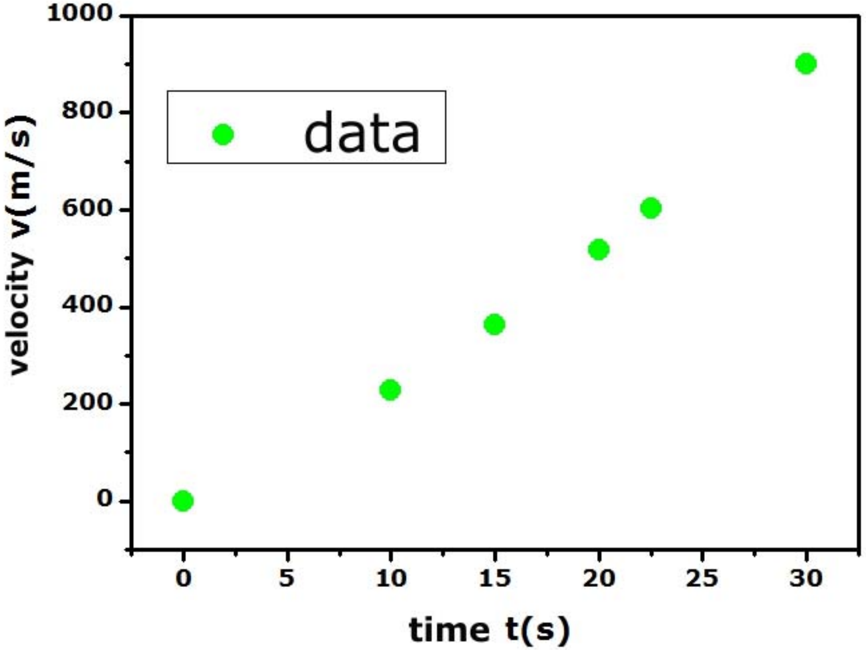
The one needed equation may take a form of $a_1 = 0$

The first spline function is linear.

Example

Table 1 Velocity v as a function of time t

t(s)	v(m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Find speed at t = 16 s using the quadratic spline interpolation

Solution

$$v(t) = a_1 t^2 + b_1 t + c_1, \qquad 0 \le t \le 10$$

$$= a_2 t^2 + b_2 t + c_2, \qquad 10 \le t \le 15$$

$$= a_3 t^2 + b_3 t + c_3, \qquad 15 \le t \le 20$$

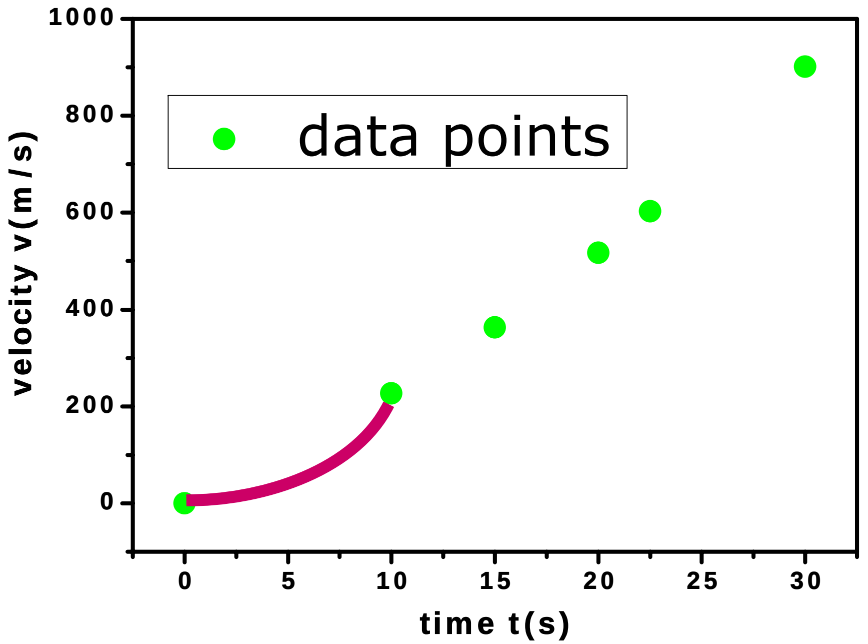
$$= a_4 t^2 + b_4 t + c_4, \qquad 20 \le t \le 22.5$$

$$= a_5 t^2 + b_5 t + c_5, \qquad 22.5 \le t \le 30$$

Each splines goes through two consecutive data points

$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \le t \le 10$$

$$\begin{cases} a_1(0)^2 + b_1(0) + c_1 = 0 \\ a_1(10)^2 + b_1(10) + c_1 = 227.04 \end{cases}$$



More equations

t(s)	v(m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

There have 10 equations,

15 coefficients to calculate

$$a_2(10)^2 + b_2(10) + c_2 = 227.04$$

$$a_2(15)^2 + b_2(15) + c_2 = 362.78$$

$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$

$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

The continuity of derivatives

$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \le t \le 10$$

$$= a_2 t^2 + b_2 t + c_2, \quad 10 \le t \le 15$$

$$\left.\frac{d}{dt}\left(a_1 t^2 + b_1 t + c_1\right)\right|_{t=10} = \left.\frac{d}{dt}\left(a_2 t^2 + b_2 t + c_2\right)\right|_{t=10}$$

$$\left.\left(2a_1 t + b_1\right)\right|_{t=10} = \left.\left(2a_2 t + b_2\right)\right|_{t=10}$$

$$2a_1(10) + b_1 = 2a_2(10) + b_2$$

$$20a_1 + b_1 - 20a_2 - b_2 = 0$$

The continuity of derivatives – cont.

for t=10s $$2a_1(10)+b_1-2a_2(10)-b_2=0$$

for t=15s $$2a_2(15)+b_2-2a_3(15)-b_3=0$$

for t=20s $$2a_3(20)+b_3-2a_4(20)-b_4=0$$

for t=22.5s $$2a_4(22.5)+b_4-2a_5(22.5)-b_5=0$$

4 additional equations

The last equation $a_1=0$

The final matrix of 15 equations and 15 unknowns

$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\
20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

b_i

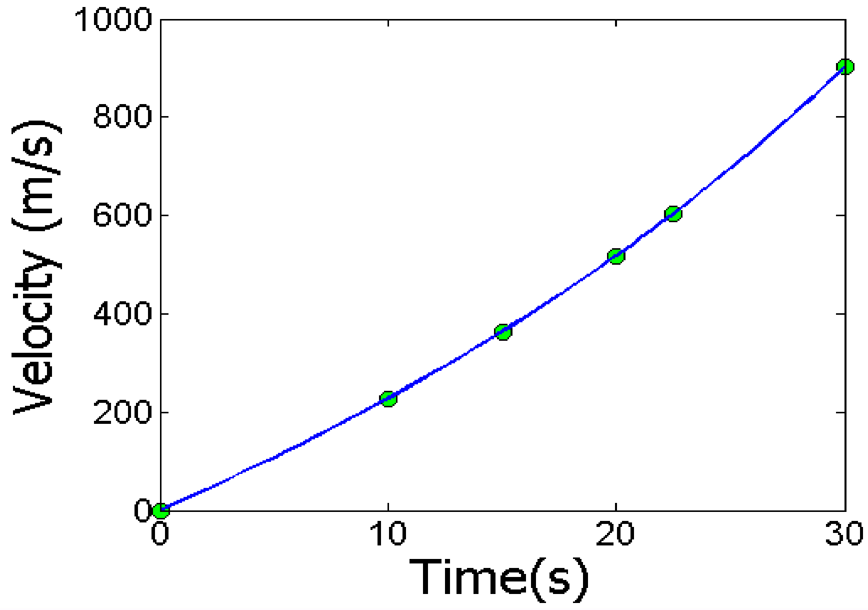
Solution

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	-0.1356	35.66	-141.61
4	1.6048	-33.956	554.55
5	0.20889	28.86	-152.13

Make sure that these values are correct

Solution

$$v(t) = 22.704t, \qquad 0 \le t \le 10$$
$$= 0.8888t^2 + 4.928t + 88.88, \qquad 10 \le t \le 15$$
$$= -0.1356t^2 + 35.66t - 141.61, \qquad 15 \le t \le 20$$
$$= 1.6048t^2 - 33.956t + 554.55, \qquad 20 \le t \le 22.5$$
$$= 0.20889t^2 + 28.86t - 152.13, \qquad 22.5 \le t \le 30$$



Velocity at a specific moment

a) Velocity at t=16s

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$

$$
\begin{aligned}
v(16) &= -0.1356(16)^2 + 35.66(16) - 141.61 \\
&= 394.24 \text{ m/s}
\end{aligned}
$$

As a homework, please compare the calculated value of the velocity with the value obtained by polynomial interpolation

Acceleration at a specific moment

b) Acceleration at t=16s

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888\,t^2 + 4.928\,t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$

$$a(16) = \frac{d}{dt} v(t)\Big|_{t=16}$$

Acceleration at a specific moment

The quadratic spline function at the point t = 16s is given as:

$$v(t) = -0.1356\,t^2 + 35.66t - 141.61, \quad 15 \le t \le 20$$

$$a(t) = \frac{d}{dt}(-0.1356t^2 + 35.66t - 141.61)$$

$$= -0.2712t + 35.66,$$

$$a(16) = -0.2712(16) + 35.66 \quad = 31.321\,\mathrm{m/s^2}$$

As a homework, please compare the calculated value of the acceleration with the value obtained by polynomial interpolation

Calculation of a distance covered

c) Find the distance covered by the rocket between t=11s to t=16s.

$$v(t) = 22.704t, \qquad 0 \le t \le 10$$

$$= 0.8888t^2 + 4.928t + 88.88, \qquad 10 \le t \le 15$$

$$= -0.1356t^2 + 35.66t - 141.61, \qquad 15 \le t \le 20$$

$$= 1.6048t^2 - 33.956t + 554.55, \qquad 20 \le t \le 22.5$$

$$= 0.20889t^2 + 28.86t - 152.13, \qquad 22.5 \le t \le 30$$

$$S(16) - S(11) = \int_{11}^{16} v(t)\,dt$$

Calculation of a distance covered

$$v(t) = 0.8888t^2 + 4.928t + 88.88, \qquad 10 \le t \le 15$$

$$= -0.1356t^2 + 35.66t - 141.61, \qquad 15 \le t \le 20$$

$$S(16) - S(11) = \int_{11}^{16} v(t)dt = \int_{11}^{15} v(t)dt + \int_{15}^{16} v(t)dt$$

$$= \int_{11}^{15} (0.8888t^2 + 4.928t + 88.88)dt$$

$$+ \int_{15}^{16} (-0.1356t^2 + 35.66t - 141.61)dt$$

$$= 1595.9 \text{ m}$$

As a homework, please compare the calculated value of the distance with the value obtained by polynomial interpolation

Error of interpolation

$$\left|f(x)-W_n(x)\right| \le \frac{\sup\limits_{x\in<a,b>}\left|f^{(n+1)}(x)\right|}{(n+1)!}\cdot\left|\prod_{i=0}^{n}(x-x_i)\right|$$

Assuming that:

$$M_{n+1} = \sup_{x\in<a,b>}\left|f^{(n+1)}(x)\right|$$

The upper limit of the module (n+1)th derivative of f(x) in <a,b>

$$\omega_n(x) = (x-x_0)(x-x_1)\ldots(x-x_n)$$

Error of interpolation

$$\left| f(x) - W_n(x) \right| \le \frac{M_{n+1}}{(n+1)!} \cdot \left| \omega_n(x) \right|$$

Example:

Assess with what accuracy you can calculate the value of ln 100.5 using Lagrange interpolation formula, if the given values are: ln 100 ln 101 ln 102 ln 103

$$f(x) = \ln(x),\ n = 3,\ a = 100,\ b = 103,\ f^{(4)}(x) = -\frac{6}{x^4}$$

$$M_4 = \sup_{x \in <100,103>} \left| f^{(4)}(x) \right| = \frac{6}{100^4}$$

$$\left| \ln 100{,}5 - W\ (100{,}5) \right| \le \frac{6}{100^4 4!} \cdot 0{,}5 \cdot 0{,}5 \cdot 1{,}5 \cdot 2{,}5 \approx 2{,}344 \cdot 10^{-9}$$

The optimum choice of interpolation nodes

$$\left| f(x) - W_n(x) \right| \le \frac{M_{n+1}}{(n+1)!} \cdot \left| \omega_n(x) \right|$$

The value of the error depends on the choice of the interpolation nodes - ω_n. We cannot affect the M_{n+1}

How to choose the interpolation nodes x_i to achieve the lowest value of:

$$\sup_{x \in <a,b>} \left| \omega_n(x) \right|$$

This issue was formulated by Russian mathematician P.L. Chebyshev as a problem of finding the best algebraic polynomial approximating zero in a given interval.

Chebyshev polynomials

Chebyshev polynomials (of the first type):

$$T_n(x) = \cos(n \; arc\cos x)$$

It can be shown that the polynomial $T_n(x)$ is identical with a certain algebraic polynomial limited to the interval<-1,1>.

$$T_0(x) = 1$$

$$T_1(x) = \cos(arc\cos x) = x$$

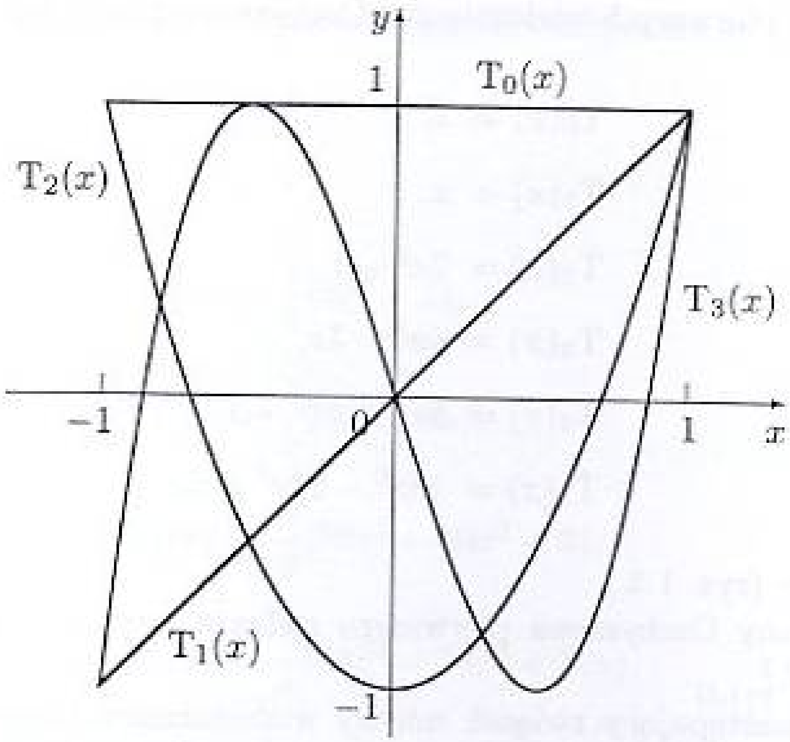
$$T_2(x) = \cos(2arc\cos x) = 2x^2 - 1$$

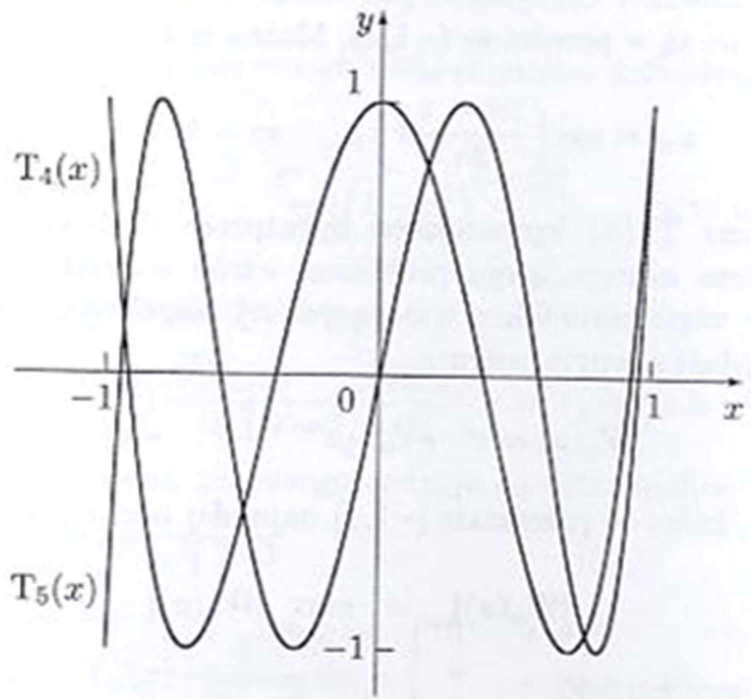
$$T_3(x) = \cos(3arc\cos x) = 4x^3 - 3x$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

Chebyshev polynomials of the first type

$$T_0(x) = 1 \qquad T_1(x) = \cos(arc\cos x) = x$$





Chebyshev polynomials

Chebyshev polynomials of the first type are the solution of the differential equation:

$$(1-x^2)\frac{d^2T_n(x)}{dx^2}-x\frac{dT_n(x)}{dx}+n^2T_n(x)=0$$

They are defined by the Rodrigues formula:

$$T_n(x)=(-1)^n\frac{\sqrt{1-x^2}}{(2n-1)!!}\frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n-1/2}\right]$$

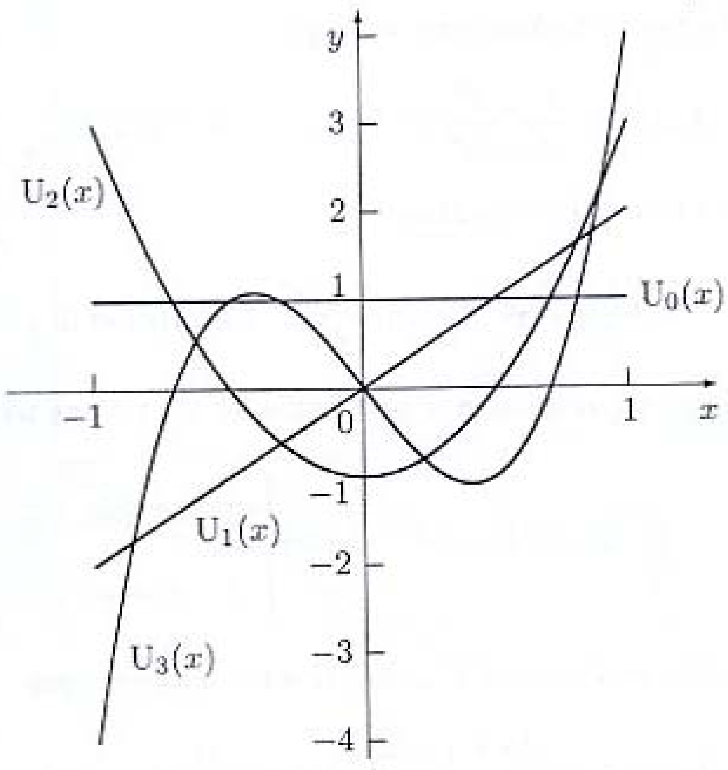
Chebyshev polynomials of the first type are orthogonal in the interval

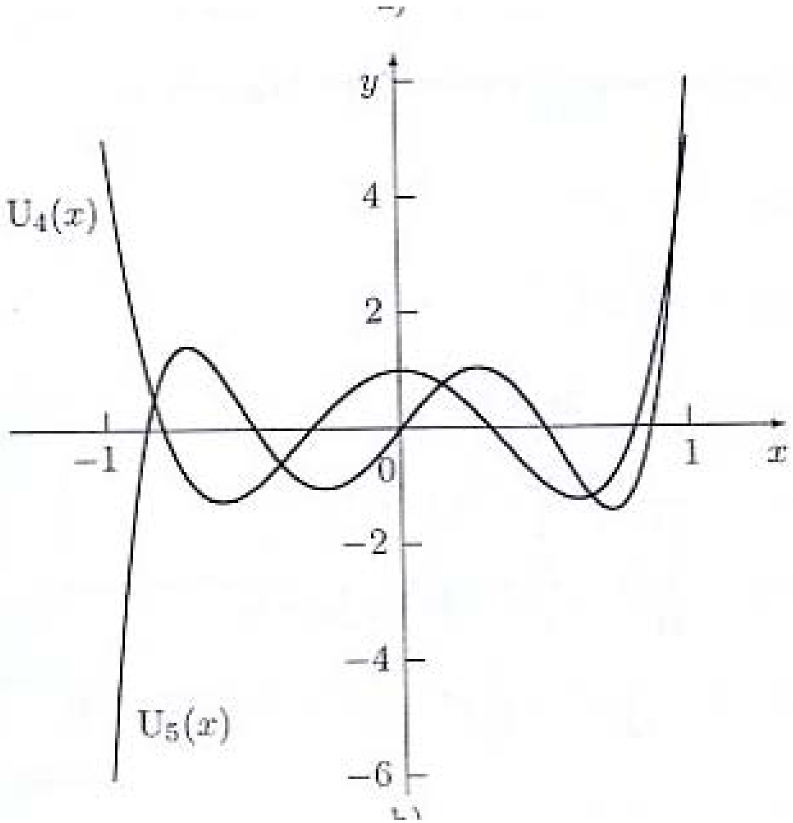
<-1,1> with a weight:

$$w(x)=\frac{1}{\sqrt{1-x^2}}$$

Chebyshev polynomials of the second type

$U_0(x) = 1$ $U_1(x) = 2x$ $U_2(x) = 4x^2 - 1$





Chebyshev polynomials

Chebyshev polynomials of the second type are the solution of the differential equation:

$$(1-x^2)\frac{d^2U_n(x)}{dx^2}-3x\frac{dU_n(x)}{dx}+n(n+2)U_n(x)=0$$

They are defined by the Rodrigues formula

$$U_n(x)=(-1)^n\frac{1}{\sqrt{1-x^2}}\frac{n+1}{(2n+1)!!}\frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n+1/2}\right]$$

Chebyshev polynomials of the second type are orthogonal in the interval <-1,1> with a weight :

$$w(x)=\sqrt{1-x^2}$$

The optimal choice of interpolation nodes

Every Chebyshev polynomial of a degree n has roots at:

$$x_m = \cos(\frac{2m+1}{2n}\pi), \ \ m = 0, 1, 2, \ldots, n-1$$

between -1 i +1

Coefficient of the highest power of T_n (x) is equal to 2^{n-1}.

We are looking for a polynomial, which has the highest power factor of unity

$$T^*_{n+1}(x) = \frac{1}{2^n} T_{n+1}(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$$

where x_m(m=0, 1, 2, …, n) are the roots of a polynomial T_{n+1}

The optimal choice of interpolation nodes

Expression:

$$\sup_{x\in<a,b>} \left|\omega_n(x)\right|$$

in the interval<-1,1> has the lowest value for the polynomial :

$$\omega_n(x) = \frac{1}{2^n} T_{n+1}(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$$

in this case:

$$\sup_{x\in<-1,1>} \left|\omega_n(x)\right| = \frac{1}{2^n}$$

When, in the interval<-1,1> we assume Chebyshev polynomial zeros as the interpolation nodes , then:

$$\left| f(x) - W_n(x) \right| \le \frac{M_{n+1}}{2^n (n+1)!}$$

The optimal choice of interpolation nodes

At any range <a,b> estimated error is:

$$\left| f(x) - W_n(x) \right| \le \frac{M_{n+1}}{(n+1)!} \frac{(b-a)^{n+1}}{2^{2n+1}}$$

the selection of nodes:

$$x_m = \frac{1}{2}\left[(b-a)\cos\frac{2m+1}{2n+2}\pi) + (b+a) \right], \quad m = 0,1,2,\dots,n$$

The new nodes x_m are not evenly spaced, but are concentrated at the ends of the interval.

Simple linear transformations convert x from the interval <a,b> to z in the interval<-1,1>

$$x = \frac{1}{2}\left[(b-a)z + (b+a) \right]$$

$$z = \frac{1}{b-a}(2x - b - a)$$

Interpolation - summary

Conclusions:

1. When calculating the values of the polynomial interpolation in one or more points the problem of selecting a interpolation formula is not important.
2. The selected formula and distribution of nodes affect only the calculation error.
3. The number of multiplications and division mainly determines the total time needed for calculations

for the Lagrange polynomial n^2+4n+2

For the Newton polynomial $1/2\ n^2+3/2\ n$