Lecture 3.

Probability and elements of combinatorics

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Outline:

- Definitions of probability
- Random and elementary events; sample space
- Relation of events
- Introduction to combinatorics and counting problems
- Conditional probability

Definitions of probability

- Classical
- Geometric
- Frequency (von Mises)
- Axiomatic (Kołmogorow)

Classical definition of probability

First (classical) definition of probability was formulated by P.S. Laplace in 1812.

Consider random experiment that results always in exactly one of N **equally possible** results.

Probability of event A is given as a ratio of number n_a of outcomes favorable to A to the number of all possible outcomes N

$$P(A) = \frac{n_a}{N}$$

A is a subset of a sure event Ω. $A \subset \Omega$

Geometric definition of probability

Introduced in order to treat the cases of infinite number of outcomes.

Consider that in r-dimensional space where there exists a region G that contains a smaller region g. A random experiment consists in a random choice of a point in G assuming that all points are equally probable.

Probability of event A that randomly chosen point will be found in a region g is given as

$$P(A) = \frac{measure(g)}{measure(G)}$$

Bertrand paradox

In a given circle one draws at random a chord.

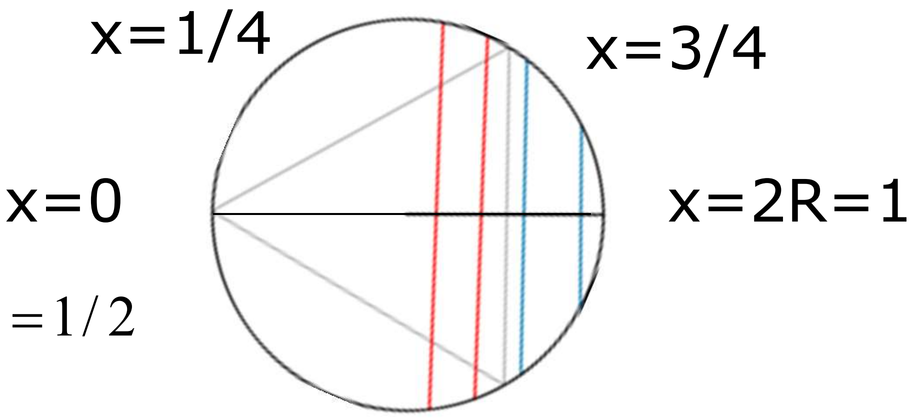
Calculate a probability that it will be longer than a side of equilateral triangle inscribed in a circle?

There exist three possible solutions and three possible answers: ½, 1/3, ¼.

The source of paradox lies in the lack of precision.
What does it mean random way in this case?

Bertrand paradox

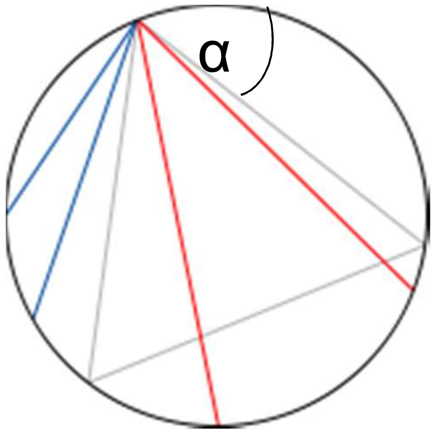
The "random radius" method: Choose a radius of the circle, choose a point on the radius and construct the chord through this point and perpendicular to the radius. To calculate the probability in question imagine the triangle rotated so a side is perpendicular to the radius. The chord is longer than a side of the triangle if the chosen point is nearer the center of the circle than the point where the side of the triangle intersects the radius. The side of the triangle bisects the radius, therefore the probability a random chord is longer than a side of the inscribed triangle is 1/2.



$$P(A)=\frac{measure\,(g)}{measure\,(G)}=\frac{3/4-1/4}{1}=2/4=1/2$$

Bertrand paradox

The "random endpoints" method: Choose two random points on the circumference of the circle and draw the chord joining them. To calculate the probability in question imagine the triangle rotated so its vertex coincides with one of the chord endpoints. Observe that if the other chord endpoint lies on the arc between the endpoints of the triangle side opposite the first point, the chord is longer than a side of the triangle. The length of the arc is one third of the circumference of the circle, therefore the probability that a random chord is longer than a side of the inscribed triangle is 1/3.

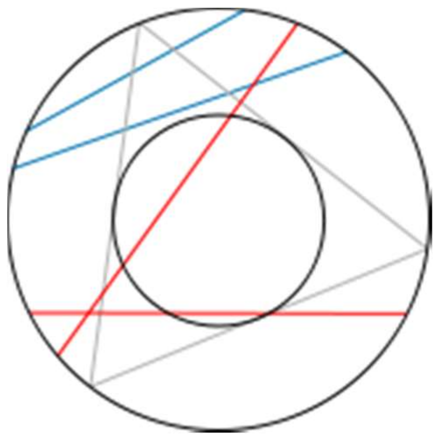


$$P(A) = \frac{measure\,(g)}{measure\,(G)} = \frac{\frac{2}{3}\pi - \frac{1}{3}\pi}{\pi} = 1/3$$

Bertrand paradox

The "random midpoint" method: Choose a point anywhere within the circle and construct a chord with the chosen point as its midpoint. The chord is longer than a side of the inscribed triangle if the chosen point falls within a concentric circle of 1/2 the radius of the larger circle. The area of the smaller circle is one fourth the area of the larger circle, therefore the probability a random chord is longer than a side of the inscribed triangle is 1/4.

$$P(A) = \frac{measure\,(g)}{measure\,(G)} = \frac{\pi\, r^2}{\pi\, R^2} = 1/4$$



Frequency definition of probability

Proposed by R. von Mises in 1931. Has no drawbacks of classical nor geometric definition. Is intuitive and agrees with the observed laws concerning frequency. However, it is unacceptable as a definition of mathematical quantity (a posteriori).

Probability of event A is a limit of frequency of this event when the number of experiments n tends to infinity

$$P(A) = \lim_{n \to \infty} \frac{n(A)}{n}$$

Axiomatic definition of probability

To each random event A we ascribe a number P(A), named a probability of this event that satisfies the following axioms:

1. $0 \leq P(A) \leq 1$.

2. Probability of a sure event equals to 1

$$P(\Omega) = 1$$

3. (countable additivity of probability) Probability of an alternative of countable disjoint (mutually exclusive) events is equal to the sum of probabilities of these events: if $A_1, A_2, \ldots \in M$, while for each pair of i, j ($i \neq j$) the following condition is fulfilled $A_i \cap A_j = \emptyset$, then

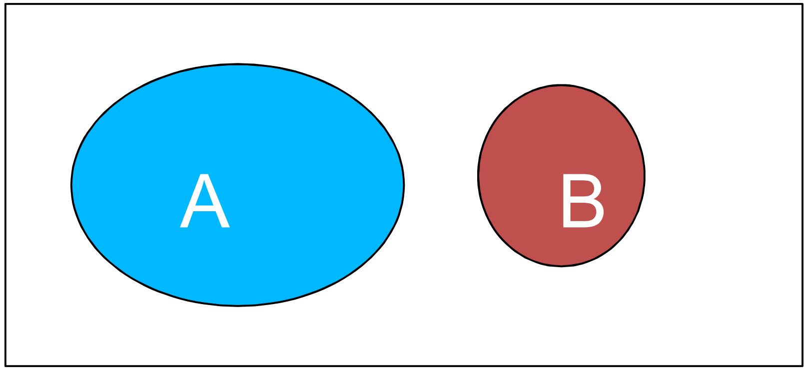
$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

Consequences of axioms

Probability of sum of the mutually exclusive random events A i B equals to the sum of probabilities of these events

(Kołmogorov, 1933)

$$P(A \cup B) = P(A) + P(B), \mathit{where} \quad A \cap B = \varnothing$$



Random or elementary events

For each random experiment we consider a set of its all possible outcomes, i.e., sample space **Ω** . These outcomes are called **random events**.

Among all random events we can distinguish some simple, irreducible ones that are characterized by a single outcome. These are **elementary events**.

Example:

All sets $\{k\}$, where $k \in \mathbf{N}$ if objects are being counted and the sample space is $S = \{0, 1, 2, 3, \ldots\}$ (the natural numbers).

Example of a random event

A coin is tossed twice. Possible outcomes are as follows:

- (T, T) – both tails
- (H, T) – head first, tail next
- (T, H) – tail first, head next
- (H, H) – both heads

Ω={(T, T); (H, T) ; (T, H); (H, H)} is a set of elementary events, i.e., the sample space

If the set of elementary events contains n-elements then the number of all random events is 2^n

Example of a random event

Here we have 2^4 random events.

For instance:

A = {(T,T); (T,H); (H,T)} – at least one tail T

B = {(T,H); (T,T)} – tail in the first two essays

G = {(T,T)} – both tails

H = {(T,H); (H,T)} – exactly one tail

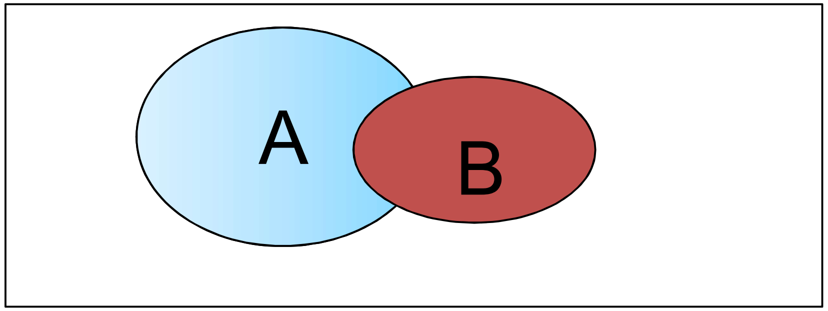
Example for individual study

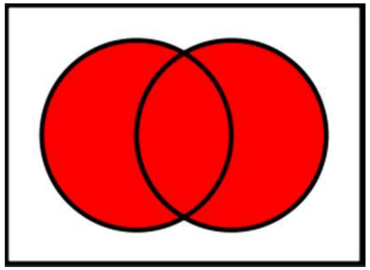
Count all random events (including sure and impossible ones) in the experiment that consists in throwing a dice. Determine the space of events.

Relations of events – Venn diagrams

Sum of events– when at least one of events A or B takes place (**union** of sets)

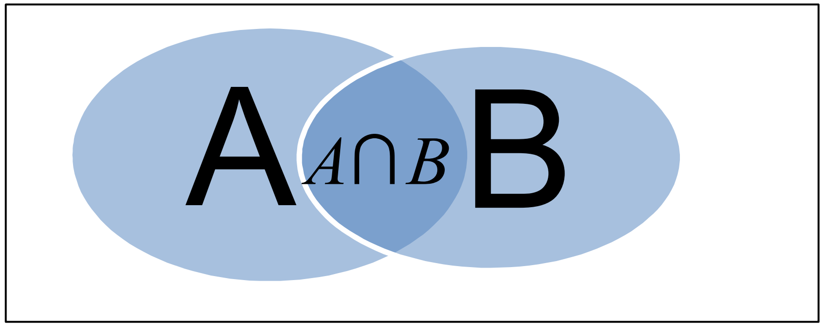
$$A \cup B$$

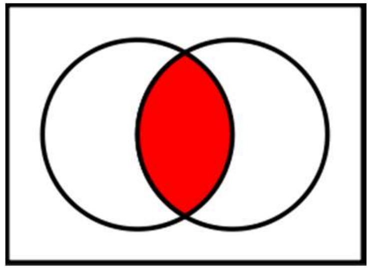




Product of events– both A and B happen (**intersection** of sets)

$$A \cap B$$

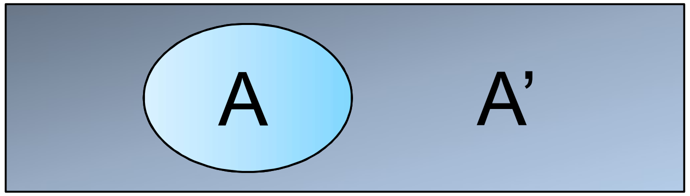




Relations of events

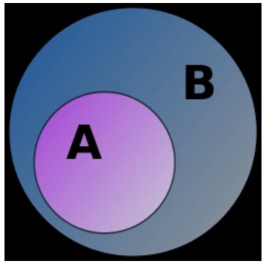
Complementary event– event A does not take place

$$A'$$



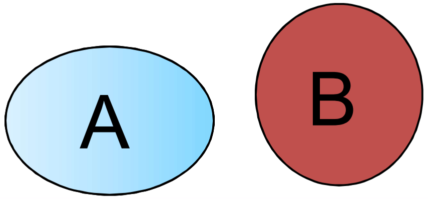
Event A **incites** B (subset A is totally included in B)

$$A \subset B$$



Events A and B are **mutually exclusive**

$$A \cap B = \varnothing$$



Introduction to combinatorics and counting problems

Combinatorics concerns itself with **finite** collections of **discrete** objects. With the growth of digital devices, especially digital computers, discrete mathematics has become more and more important.

Counting problems arise when the combinatorial problem is to count the number of different arrangements of collections of objects of a particular kind. Such counting problems arise frequently when we want to calculate probabilities and so they are of wider application than might appear at first sight. Some counting problems are very easy, others are extremely difficult.

Alan Slomson, An Introduction to Combinatorics, Chapman and Hall Mathematics, 1991

Introduction to combinatorics and counting problems

Problem I A café has the following menu

Tomato soup

Fruit juice

Lamb chops

Baked cod

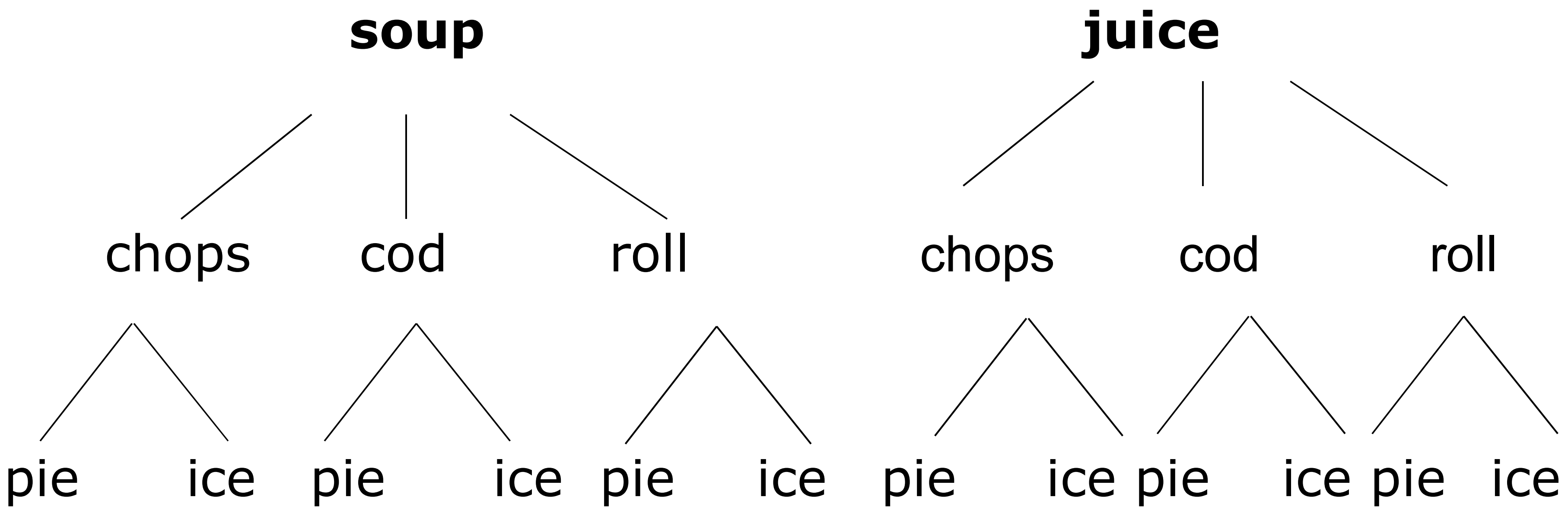
Nut roll

Apple pie

Strawberry ice

How many different three course meals could you order?

Solution to problem I



We would obtain 2x3x2=12 as the total of possible meals.

Problem II In a race with 20 horses, in how many ways the first three places can be filled?

Solution

There are 20 horses that can come first. Whichever horse comes first, there are 19 horses left that can come second. So there are 20x19=380 ways in which the first two places can be filled. In each of these 380 cases there are 18 horses which can come third. So there are:

20x19x18=380x18=6840 ways in which the first three positions can be filled.

What is a difference between these two problems?

Introduction to combinatorics and counting problems

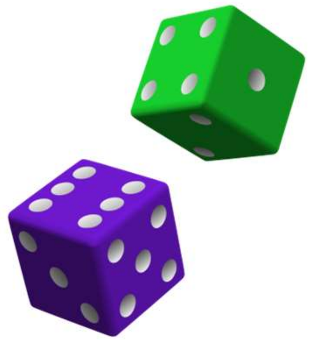
In many situations it is necessary to determine the number of elements of the set under considerations.

We use simple arithmetic methods:

- sum rule
- product rule

coin toss

drawing cards from a deck



dice throw

Sum Rule

If two events <u>are</u> mutually exclusive, that is, they cannot occur at the same time, then we must apply the sum rule

Theorem:

If an event e_1 can be realized in n_1 ways,

an event e_2 in n_2 ways, and

e_1 and e_2 are mutually exclusive

then the number of ways of both events occurring is

$$n_1 + n_2$$

Sum Rule

There is a natural generalization to any sequence of m tasks; namely the number of ways m mutually exclusive events can occur

$$n_1 + n_2 + \ldots + n_{m-1} + n_m$$

We can give another formulation in terms of sets. Let $A_1, A_2, \ldots, A_m$ be pairwise disjoint sets. Then

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| \cup |A_2| \cup \ldots \cup |A_m|$$

Principle of Inclusion-Exclusion (PIE)

Say there are two events, e_1 and e_2, for which there are n_1 and n_2 possible outcomes respectively.

Now, say that only <u>one</u> event can occur, not both

In this situation, we cannot apply the sum rule. Why?

… because we would be overcounting the number of possible outcomes.

- Instead we have to count the number of possible outcomes of e_1 and e_2 minus the number of possible outcomes in common to both; i.e., the number of ways to do both tasks
- If again we think of them as sets, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Product Rule

If two events are not mutually exclusive (that is we do them separately), then we apply the product rule

Theorem:

Suppose a procedure can be accomplished with two disjoint subtasks. If there are

n_1 ways of doing the first task and

n_2 ways of doing the second task,

then there are $n_1 . n_2$ ways of doing the overall procedure

Application of sum and product rules

There are two towers at the entrance to the castle,. The first is protected by a two-digits „even” code while the second by a two-digits „odd” code. It is sufficient to break one code in order to enter. How many ways there are to the castle?

Even code.
Possible tens: 2,4,6,8
Possible units: 0,2,4,6,8
Product rule: $5 \times 4 = 20$
Odd code.
Possible tens: 1,3,5,7,9
Possible units: 1,3,5,7,9
Product rule $5 \times 5 = 25$
Sum rule: $25 + 20 = 45$

Counting problems and introduction to combinatorics

- Ordered arrangement (**sequence**) = **permutation**

(1,2,3); (2,1,3); (3,1,2) etc.

- Order is not important (**set, subset**) = **combination**

{1,2,3}

In both cases we have to distinguish:

with or without **replacement**

Permutation

An ordered arrangement of k elements of a set of n elements is called an <u>k-permutation</u>

Number of permutations depends on whether the elements of sequence can be repeated or not. The method of sampling is important: without replacement = no repetitions; with replacement = repetitions of elements are possible

Permutations without replacement

Example: Take into account a set of n=3-elements Z={a,b,c} and write down all possible k=2 permutations without replacement:

(a,b) (b,a) (a,c) (c,a) (b,c) (c,b)

The number of these permutations can be calculated as:

3x2=6

In general:

$$V_n^{(k)} = \prod_{i=0}^{k-1}(n-i) = n(n-1)(n-2)...(n-k+1)$$

Permutations without replacement

Number of k-permutations without replacement drawn from a set of n elements can be calculated from the following formula:

$$V_n^{(k)} = \frac{n!}{(n-k)!}$$

When k=n,

$$V_n^{(k)} = n!$$

Example: (abc) (acb) (bac) (bca) (cab) (cba)

Permutations—without replacement

Think cards (w/o reshuffling) and seating arrangements.

Example: You are moderating a debate of gubernatorial candidates. How many different ways can you seat the panelists in a row? Call them Arianna, Buster, Camejo, Donald, and Eve.

Permutations—without replacement

→ ***"Trial and error" method:***

<u>Systematically write out all possibilities:</u>

A B C D E

A B C E D

A B D C E

A B D E C

A B E C D

A B E D C

.

.

.

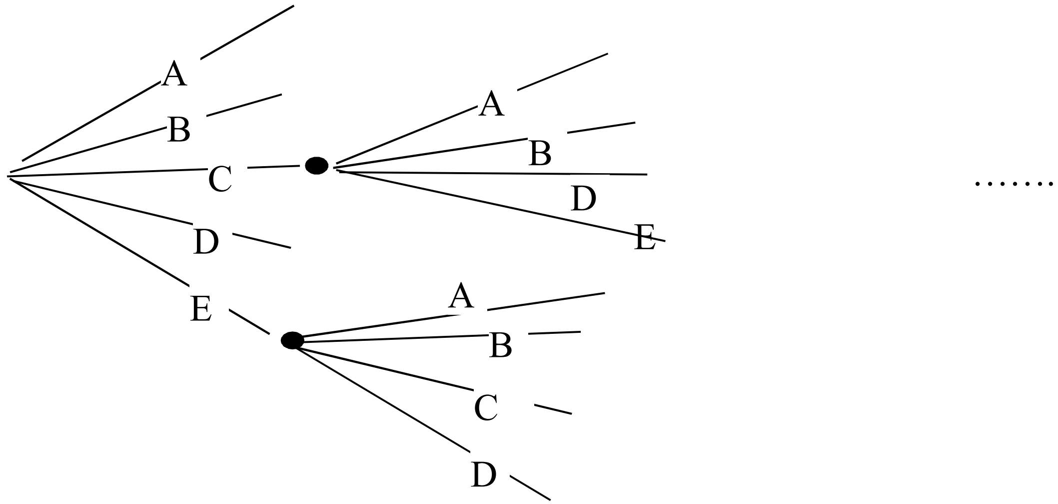
Quickly becomes a pain!
Easier to figure out patterns using a probability tree!

Permutations—without replacement

Seat One:
5 possible

Seat Two:
only 4 possible

Etc….



# of permutations = 5 x 4 x 3 x 2 x 1 = 5!

There are 5! ways to order 5 people in 5 chairs (since a person cannot repeat)

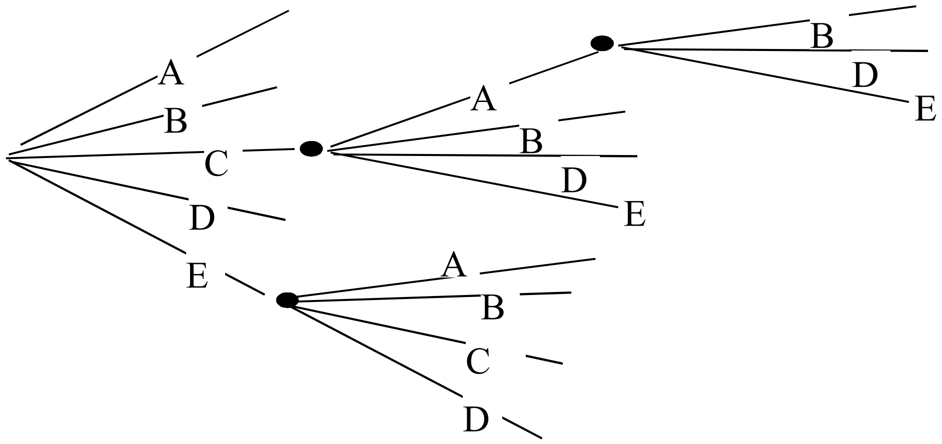
Permutations—without replacement

What if you had to arrange 5 people in only 3 chairs (meaning 2 are out)?

Seat One:
5 possible

Seat Two:
Only 4 possible

Seat Three:
only 3 possible



$$5x4x3 =$$

$$\frac{5x4x3x2x1}{2x1} = \frac{5!}{2!} =$$

$$\frac{5!}{(5-3)!}$$

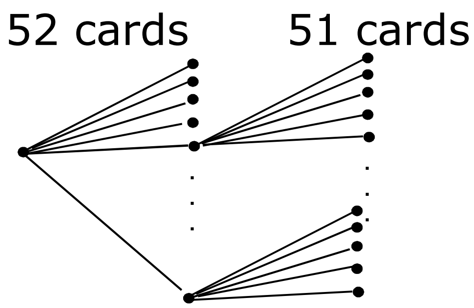
Permutations—without replacement

Note this also works for 5 people and 5 chairs:

$$\boxed{\frac{5!}{(5-5)!} = \frac{5!}{0!} = 5!}$$

Permutations—without replacement

How many two-card hands can I draw from a deck when order matters (e.g., ace of spades followed by ten of clubs is different than ten of clubs followed by ace of spades)



$$\boxed{\frac{52!}{(52-2)!} = 52x51}$$

Permutations—without replacement

Summary: order matters, without replacement

Formally, "order matters" and "without replacement"→ use factorials→

$$\frac{(n \text{ people or cards})!}{(n \text{ people or cards} - k \text{ chairs or draws})!} = \frac{n!}{(n-k)!}$$

$$\text{or} \quad n(n-1)(n-2)\ldots(n-k+1)$$

Permutations—with replacement

Example: Take into account a set of n=3-elements Z={a,b,c} and write down all possible k=2 permutations with replacement:

(a,a)	(b,a)	(c,a)
(a,b)	(b,b)	(c,b)
(a,c)	(b,c)	(c,c)

Calculate the number of possible permutations:

$$3x3=3^2 = 9$$

Permutations—with replacement

Number of k-permutations with replacement drawn from a set of n elements can be calculated from the following formula:

$$W_n^{(k)} = n^k$$

Problem: Electronic devices usually require a personal code to operate. This particular device uses 4-digits code. Calculate how many codes are possible.

Solution: Each code is represented by k=4 permutations with replacement of a set of 10 digits {0,1,2,3,4,5,6,7,8,9}

$$W_{10}^{(4)} = 10^4 = 10\,000$$

Permutations—with replacement

When you roll a pair of dice (or 1 die twice), what's the probability of rolling 6 twice?

$$P(6,6) = \frac{1 \text{ way to roll } 6, 6}{6^2} = \frac{1}{36}$$

What's the probability of rolling a 5 and a 6?

$$P(5 \,\&\, 6) = \frac{2 \text{ ways} : 5,6 \text{ or } 6,5}{6^2} = \frac{2}{36}$$

Permutations—with replacement

Summary: order matters, with replacement

Formally, "order matters" and "with replacement"→ use powers→

$$\boxed{(\#\text{ possible outcomes per event})^{\text{the \# of events}} = n^k}$$

Combination

Combination containing k elements drawn from a set of n elements is a k-elemental subset (order does not matter) composed of the elements of the set.

Number of combinations depends on whether the elements of subset can be repeated or not. The method of sampling is important: without replacement = no repetitions; with replacement = repetitions of elements are possible

Combination without replacements

Example: Take into account a set of n=3-elements Z={a,b,c} and write down all possible k=2 combinations without replacement:

{a,b} {a,c} {b,c}

Calculate the number of subsets 6/2 = 3

In general:

$$C_n^{(k)} = \frac{V_n^{(k)}}{k!}$$

Combination without replacements

Number of k-combinations without replacement of a set containing n elements can be calculated from the following formula:

$$C_n^{(k)} = \frac{n!}{k!(n-k)!}$$

Or:

$$C_n^{(k)} = \binom{n}{k}$$

Combinations—order doesn't matter

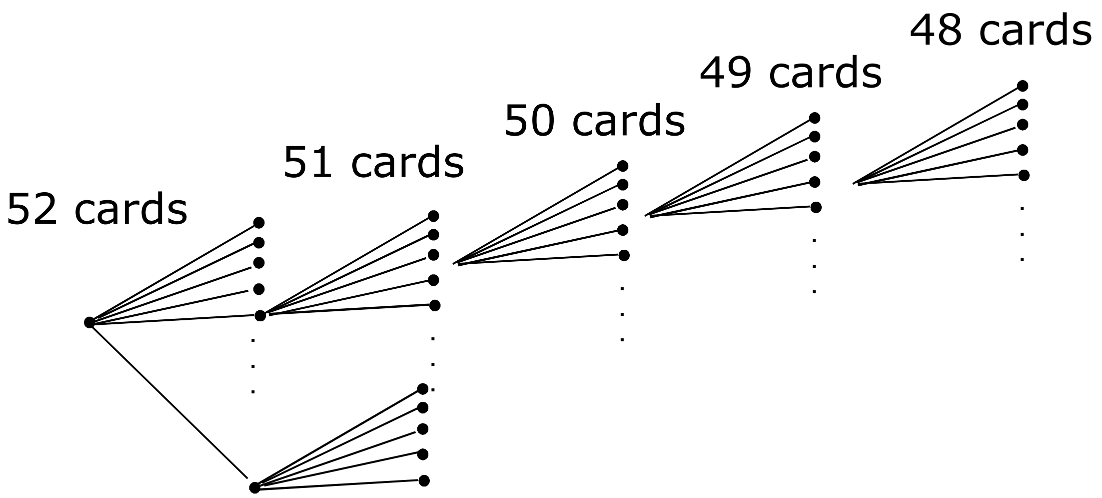
Combination function, or "choosing"

Written sometimes as: $${}_nC_k \text{ or } \binom{n}{k}$$

Spoken: "*n* choose *k*"

Combinations

How many five-card hands can I draw from a deck when order does **not** matter?



$$\boxed{\frac{52x51x50x49x48}{?}}$$

Combinations

Denominator is a number of permutations without replacement.

$$5! = 120$$

$$\text{total \# of 5 - card hands} = \frac{52x51x50x49x48}{5!} = \frac{52!}{(52-5)!5!}$$

Combinations

How many unique 2-card sets out of 52 cards?

$$\frac{52x51}{2} = \frac{52!}{(52-2)!2!}$$

5-card sets?

$$\frac{52x51x50x49x48}{5!} = \frac{52!}{(52-5)!5!}$$

r-card sets?

$$\frac{52!}{(52-r)!r!}$$

r-card sets out of n-cards?

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Summary: combinations

If r objects are taken from a set of n objects without replacement and disregarding order, how many different samples are possible?

Formally, "order doesn't matter" and "without replacement"→ use choosing→

$$\boxed{\binom{n}{r} = \frac{n!}{(n-r)!r!}}$$

Examples—Combinations

A lottery works by picking 6 numbers from 1 to 49. How many combinations of 6 numbers could you choose?

$$\binom{49}{6} = \frac{49!}{43!6!} = 13{,}983{,}816$$

Which of course means that your probability of winning is 1/13,983,816!

Combinations with replacement

Example: Take into account a set of n=3-elements Z={a,b,c} and write down all possible k=2 combinations with replacement:

{a,a} {a,b} {a,c} {b,b} {b,c} {c,c}

Calculate the number of combinations with replacements

6

In general:

$$c_n^{(k)} = \binom{n+k-1}{k}$$

Summary of Counting Methods

Counting methods for computing probabilities

- **Permutations—order matters!**
  - **With replacement**
  - **Without replacement**
- **Combinations—Order doesn't matter**
  - **Without replacement**
  - **With replacement**

Conditional probability

General definition:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

under assumption that P(B) > 0 (event B has to be possible)

Conditional probability

Useful expressions:

$$P(A) = P(A|\Omega)$$ for any event A

$$A \cap B = \varnothing \Rightarrow P(A|B) = 0$$

$$A \subset B \Rightarrow P(A|B) = \frac{P(A)}{P(B)}$$

$$B \subset A \Rightarrow P(A|B) = 1$$

Example

We are throwing a 6-sided die three times. Each time we have got a different number of dots. Calculate a probability that once we get a „5” assuming that each attempt gives different number.

$$P(A \cap B) = \frac{5 \cdot 4 \cdot 3}{\Omega}$$

$$P(B) = \frac{6 \cdot 5 \cdot 4}{\Omega}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{5 \cdot 4 \cdot 3\Omega}{6 \cdot 5 \cdot 4 \cdot \Omega}$$