Lecture 5.

Numerical integration

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Outline

- Trapezoidal rule
- Multi-segment trapezoidal rule
- Richardson extrapolation
- Romberg's method
- Simpson's rule
- Gaussian method

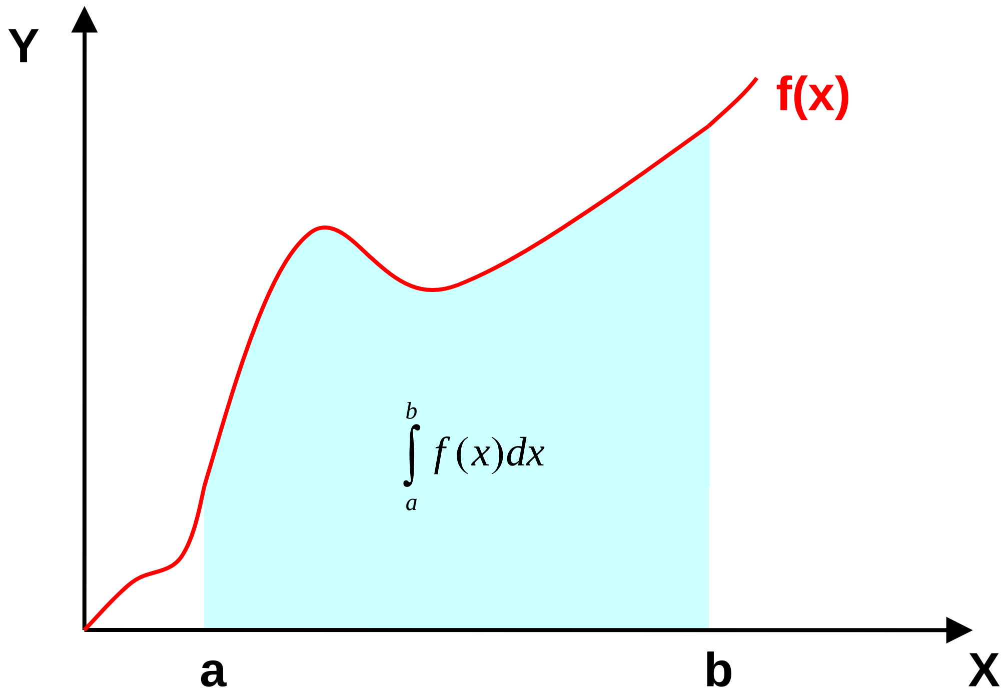
Numerical integration - idea

$$I = \int_a^b f(x)dx$$

The integral can be approximated by:

$$S = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$x_i \le c_i \le x_{i+1}$$



Newton–Cotes methods

Newton – Cotes integration belongs to a class of methods with fixed nodes: function f(x) is interpolated by a polynomial (e.g. Lagrange polynomial)

$$f(x) \approx f_n(x)$$

where:

$$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n$$

Then, the integral of f(x) can be approximated as an integral of the interpolated function $f_n(x)$

$$I = \int_a^b f(x)\,dx \approx \int_a^b f_n(x)\,dx$$

Trapezoidal rule

The trapezoidal rule assumes: n = 1, thus:

$$f_1(x) = a_0 + a_1 x$$

$$I = \int_a^b f(x)\,dx \approx \int_a^b f_1(x)\,dx = \int_a^b (a_0 + a_1 x)dx$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2}$$

But what is a_0 and a_1 ?

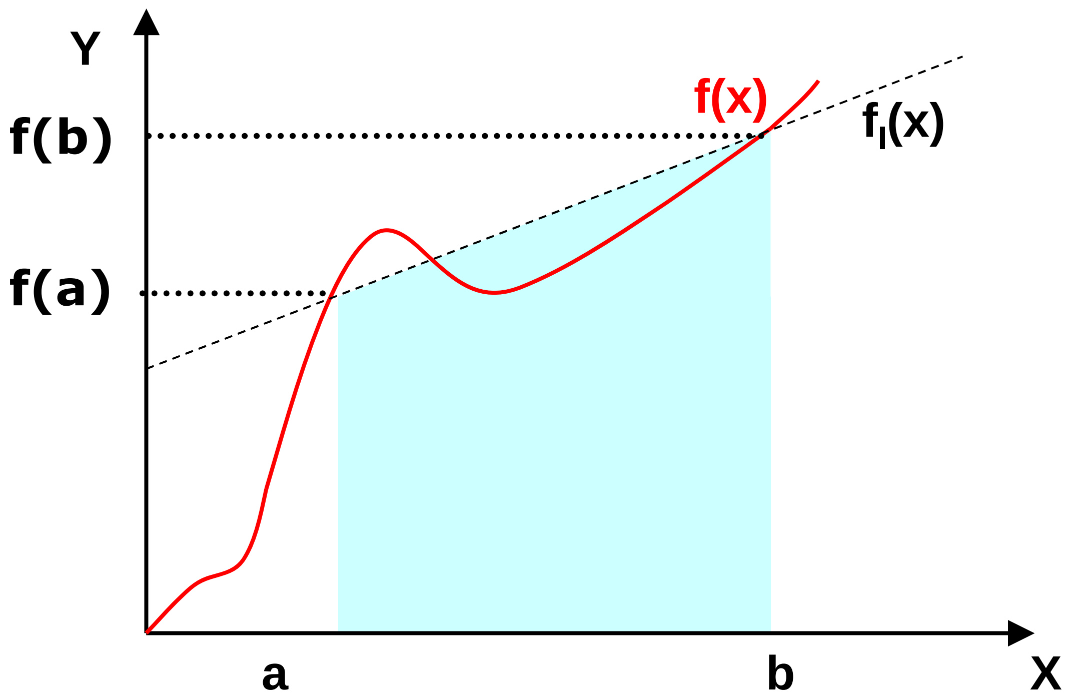
Now if one chooses, (a,f(a)) i (b,f(b)) as the two points to approximate f(x) by a straight line from a to b. It follows that:

$$\begin{cases} f(a) = f_1(a) = a_0 + a_1 a & \Longrightarrow \quad a_0 = \dfrac{f(a)b - f(b)a}{b-a} \\ f(b) = f_1(b) = a_0 + a_1 b & \Longrightarrow \quad a_1 = \dfrac{f(b) - f(a)}{b-a} \end{cases}$$

Trapezoidal rule

$$\int_a^b f(x)dx = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

$$\int_a^b f(x)dx \approx area\, of\, trapezoid$$



Trapezoidal rule

Example 1:

Velocity v(t) of a rocket from $t_1=8$ s to $t_2=30$ s is given by:

$$v(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

a) Use the single segment trapezoidal rule to find the distance covered by the rocket from $t_1=8$ s to $t_2=30$ s
b) Find the true relative error.

Trapezoidal rule

a)

$$I \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right] \qquad a = 8\ s \qquad b = 30\ s$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

$$f(8) = 2000\ln\left[\frac{140000}{140000-2100(8)}\right] - 9.8(8) \quad = 177.27\ m/s$$

$$f(30) = 2000\ln\left[\frac{140000}{140000-2100(30)}\right] - 9.8(30) \quad = 901.67\ m/s$$

$$I = (30-8)\left[\frac{177.27+901.67}{2}\right] \quad = 11868\ m$$

Trapezoidal rule

b) The true value

$$\Delta x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt = 11061\, m$$

The relative error:

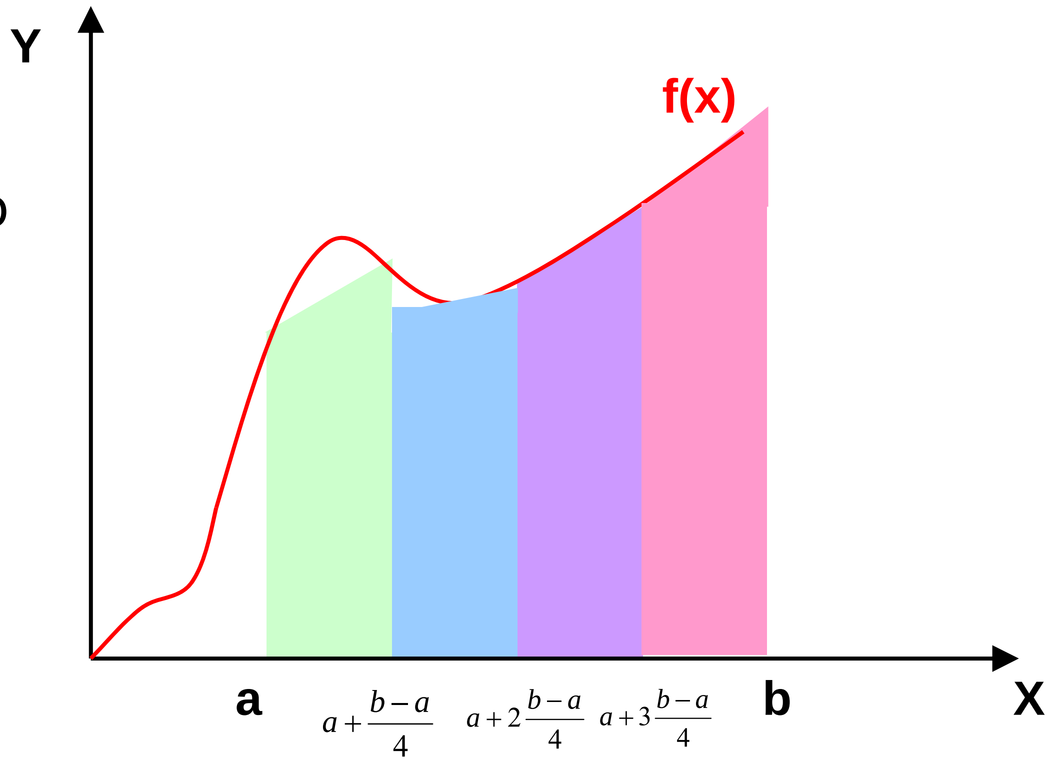
$$\left| \in_t \right| = \left| \frac{11061 \quad - 11868}{11061} \right| \times 100 \quad = 7.2959\ \%$$

Multi-segment trapezoidal rule

The true error using a single segment trapezoidal rule was large. We can divide the interval from a to b into smaller n segments of equal length – h and apply the trapezoidal rule over each segment:

$$h = \frac{b-a}{n}$$

for n=4



$$I = \int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \int_{a+2h}^{a+3h} f(x)dx + \int_{a+3h}^{a+4h} f(x)dx$$

Multi-segment trapezoidal rule

$$\int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^{b} f(x)dx$$

$$= h\left[\frac{f(a)+f(a+h)}{2}\right] + h\left[\frac{f(a+h)+f(a+2h)}{2}\right] + \ldots$$

$$\ldots + [b-(a+(n-1)h]\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

Multi-segment trapezoidal rule

Example 2:

Velocity v(t) of a rocket from t_1=8 s to t_2=30 s is given by:

$$v(t) = 2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

a) Use the complex segment trapezoidal rule to find the distance covered from t_1=8 s to t_2=30 s for n = 2
b) Find the true relative error.

Multi-segment trapezoidal rule

a) $$I=\frac{b-a}{2n}\left[f(a)+2\left\{\sum_{i=1}^{n-1}f(a+ih)\right\}+f(b)\right]$$

n = 2 a = 8 s b = 30 s $$h=\frac{b-a}{n}=\frac{30-8}{2}=11s$$

$$I=\frac{30-8}{2(2)}\left[f(8)+2\left\{\sum_{i=1}^{2-1}f(a+ih)\right\}+f(30)\right]$$

$$=\frac{22}{4}\left[f(8)+2f(19)+f(30)\right]=\frac{22}{4}\left[177.27+2(484.75)+901.67\right]$$

$$=11266\ m$$

Multi-segment trapezoidal rule

b) true value:

$$x=\int_{8}^{30}\left(2000ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt \quad =11061\,m$$

The relative error:

$$\left|\in_t\right|=\left|\frac{11061-11266}{11061}\right|\times 100 \quad =1.8534\%$$

Multi-segment trapezoidal rule

n	Δx	E_t	$\left|\epsilon_t\right|\%$	$\left|\epsilon_a\right|\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

Estimation of error

The relative error for a simple trapezoidal rule is

$$E_t = \frac{(b-a)^3}{12} f''(\zeta), \quad a < \zeta < b$$

The relative error in the multi-segment trapezoidal rule is a sum of errors for each segment. The relative error within the first segment is given by:

$$E_1 = \frac{[(a+h)-a]^3}{12} f''(\zeta_1), \quad a < \zeta_1 < a+h$$

$$= \frac{h^3}{12} f''(\zeta_1)$$

Estimation of error

By analogy:

$$E_i = \frac{\left[(a+ih)-(a+(i-1)h)\right]^3}{12} f''(\zeta_i), \quad a+(i-1)h < \zeta_i < a+ih$$

$$= \frac{h^3}{12} f''(\zeta_i)$$

for n-th segment :

$$E_n = \frac{\left[b-\{a+(n-1)h\}\right]^3}{12} f''(\zeta_n), \quad a+(n-1)h < \zeta_n < b$$

$$= \frac{h^3}{12} f''(\zeta_n)$$

Estimation of error

The total error in the complex trapezoidal rule is a sum of the errors for a single segment:

$$E_t = \sum_{i=1}^{n} E_i = \frac{h^3}{12}\sum_{i=1}^{n} f''(\zeta_i) = \frac{(b-a)^3}{12n^2}\,\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

Formula: $$\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

yields an approximate average value of the second derivative in the range of

$$a < x < b$$

$$E_t \propto \alpha \frac{1}{n^2}$$

Estimation of error

Table below presents the results for the integral

$$\int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

as a function of the number of segments n. When n twice increases, the absolute error E_t decreases four times!

n	Value	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
2	11266	-205	1.854	5.343
4	11113	-51.5	0.4655	0.3594
8	11074	-12.9	0.1165	0.03560
16	11065	-3.22	0.02913	0.00401

Richardson’s extrapolation and Romberg’s method of integration

Richardson’s extrapolation and Romberg’s method of integration constitute an extension of the trapezoidal method and give better approximation of the integral by reducing the true error.

Richardson extrapolation

The true error obtained when using the multi-segment trapezoidal rule with n segments to approximate an integral is given by:

$$E_t \cong \frac{C}{n^2}$$

where: C is an approximate constant of proportionality

Since:

$$E_t = TV - I_n$$

true value

approximate value using n-segments

Richardson extrapolation

It can be shown that:

$$\frac{C}{(2n)^2} \cong TV - I_{2n}$$

If the number of segments is doubled from n to 2n:

$$\begin{cases} \dfrac{C}{(n)^2} \cong TV - I_n \\ \dfrac{C}{(2n)^2} \cong TV - I_{2n} \end{cases}$$

We get:

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

Richardson extrapolation

Example 3:

The velocity v(t) of a rocket from t_1=8 s to t_2=30 is given by:

$$v(t) = 2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

a) Use Richardson extrapolation rule to find the distance covered for n = 2
b) Find the relative true error

Table of results for n = 8 segments (trapezoidal rule)

n	Δx	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

Richardson extrapolation

a)

$$I_2 = 11266m \qquad I_4 = 11113m$$

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$ for *n=2*

$$TV \cong I_4 + \frac{I_4 - I_2}{3} = 11113 + \frac{11113 - 11266}{3}$$

$$= 11062m$$

b)

The true value:

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt \;= 11061\ m$$

The absolute true error:

$$E_t = 11061 - 11062 \;= -1\ m$$

Richardson extrapolation

c) The relative error:

$$|\in_t| = \left|\frac{11061 - 11062}{11061}\right| \times 100 = 0.00904\%$$

Comparison of different methods:

n	**Δx (m) Trapezoidal rule**	$\|\in_t\|\%$ **Trapezoidal rule**	**Δx (m) Richardson extrapolation**	$\|\in_t\|\%$ **Richardson extrapolation**
1	11868	7.296	--	--
2	11266	1.854	11065	0.03616
4	11113	0.4655	11062	0.009041
8	11074	0.1165	11061	0.0000

Romberg's method

Romberg’s method uses the same pattern as Richardson extrapolation. However, Romberg used a recursive algorithm for the extrapolation as follows:

$$TV \cong I_{2n} + \frac{I_{2n} - I_n}{3}$$

The true value TV is replaced by the result of the Richardson extrapolation

$$(I_{2n})_R$$

Note also that the sign $\cong$ is replaced by the sign $=$

$$(I_{2n})_R = I_{2n} + \frac{I_{2n} - I_n}{3} \quad = I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1}$$

Romberg's method

Esimated true value is given by: $TV \cong (I_{2n})_R + Ch^4$

where: Ch^4 is the value of the error of approximation

Another value of integral obtained while doubling the number of segments from 2n to 4n:

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

Estimated true value is given by:

$$TV \cong (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{15}$$

$$= (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1}$$

Romberg's method

A general expression for Romberg integration can be written as:

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1}, k \geq 2$$

The index k represents the order of extrapolation

k=1 represents the values obtained from the regular trapezoidal rule

k=2 represents the values obtained using the true error estimate as $O(h^2)$

The value of an integral with for *j + 1* is more accurate than the value of the integral for *j* index

Example 4:

Velocity v(t) of a rocket from t_1=8 s to t_2=30 s is given by:

$$v(t) = 2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

a) Use Romberg’s method to find the distance covered. Use n = 1, 2, 4, and 8
b) Find the absolute true error and the relative approximate error

Table of results for n = 8 segments (trapezoidal rule)

n	Δx	E_t	$\|\in_t\|\%$	$\|\in_a\|\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
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8	11074	-12.9	0.1165	0.03560

Romberg's method

From this table, the initial values from the trapezoidal rule are:

$$I_{1,1} = 11868 \qquad I_{1,2} = 11266$$

$$I_{1,3} = 11113 \qquad I_{1,4} = 11074$$

To get the first order extrapolation values:

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$

$$= 11266 + \frac{11266 - 11868}{3}$$

Romberg's method

Similarly,

$$I_{2,2} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3}$$

$$= 11113 + \frac{11113 - 11266}{3}$$

$$= 11062$$

$$I_{2,3} = I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3}$$

$$= 11074 + \frac{11074 - 11113}{3}$$

$$= 11061$$

Romberg's method

For the second order extrapolation:

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

$$= 11062 + \frac{11062 - 11065}{15}$$

$$= 11062$$

Similarly,

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$

$$= 11061 + \frac{11061 - 11062}{15}$$

$$= 11061$$

Romberg's method

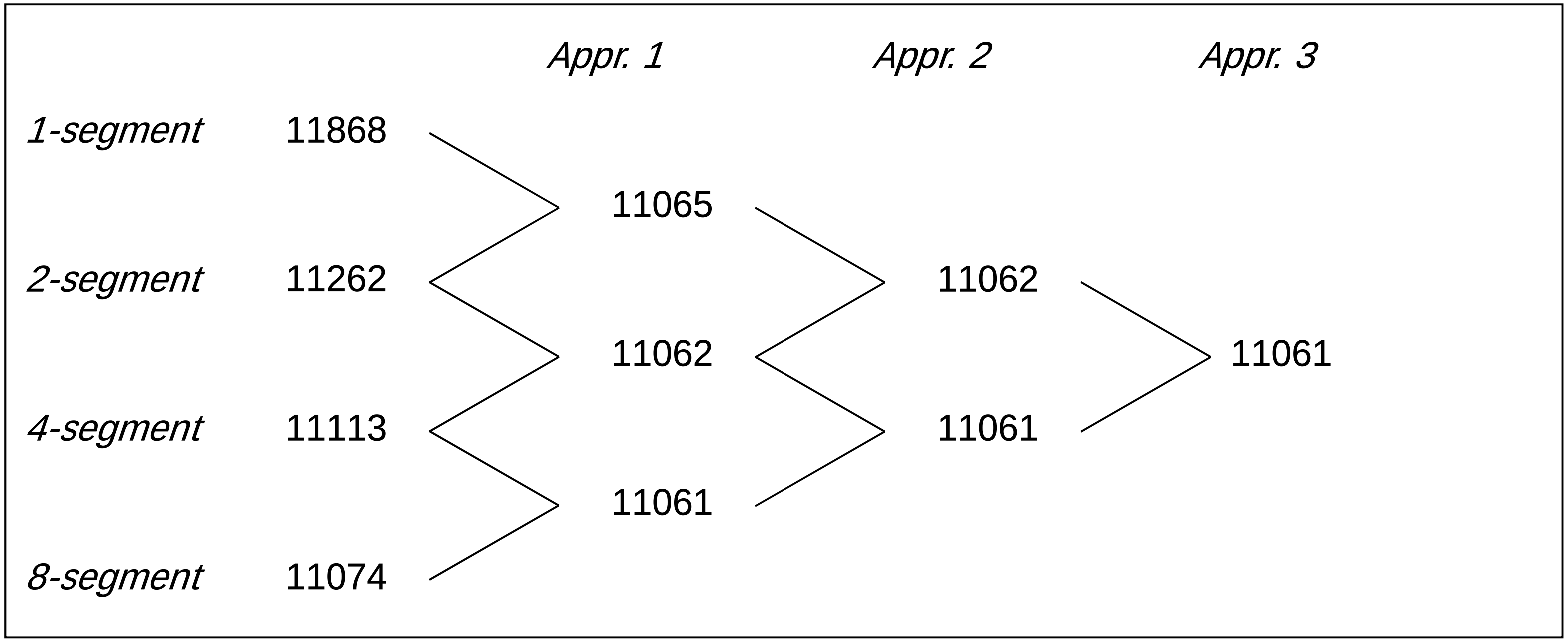
For the third order extrapolation

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$

$$= 11061 + \frac{11061 - 11062}{63}$$

$$= 11061 m$$

Romberg's method



Improved estimates of the value of an integral using Romberg integration

Simpson's rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over an interval from a to b. Simpson’s rule assumes that the integrand can be approximated by a second order polynomial.

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

where:

$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

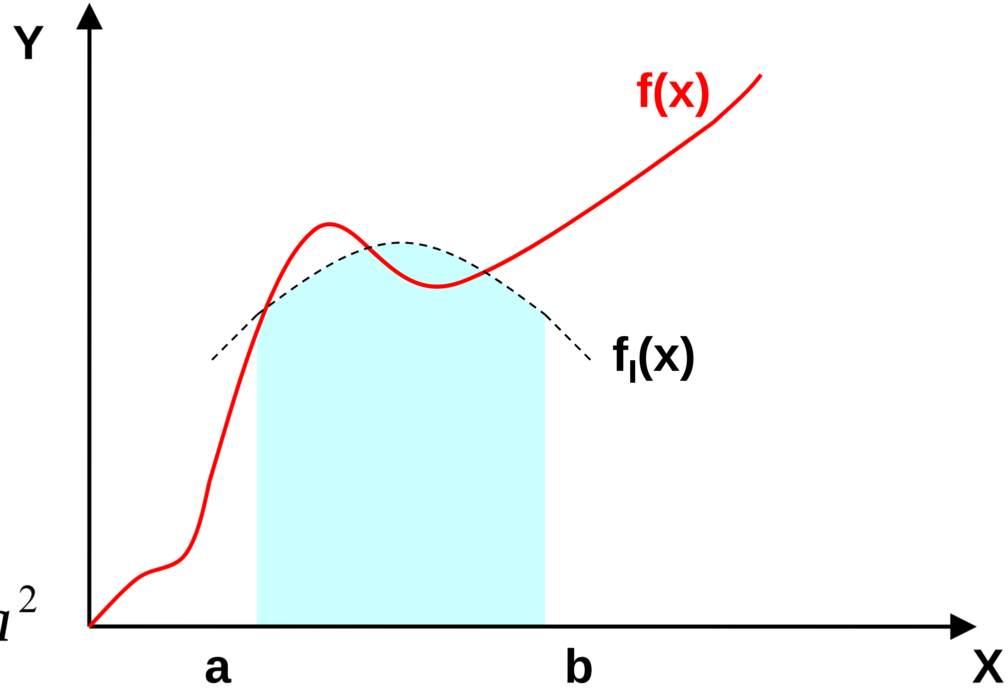
Simpson's rule

A parabola passing through three points :

$$(a, f(a)),$$

$$\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right),$$

$$(b, f(b))$$



$$\begin{cases} f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2 \\ f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2 \\ f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2 \end{cases}$$

Simpson's rule

Coefficients a_0, a_1, a_2 are:

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Simpson's rule

Since:

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Simpson's rule

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$h = \frac{b-a}{2}$$

It follows that:

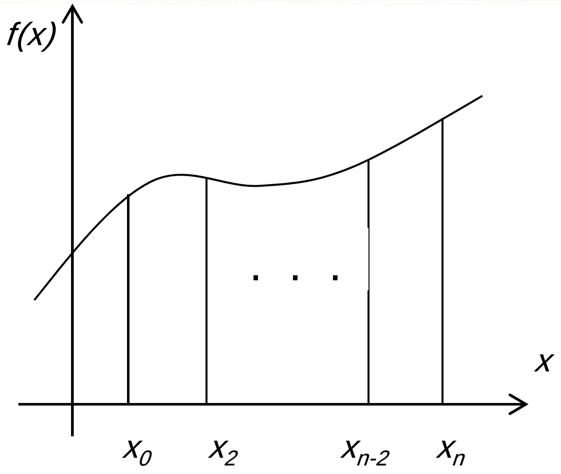
$$\int_a^b f_2(x)dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

it is called Simpson's 1/3 rule

Multi-segment Simpson's rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \dots..$$

$$\dots. + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$



$$\int_a^b f(x)dx = (x_2 - x_0)\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + \dots$$

$$+ (x_4 - x_2)\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + \dots$$

$$\dots + (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6}\right] + \dots$$

$$+ (x_n - x_{n-2})\left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right]$$

$$x_i - x_{i-2} = 2h$$

$$i = 2, 4, \dots, n$$

Simpson's rule

$$\int_a^b f(x)dx = 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Simpson's rule

$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0)+4\{f(x_1)+f(x_3)+\ldots+f(x_{n-1})\}+\ldots]$$

$$\ldots+2\{f(x_2)+f(x_4)+\ldots+f(x_{n-2})\}+f(x_n)\}]$$

$$=\frac{h}{3}\left[f(x_0)+4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i)+f(x_n)\right]$$

$$=\frac{b-a}{3n}\left[f(x_0)+4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i)+f(x_n)\right]$$

Estimation of errors in Simpson's rule

Approximate values of the integral,
using Simpson's rule with multiple segments

n	Approximate values	E_t	$\|\epsilon_t\|$
2	11065.72	4.38	0.0396%
4	11061.64	0.30	0.0027%
6	11061.40	0.06	0.0005%
8	11061.35	0.01	0.0001%
10	11061.34	0.00	0.0000%

Simpson's rule errors

Error for one segment
$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

Error for the multi-segment

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1) \quad = -\frac{h^5}{90} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2) \quad = -\frac{h^5}{90} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$

$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i) = -\frac{h^5}{90} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$

Simpson's rule errors

True error

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i = -\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) = -\frac{(b-a)^5}{90n^5}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^4}\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

$$E_t = -\frac{(b-a)^5}{90n^4}\overline{f}^{(4)}$$

the average value of the derivative

Gauss-Quadrature Method

Gaussian integral is given by:

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

constant coefficients

Points x_1 and x_2, which define the value of the integrand are not fixed (as before), but there are a priori distributed randomly within <a,b>.

Gauss-Quadrature Method

There are four unknowns x_1, x_2, c_1, c_2. These are found by assuming that the formula gives exact results for integrating a general third order polynomial:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

Gauss-Quadrature Method

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

Hence:

$$\int_a^b f(x)dx = c_1\left(a_0 + a_1 x_1 + a_2 x_1^{\,2} + a_3 x_1^{\,3}\right) + c_2\left(a_0 + a_1 x_2 + a_2 x_2^{\,2} + a_3 x_2^{\,3}\right)$$

$$= a_0(c_1 + c_2) + a_1(c_1 x_1 + c_2 x_2) + a_2\left(c_1 x_1^{\,2} + c_2 x_2^{\,2}\right) + a_3\left(c_1 x_1^{\,3} + c_2 x_2^{\,3}\right)$$

The formula would then give:

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

Gauss-Quadrature Method

$$a_0(c_1+c_2)+a_1(c_1x_1+c_2x_2)+a_2\left(c_1x_1^{\;2}+c_2x_2^{\;2}\right)+a_3\left(c_1x_1^{\;3}+c_2x_2^{\;3}\right)$$

$$=a_0(b-a)+a_1\left(\frac{b^2-a^2}{2}\right)+a_2\left(\frac{b^3-a^3}{3}\right)+a_3\left(\frac{b^4-a^4}{4}\right)$$

$$\Downarrow$$

$$b-a=c_1+c_2$$

$$\frac{b^2-a^2}{2}=c_1x_1+c_2x_2$$

$$\frac{b^3-a^3}{3}=c_1x_1^{\;2}+c_2x_2^{\;2}$$

$$\frac{b^4-a^4}{4}=c_1x_1^{\;3}+c_2x_2^{\;3}$$

Gauss-Quadrature Method

This gives us four equations as follows:

$$
\begin{cases}
b - a = c_1 + c_2 \\
\dfrac{b^2 - a^2}{2} = c_1 x_1 + c_2 x_2 \\
\dfrac{b^3 - a^3}{3} = c_1 {x_1}^2 + c_2 {x_2}^2 \\
\dfrac{b^4 - a^4}{4} = c_1 {x_1}^3 + c_2 {x_2}^3
\end{cases}
$$

we can find that the above four simultaneous nonlinear equations have only one acceptable solution

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2} \qquad x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$c_1 = \frac{b-a}{2} \qquad c_2 = \frac{b-a}{2}$$

Gauss-Quadrature Method

Hence:

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$= \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)$$

General n-point rules would approximate the integral:

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \dots\dots\dots + c_n f(x_n)$$

Gauss-Quadrature Method

The coefficients and arguments for n-point Gauss method are given within the range of <-1,1> :

$$\int_{-1}^{1} g(x)dx \cong \sum_{i=1}^{n} c_i g(x_i)$$

n	Coefficients	Function arguments
2	c_1 = 1.000000000 c_2 = 1.000000000	x_1 = -0.577350269 x_2 = 0.577350269
3	c_1 = 0.555555556 c_2 = 0.888888889 c_3 = 0.555555556	x_1 = -0.774596669 x_2 = 0.000000000 x_3 = 0.774596669
4	c_1 = 0.347854845 c_2 = 0.652145155 c_3 = 0.652145155 c_4 = 0.347854845	x_1 = -0.861136312 x_2 = -0.339981044 x_3 = 0.339981044 x_4 = 0.861136312

Gauss-Quadrature Method

n	Coefficients	Function arguments
5	c_1 = 0.236926885 c_2 = 0.478628670 c_3 = 0.568888889 c_4 = 0.478628670 c_5 = 0.236926885	x_1 = -0.906179846 x_2 = -0.538469310 x_3 = 0.000000000 x_4 = 0.538469310 x_5 = 0.906179846
6	c_1 = 0.171324492 c_2 = 0.360761573 c_3 = 0.467913935 c_4 = 0.467913935 c_5 = 0.360761573 c_6 = 0.171324492	x_1 = -0.932469514 x_2 = -0.661209386 x_3 = -0.2386191860 x_4 = 0.2386191860 x_5 = 0.661209386 x_6 = 0.932469514

Gauss-Quadrature Method

So if the table is given for: $\int_{-1}^{1} g(x)dx$ integrals, how does one solve

$$\int_{a}^{b} f(x)dx \quad ?$$

The answer lies in that any integral with limits of $[a, b]$

can be converted into an integral with limits: $[-1, 1]$

Let, $x = mt + c$

if $x = a, \quad t = -1$

ir $x = b, \quad t = 1$

It follows that:

$$\begin{cases} m = \dfrac{b-a}{2} \\ c = \dfrac{b+a}{2} \end{cases}$$

Gauss-Quadrature Method

Hence: $$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

$$dx = \frac{b-a}{2}dt$$

Substituting our values of x and dx into the integral gives us:

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\frac{b-a}{2}dt$$

Gauss-Quadrature Method

Example 5:

Use two-point Gauss quadrature rule to approximate the distance covered by a rocket from t = 8 s to t = 30 s if the velocity is given by:

$$v(t) = 2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

a) Use the Gauss-Quadrature Mmethod to find the distance covered from t_1=8 s to t_2=30 s
b) Find the true error.

Gauss-Quadrature Method

First, change the limits of integration from from [8,30] to [-1,1]

$$\int_{8}^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right)dx$$

$$= 11\int_{-1}^{1} f(11x+19)dx$$

The weighting factors and function argument values are (n=2):

$$c_1 = 1.000000000$$

$$x_1 = -0.577350269$$

$$c_2 = 1.000000000$$

$$x_2 = 0.577350269$$

Gauss-Quadrature Method

The formula is:

$$11\int_{-1}^{1} f(11x+19)dx \approx 11c_1 f(11x_1+19)+11c_2 f(11x_2+19)$$

$$=11f(11(-0.5773503)+19)+11f(11(0.5773503)+19)$$

$$=11f(12.64915)+11f(25.35085)$$

$$=11(296.8317)+11(708.4811)$$

$$=11058.44\ m$$

Since:

$$f(12.64915) = 2000\,ln\left[\frac{140000}{140000 - 2100(12.64915)}\right] - 9.8(12.64915)$$

$$= 296.8317$$

$$f(25.35085) = 2000\,ln\left[\frac{140000}{140000 - 2100(25.35085)}\right] - 9.8(25.35085)$$

$$= 708.4811$$

Gauss-Quadrature Method

b) The absolute true error:

$$E_t = 11061.34 - 11058.44$$

$$= 2.9000\ m$$

c) The relative error: $\left|\in_t\right|$

$$\left|\in_t\right| = \left|\frac{11061.34 - 11058.44}{11061.34}\right| \times 100\%$$

$$= 0.0262\%$$