Lecture 4.

Nonlinear equations

dr hab.inż. Katarzyna Zakrzewska, prof.AGH

Solving nonlinear equations with one unknown

You should find root of non-linear equations that is to solve the equation

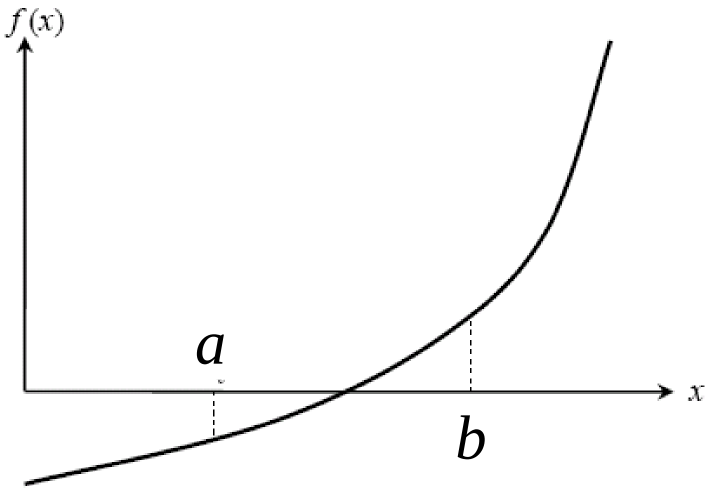
$$f(x) = 0$$

Theorem:

If the function f(x) is defined and continuous in the range <a,b> and function changes sign at the ends of the interval

$$f(a)f(b) \le 0$$

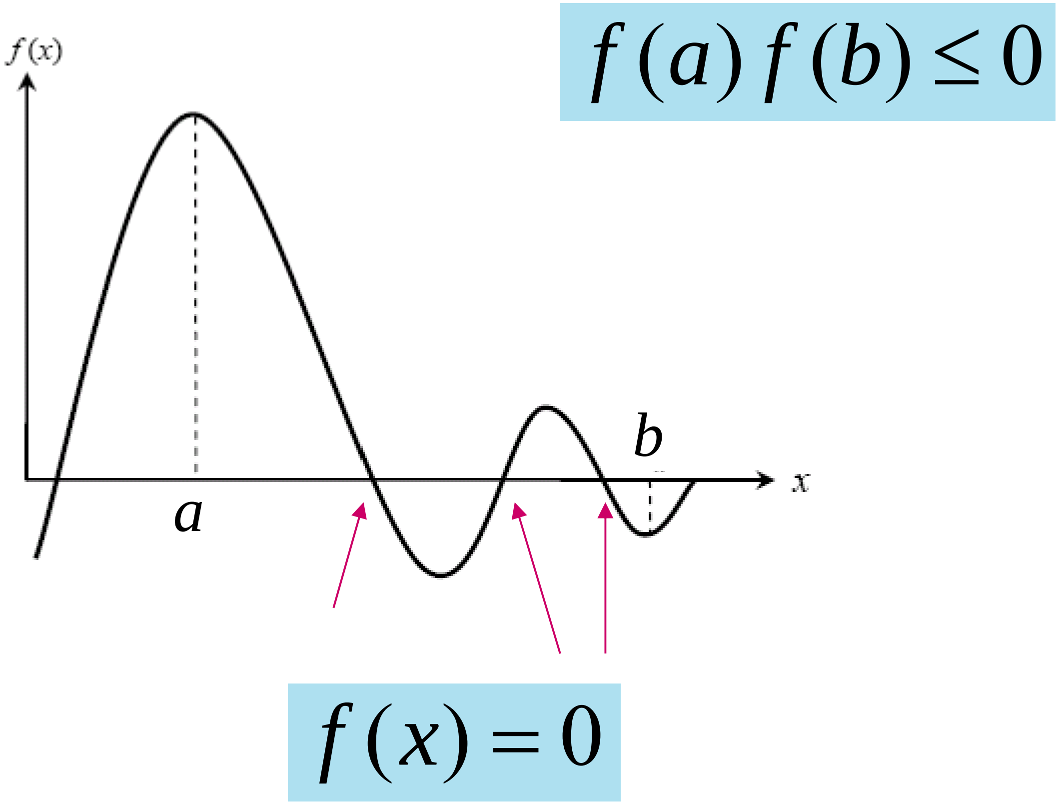
then contains at least one single root in the range <a,b>



The range <a,b>, in which the single root of the equation exists is called root isolation interval.

Solving nonlinear equations with one unknown

If the function changes sign between two points, more than one root for the equation may exist between these two points.

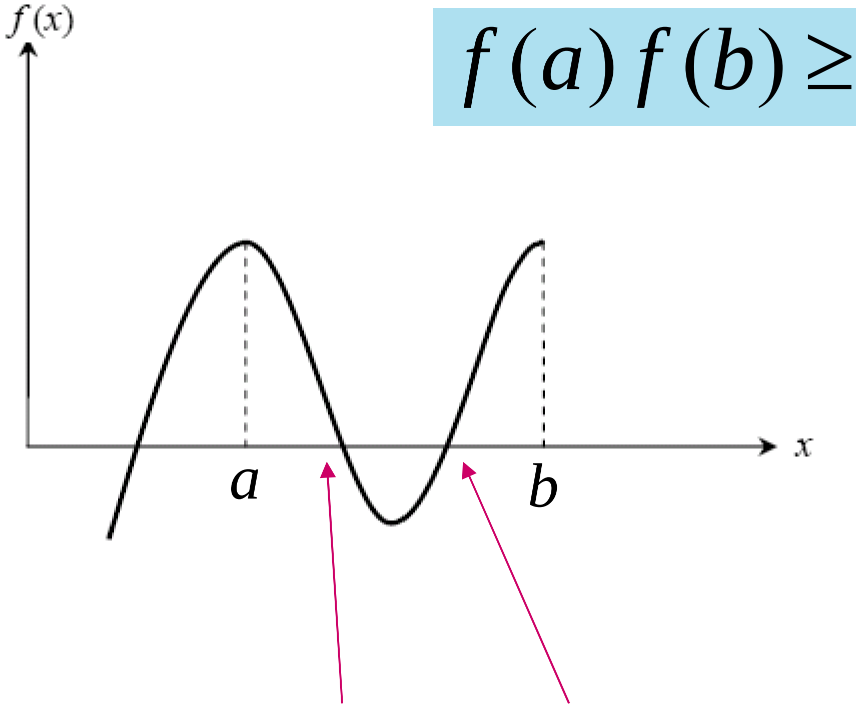


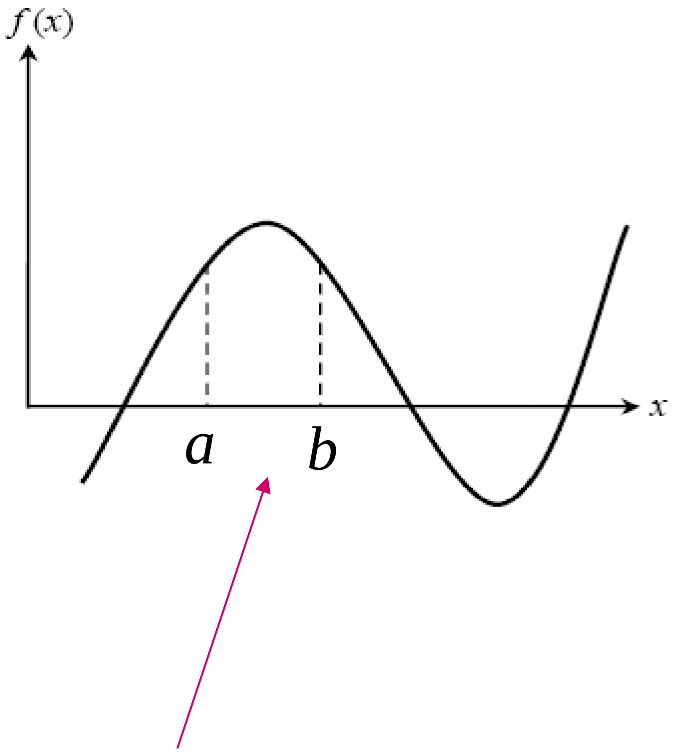
$$f(a)\,f(b) \le 0$$

$$f(x) = 0$$

Solving nonlinear equations with one unknown

$$\boxed{f(a)f(b) \geq 0}$$





If the function does not change the sign between two points, there may not be or there may exist roots for this equation between the two points.

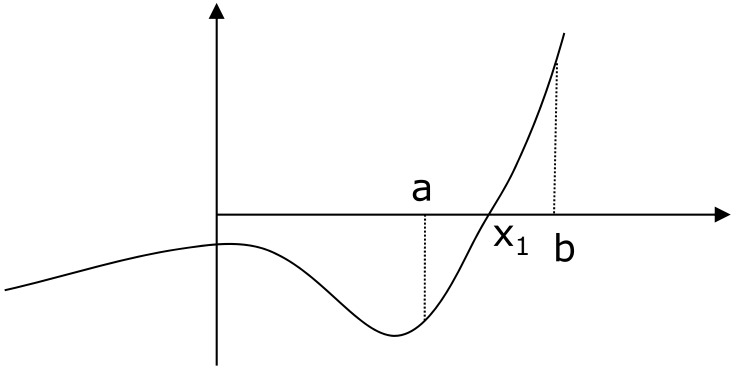
Numerical methods of solving nonlinear equations with one unknown

Methods:

- Bisection Method
- Newton-Raphson Method
- False-Position Method
- Secant Method

Bisection Method

<a,b> range is divided in half at a point: $x_1 = \frac{a+b}{2}$



If $f(x_1)=0$, then the root is x_1.
If $f(x_1) \neq 0$ then with $\langle a, x_1 \rangle$ and $\langle x_1, b \rangle$ we choose the one at the end of which the function f (x) has different signs:

$$f(a) \cdot f(x_1) < 0 \quad \text{or} \quad f(x_1) \cdot f(b) < 0$$

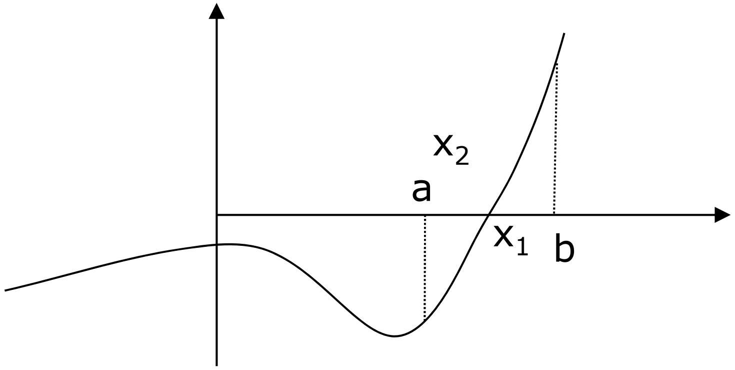
Bisection Method

The resulting ranges <a,x_1> or <x_1,b> are again divided in half at a point:

$$x_2 = \frac{a + x_1}{2}$$

or

$$x_2 = \frac{b + x_1}{2}$$



If f(x_2)=0, then the root is x_2. .
If f(x_2)≠0 then select a new range and check function sign at the ends. Repeat this process until one gets the exact solution or the desired accuracy of the solution is reached.

Bisection Method

As a result of this procedure after some number of steps, we get the exact root f $(x_n) = 0$, or a sequence of intervals such that:

$$f(x_i)\, f(x_{i+1}) < 0$$

where x_i and x_i+1 are respectively the beginning and the end of the i-th interval, and its length:

$$\left| x_i - x_{i+1} \right| = \frac{b-a}{2^i}$$

Since the left ends of the intervals form a non-decreasing sequence bounded from above, and the right ends form a non-increasing sequence bounded from below so their common border exists.

The algorithm for the bisection method

At each step, we calculate the absolute relative approximate error

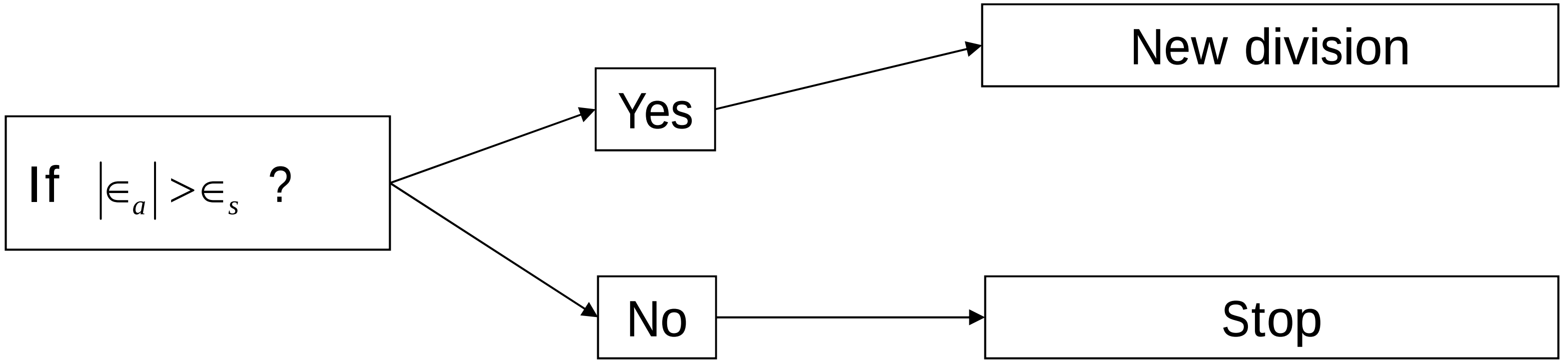
$$\left|\in_a\right| = \left|\frac{x_m^i - x_m^{i-1}}{x_m^i}\right| \times 100$$

where:

x_m^{i-1}	previous estimate of root
x_m^i	current estimate of root

The algorithm for the bisection method

Compare the absolute relative approximate error $\left|\in_a\right|$ with the pre-specified error tolerance $\in_s$.



If $\left|\in_a\right| > \in_s$?

Yes

New division

No

Stop

Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Example of the bisection method

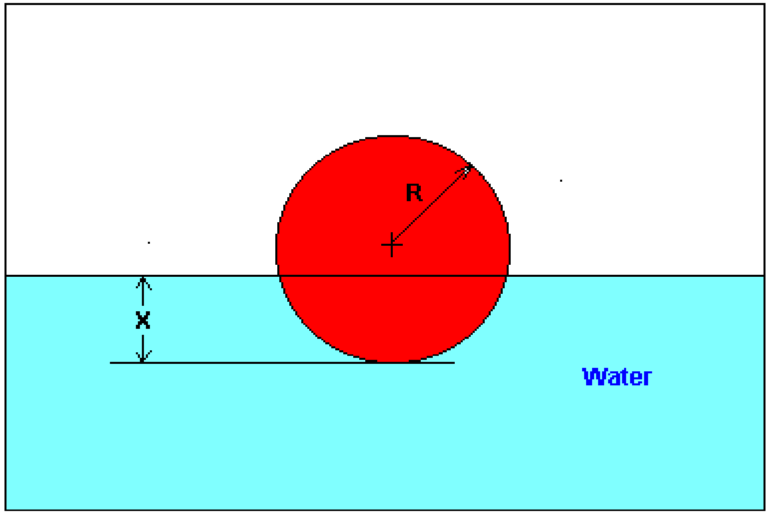
Floating ball

From the laws of physics the ball will be submerged to a depth of x such as

$$0 \le x \le 2R$$

$$0 \le x \le 2(0.055)$$

$$0 \le x \le 0.11$$



$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

Example of the bisection method

Task:

a) Use the bisection method to find the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the equation

$$x^3 - 0.165x^2 + 3.993\times 10^{-4} = 0$$

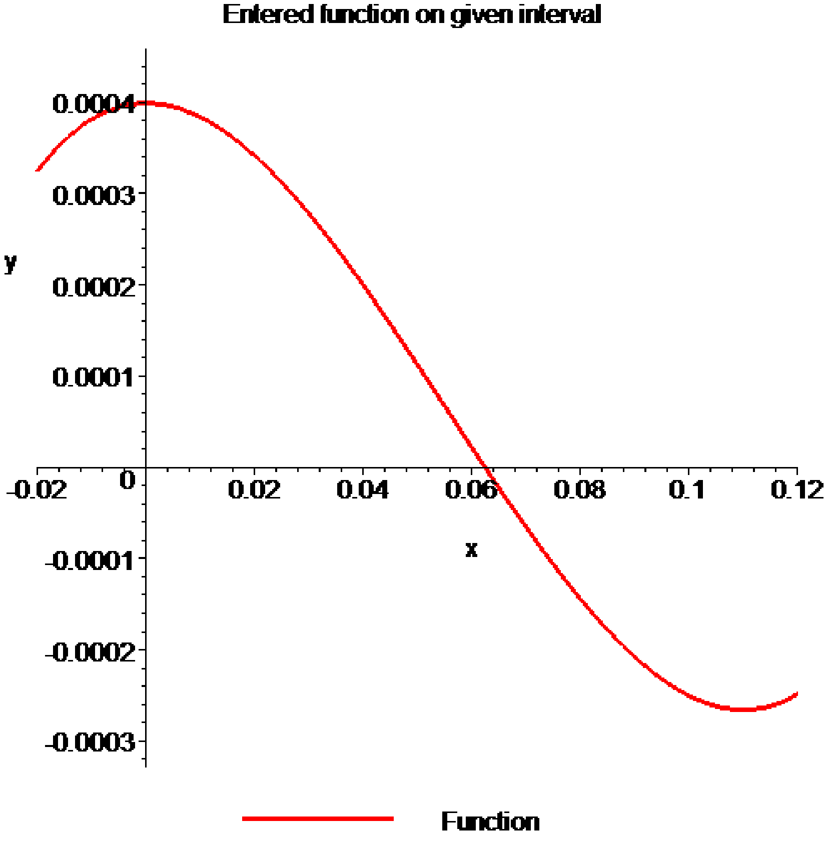
b) Find the absolute relative approximate error at the end of each iteration, and the number of significant digits at the end of each iteration.

Example of the bisection method

Solution

A graph of f(x) is useful

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$



Example of the bisection method

Let us assume $x_\ell = 0.00$

$$x_u = 0.11$$

Check if the function changes sign between x_l and x_u

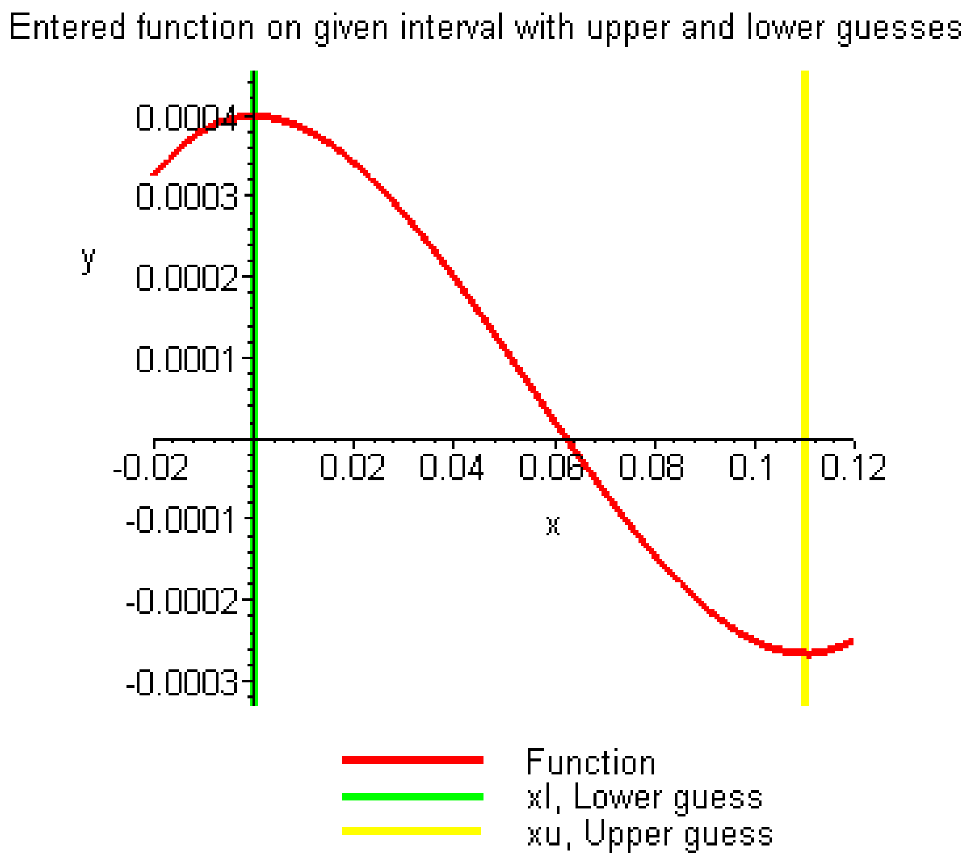
$$f(x_l) = f(0) = (0)^3 - 0.165(0)^2 + 3.993\times10^{-4} = 3.993\times10^{-4}$$

$$f(x_u) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times10^{-4} = -2.662\times10^{-4}$$

Hence $f(x_l)f(x_u) = f(0)f(0.11) = \left(3.993\times10^{-4}\right)\left(-2.662\times10^{-4}\right) < 0$

So there is at least on root between x_l and x_u, i.e., between 0 and 0.11

Example of the bisection method



Example of the bisection method

Iteration 1

The estimate of the root is $x_m = \frac{x_\ell + x_u}{2} = \frac{0+0.11}{2} = 0.055$

$$f(x_m) = f(0.055) = (0.055)^3 - 0.165(0.055)^2 + 3.993\times10^{-4} = 6.655\times10^{-5}$$

$$f(x_l)f(x_m) = f(0)f(0.055) = (3.993\times10^{-4})(6.655\times10^{-5}) > 0$$

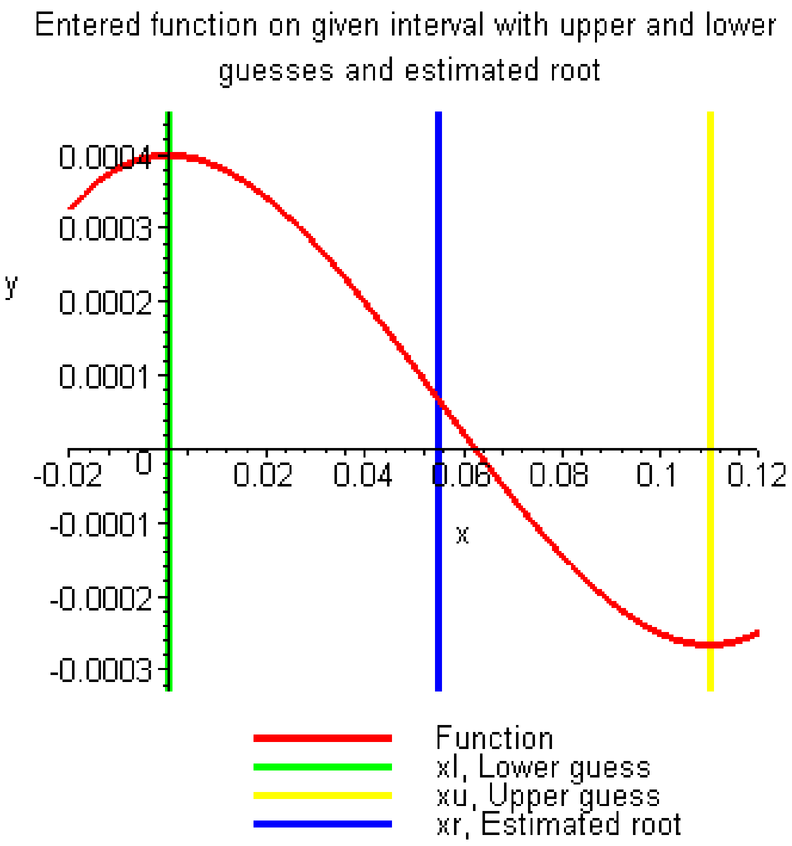
Hence the root is bracketed between x_m and x_u, that is, between 0.055 and 0.11. So, the lower and upper limits of the new bracket are :

$$x_l = 0.055,\ x_u = 0.11$$

At this point, the absolute relative approximate error cannot be calculated as we do not have a previous approximation

$$|\in_a|$$

Example of the bisection method



After first iteration

Example of the bisection method

Iteration 2

The estimated o root is $$x_m = \frac{x_\ell + x_u}{2} = \frac{0.055 + 0.11}{2} = 0.0825$$

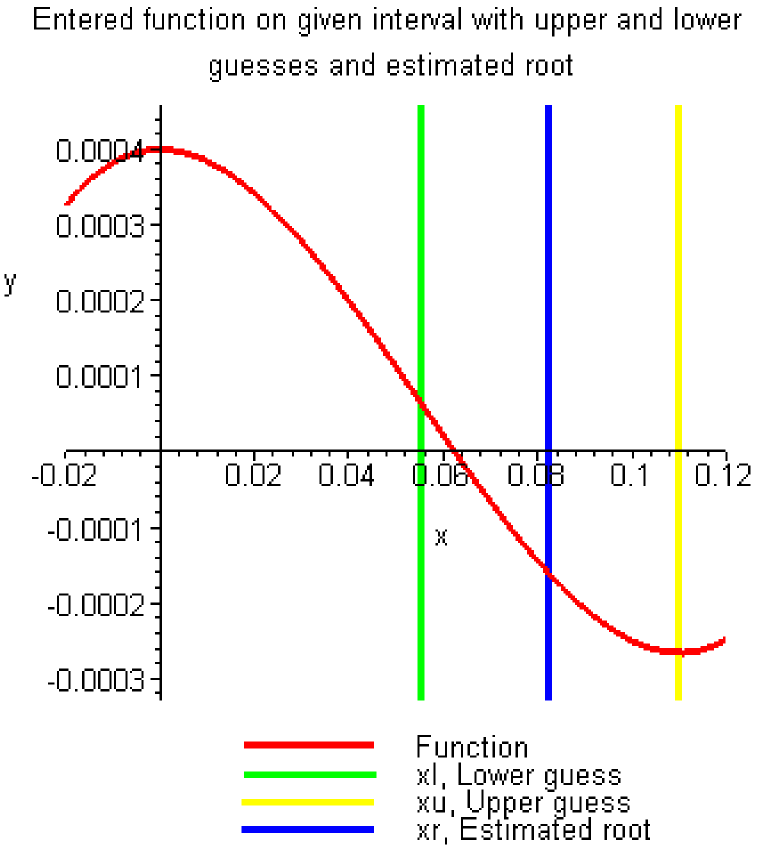
$$f(x_m) = f(0.0825) = (0.0825)^3 - 0.165(0.0825)^2 + 3.993 \times 10^{-4} = -1.622 \times 10^{-4}$$

$$f(x_l)f(x_m) = f(0)f(0.055) = (6.655 \times 10^{-5})(-1.622 \times 10^{-4}) < 0$$

Hence, the root is bracketed between x_l and x_m, i.e, between 0.055 and 0.0825. Thus, the lower and upper limits of the new bracket are:

$$x_l = 0.055, \; x_u = 0.0825$$

Example of the bisection method



After second iteration

Example of the bisection method

The absolute relative approximate error at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_m^i - x_m^{i-1}}{x_m^i}\right| \times 100$$

$$= \left|\frac{0.0825 - 0.055}{0.0825}\right| \times 100$$

$$= 33.333\%$$

None of the significant digits are correct in the estimated root of x_m= 0.0825 because the absolute relative approximate error is greater than 5%.

$$|\varepsilon_a| \leq 0.5 \times 10^{2-m}$$

Example of the bisection method

Iteration 3

The estimated root is $x_m = \frac{x_\ell + x_u}{2} = \frac{0.055 + 0.0825}{2} = 0.06875$

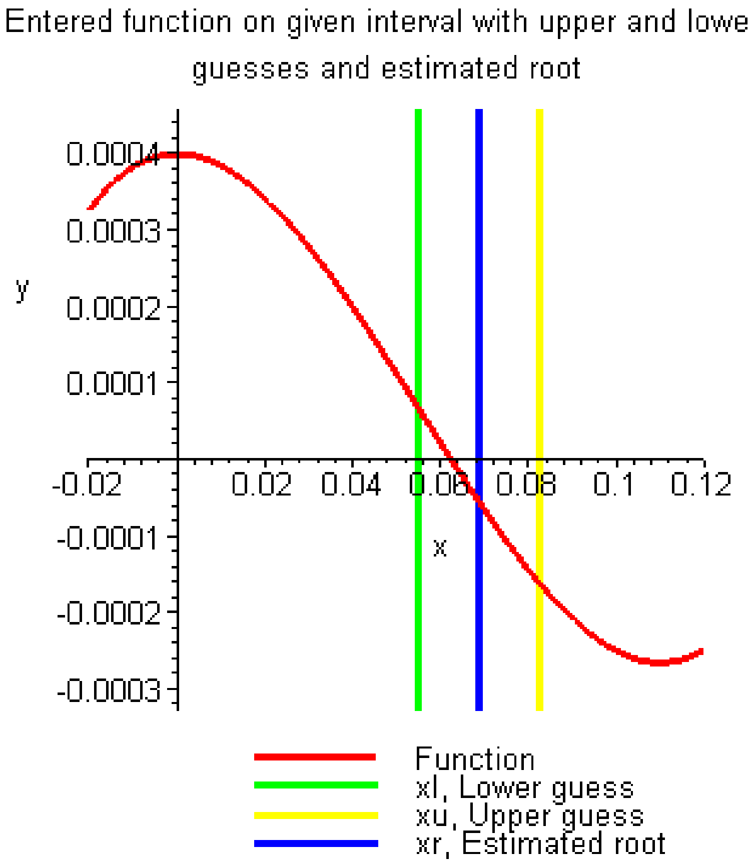
$$f(x_m) = f(0.06875) = (0.06875)^3 - 0.165(0.06875)^2 + 3.993\times10^{-4} = -5.563\times10^{-5}$$

$$f(x_l)f(x_m) = f(0.055)f(0.06875) = (6.655\times10^{-5})(-5.563\times10^{-5}) < 0$$

Hence, the root is bracketed between x_l and x_m, i.e, between 0.055 and 0.06875. Thus, the lower and upper limits of the new bracket are:

$$x_l = 0.055,\ x_u = 0.06875$$

Example of the bisection method



After third iteration

Example of the bisection method

The absolute relative approximate error at the end of Iteration 3 is

$$
\begin{aligned}
\left|\in_a\right| &= \left|\frac{x_m^i - x_m^{i-1}}{x_m^i}\right| \times 100 \\
&= \left|\frac{0.06875 - 0.0825}{0.06875}\right| \times 100 \\
&= 20\%
\end{aligned}
$$

Still none of the significant digits are correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%.

Example of the bisection method

Root of f(x)=0 as a function of number of iterations for bisection method.

Iteration	x_ℓ	x_u	x_m	$\lvert\in_a\rvert\%$	$f(x_m)$
1	0.00000	0.11	0.055	----------	6.655×10^{-5}
2	0.055	0.11	0.0825	33.33	-1.622×10^{-4}
3	0.055	0.0825	0.06875	20.00	-5.563×10^{-5}
4	0.055	0.06875	0.06188	11.11	4.484×10^{-6}
5	0.06188	0.06875	0.06531	5.263	-2.593×10^{-5}
6	0.06188	0.06531	0.06359	2.702	-1.0804×10^{-5}
7	0.06188	0.06359	0.06273	1.370	-3.176×10^{-6}
8	0.06188	0.06273	0.0623	0.6897	6.497×10^{-7}
9	0.0623	0.06273	0.06252	0.3436	-1.265×10^{-6}
10	0.0623	0.06252	0.06241	0.1721	-3.0768×10^{-7}

Example of the bisection method

Hence the number of significant digits is given by the largest value of m for which

$$\left|\in_a\right| \leq 0.5 \times 10^{2-m}$$

$$0.1721 \leq 0.5 \times 10^{2-m}$$

$$0.3442 \leq 10^{2-m}$$

$$\log(0.3442) \leq 2-m$$

$$m \leq 2-\log(0.3442) = 2.463$$

thus $m = 2$

The number of significant digits in the estimated root of 0.06241 at the end of the 10th iteration is 2.

Advantages of bisection method

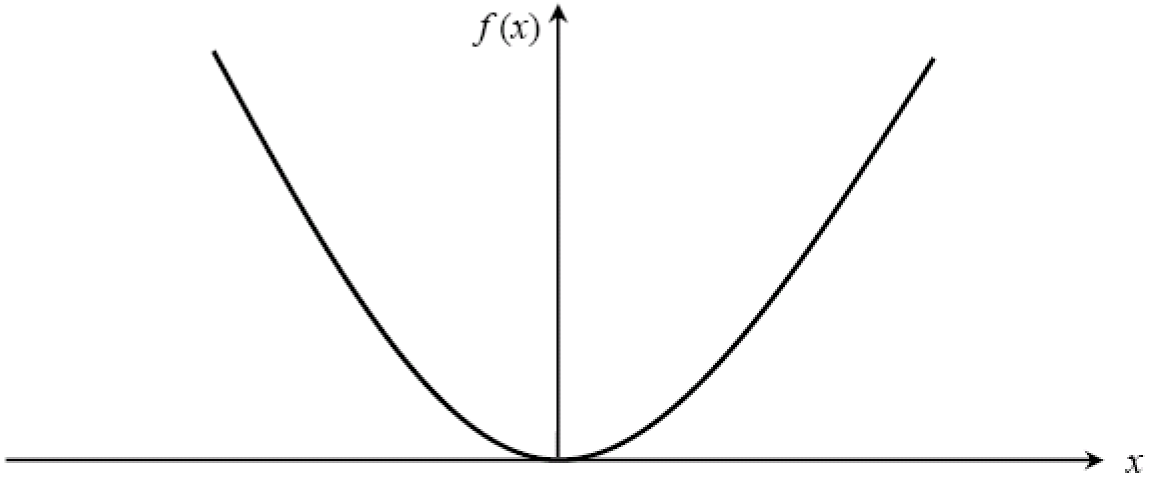
- Always convergent
- The root bracket gets halved with each iteration -guaranteed.

Drawbacks of bisection method

- Slow convergence
- If one of the initial guesses is close to the root, the iteraction slows down

Bisection drawbacks

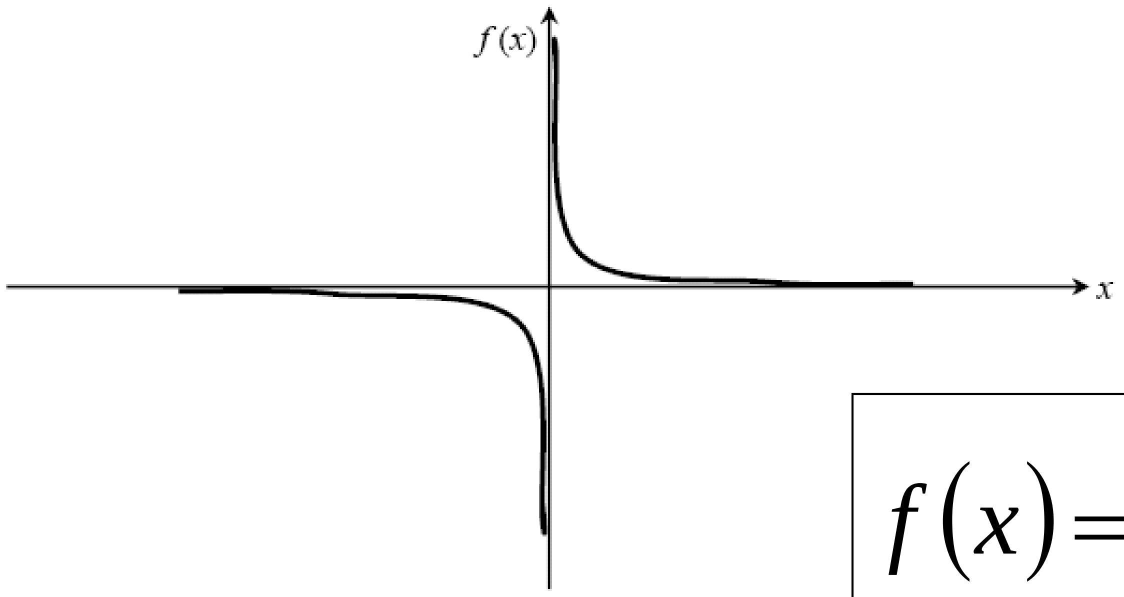
If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



$$\boxed{f(x) = x^2}$$

Bisection drawbacks

The function changes sign but root does not exist



$$f(x) = \frac{1}{x}$$

False-Position Method (regula falsi)

regula – line; falsus- false

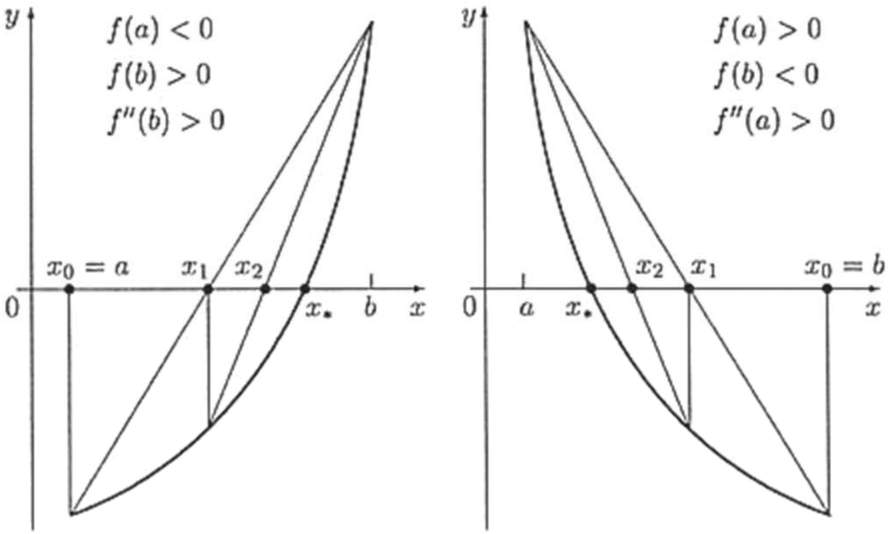
The method is called after the false assumption of the function linearity

Assumptions:

- in the <a,b> range f (x)=0 equation has exactly one root
- it is a single root
- $f(a)f(b)<0$
- f(x) in the range<a,b> is a function of C^2 class
- df/dx and d^2f/dx^2 have a constant sign in this range

needed to determine the error and a fixed point of iteration

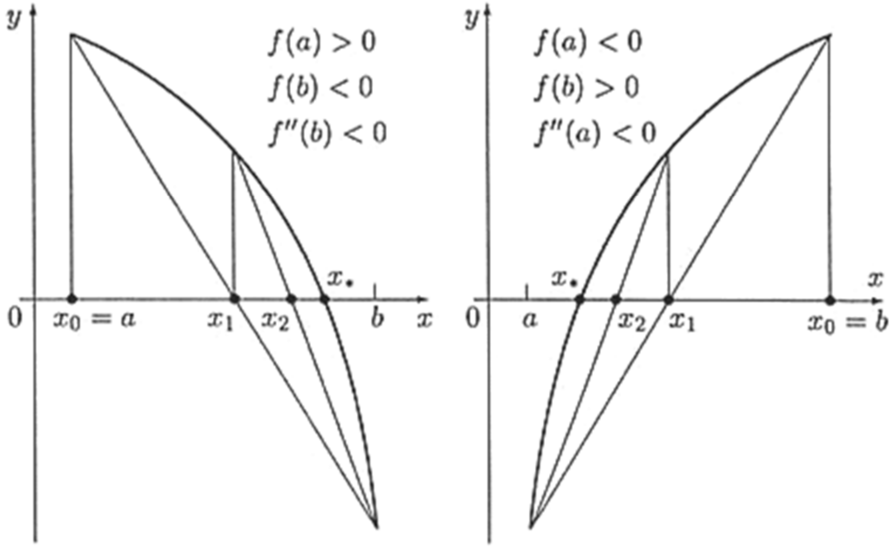
False-Position Method



With these assumptions, only the following cases are possible:

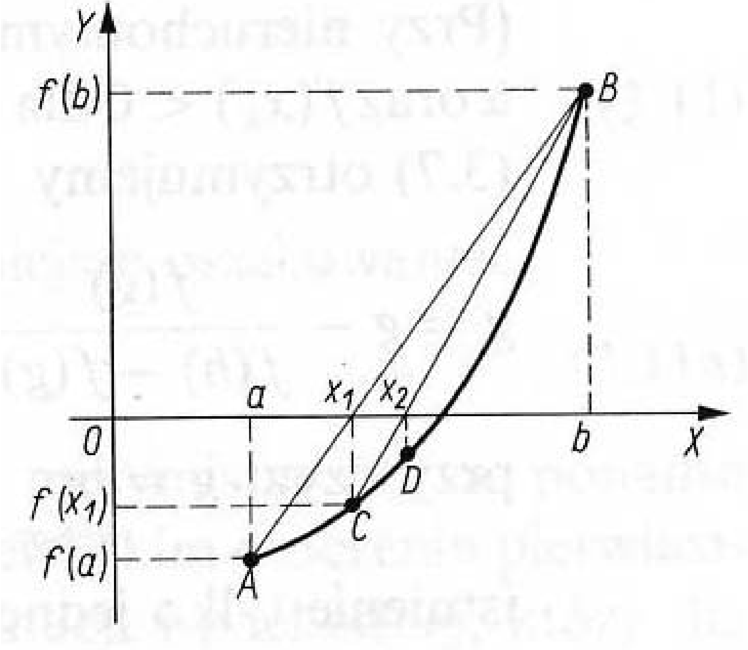
This method has a fixed point, it is the point at which the following condition is satisfied:

$$\boxed{f''f > 0}$$



False-Position Method

Consider the case:



Through the points A(a, f(a)) and B(b, f(b)) we draw a line (a secant) given by the equation:

$$\boxed{y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)}$$

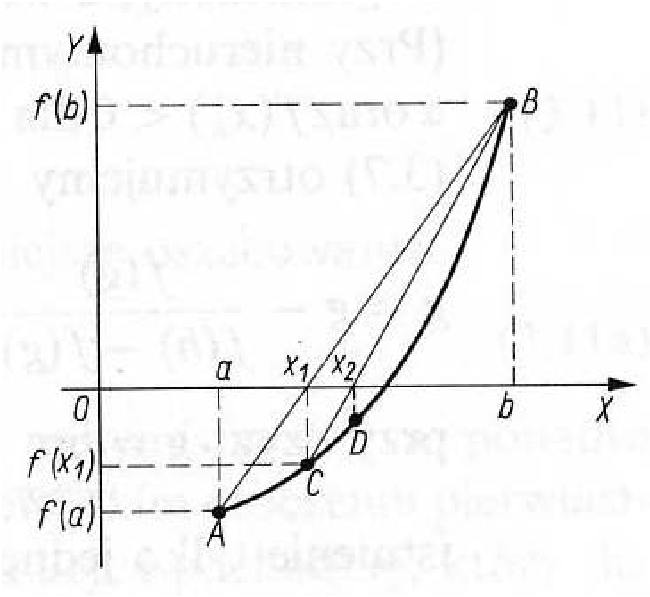
Point x_1 at which the line crosses the axis OX, there is a first approximation of the root.

$$x_1 = a - \frac{f(a)}{f(b) - f(a)}(b - a)$$

False-Position Method

If $f(x_1)=0$, then x_1 is the root we were looking for.

If the approximated value obtained this way is not accurate enough, then through the point $C = (x_1, f(x_1))$ and this one (A or B) for which we get the opposite sign, we conduct next secant. x_2 is the next point where the lines intersects the axis OX and is the next approximation. The iterative process terminates when we get a solution with the required accuracy. A sequence of : $x_1, x_2, \ldots X_n$ is created



$$x_0 = 0 \qquad x_{k+1} = x_k - \frac{f(x_k)}{f(b) - f(x_k)}(b - x_k) \quad k = 1,2,\ldots,n$$

False-Position Method

It can be shown that the adopted sequence x_1, x_2, ... x_n is increasing and bounded therefore convergent. It converges to the root α thus f(α) = 0

Error of the n-th approximation can be estimated based on:

$$f(x_n) - f(\alpha) = f'(c)(x_n - \alpha)$$

where c is in the range from α to x

$$|x_n - \alpha| \le \frac{|f(x_n)|}{m}$$

$$m = \inf_{x \in <a,b>} |f'(x)|$$

False-Position Method

Example: Find the positive root of the equation:

$$f(x) = x^3 + x^2 - 3x - 3$$

in the range of (1,2) and evaluate an error of the approximation.

We check the assumptions:

$$f'(x) = 3x^2 + 2x - 3 \qquad f''(x) = 6x + 2$$

$$f'(x) > 0 \;\; i \;\; f''(x) > 0 \;\; dla \;\; x > 1$$

$$f(1) = -4 \quad f(2) = 3$$

False-Position Method

The equation of the line passing through points A(1,-4) and B(2,3)

$$y + 4 = \frac{3+4}{2-1}(x-1)$$

y=0 for x_1=1,57142

We find f(x_1)=-1.36449. Because f(x_1)<0, we conduct a line through points B(2,3) and C(1,57142,-1,36449)

In the second step x_2=1,70540

False-Position Method

The assessment of the error of approximation in this example:

$$m = \inf_{x \in <a,b>} |f'(x)|$$

$$m = \inf_{x \in <1,2>} |3x^2 + 2x - 3| = 2$$

$f(x_2)=-0,24784$

$$|x_n - \alpha| < \frac{0,24784}{2} < 0,124$$

Because the sequence of x_n is increasing, so

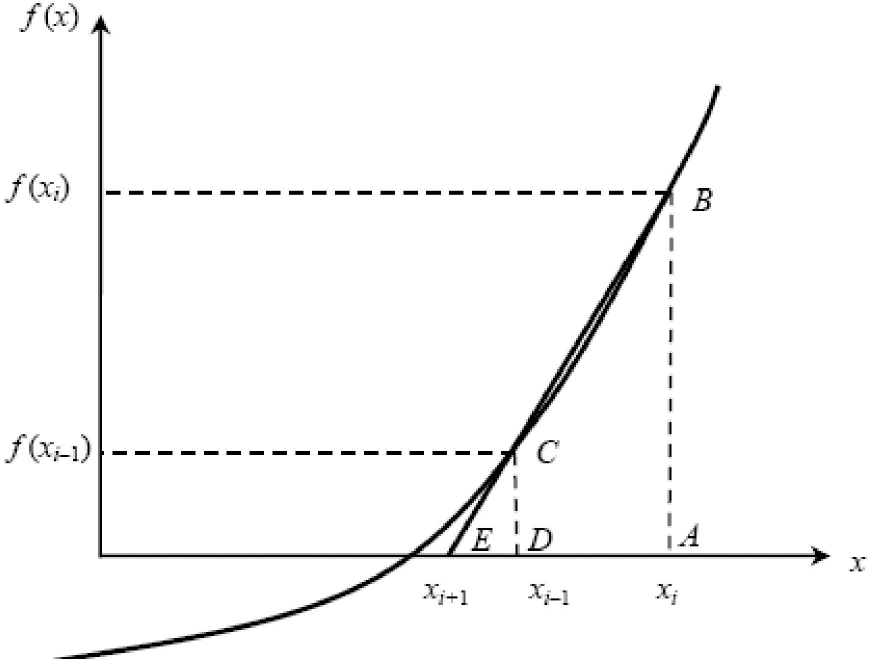
$$1,70540 < \alpha < 1,8294$$

False-Position Method and Secant Method

The drawback this method is its relatively **slow convergence**.

False-Position method can greatly improve its convergence, if we resign from the demand that the function f (x) has to have different signs at points delimiting the next line (except for the first iteration).

This is the Secant Method



Secant Method

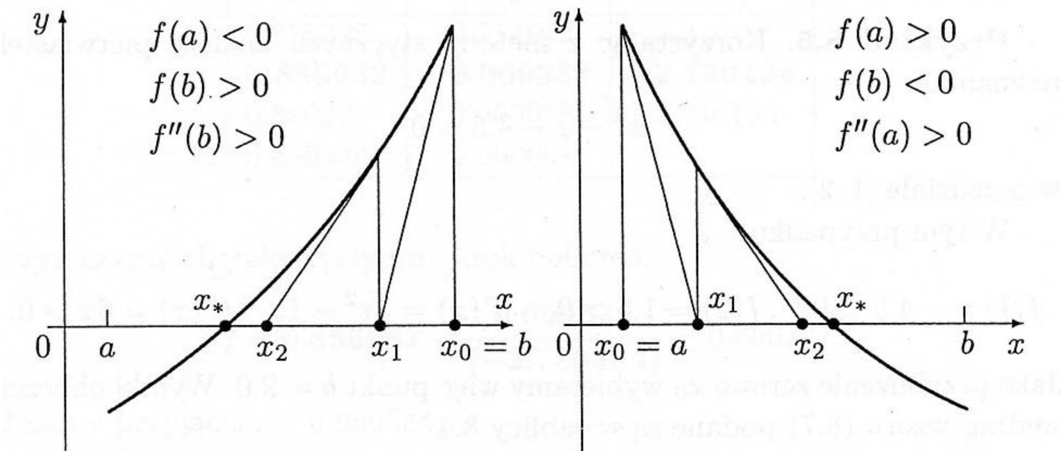
In order to calculate the approximation x_{i+1} we use two previously found points: x_i and x_{i-1}. Formula which determines the sequence of approximations is as follows:

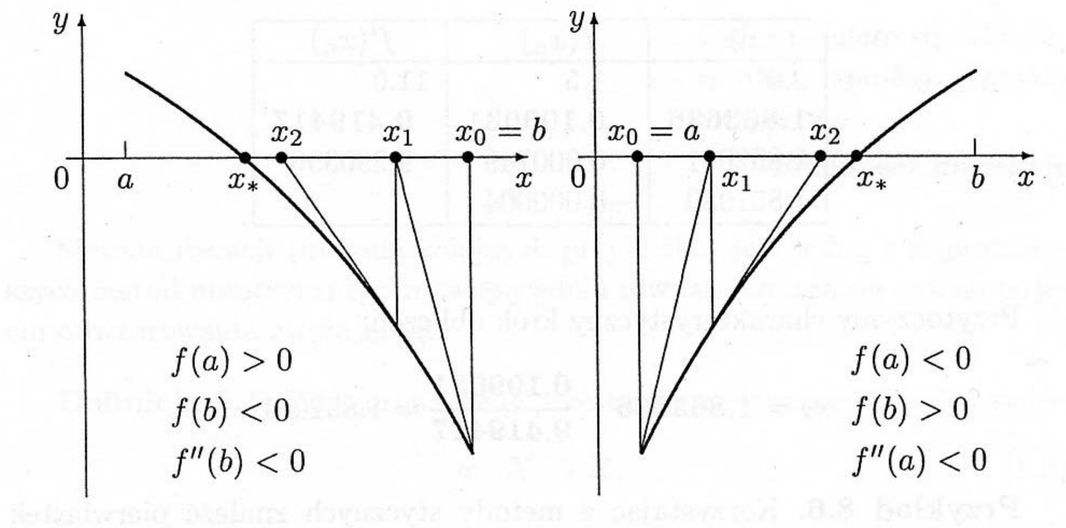
$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

A disadvantage of the secant method is that it may not be convergent to the root (for example, when the initial guess is not near the root). In addition, a sequence of approximations should be decreasing (if the distance between successive approximations is of the same order of magnitude as the estimated error, the next approximation may be completely wrong).

Secant Method
– Newton-Raphson Method

We assume that f (x) has different signs at the ends of the interval <a,b> and f’(x) and f”(x) do not change a sign.

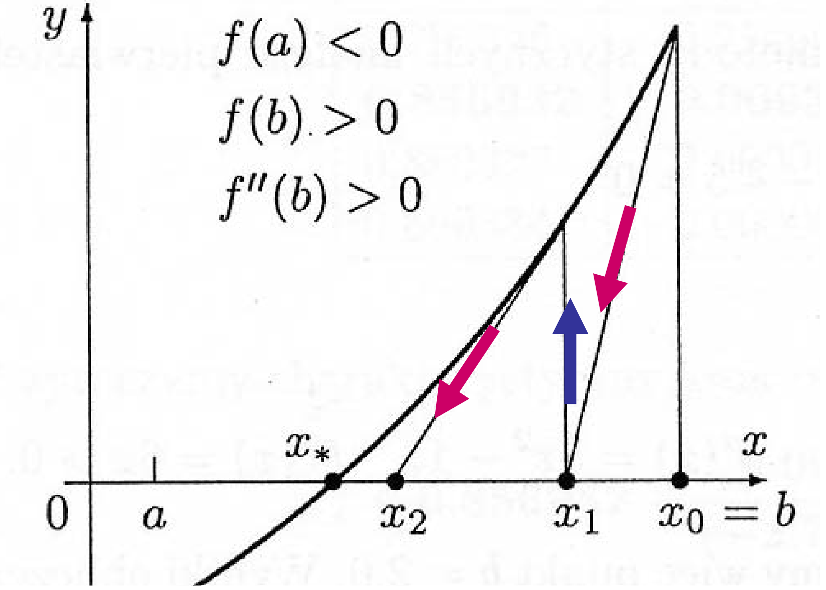




As a first root we assume the end of the range in which the **function f and their second derivative have the same sign**, ie. if $f(x_0) \cdot f''(x_0) \geq 0$, where $x_0 = a$ or $x_0 = b$.

Secant Method – Newton-Raphson Method

From the selected end we conduct the tangent to the graph of y=f(x). Point x_1, which is the point of intersection of the tangent with the axis OX is another root approximation. If the approximation obtained this way is not accurate enough, from the point $(x_1, f(x_1))$ we conduct the next tangent. The point x_2 where the tangent intersects the axis OX is the next approximation. The iterative process terminates when we get a solution with the required accuracy.



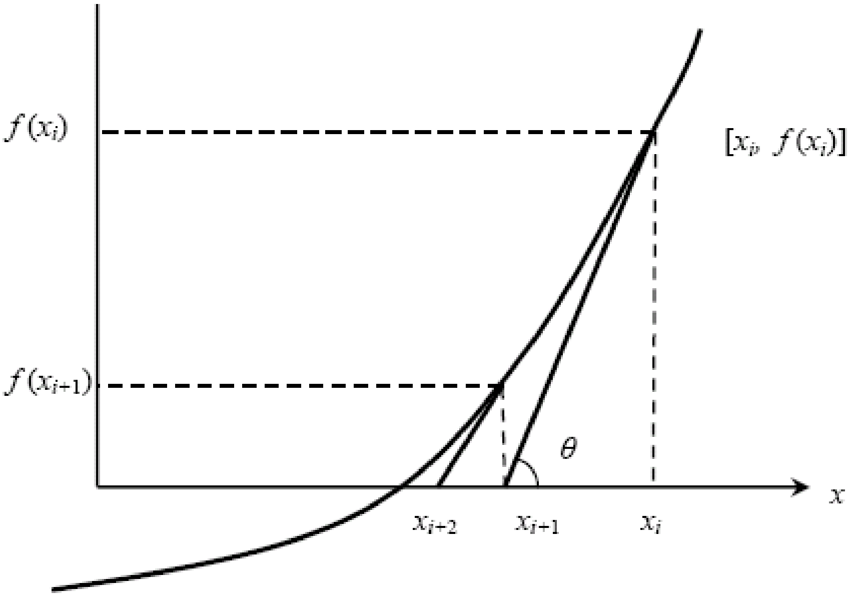
$$y - f(b) = f'(b)(x - b)$$

$$x_1 = b - \frac{f(b)}{f'(b)}$$

Secant Method
– Newton-Raphson Method

Formula for the next approximation:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



It is a convergent diminishing sequence of approximations ($x_{n+1} < x_n$) or increasing ($x_{n+1} > x_n$) and limited from below or from the top.

Error of the n-th approximation can be estimated similarly as in the False-Position Method:

$$|x_{n+1} - \alpha| \approx \left| \frac{f(x_n)}{f'(x_n)} \right|$$

Secant Method
– Newton-Raphson Method

A well-known example of the application of the secant method is **the algorithm of finding the root of the quadratic equation.**

The square root of a positive number c is a positive root of the equation:

$$x^2 - c = 0$$

Calculations: $f(x) = x^2 - c$ $\quad f'(x) = 2x$

Using the secant method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - c}{2x_n}$$

We get quite useful formula:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$