A generalization of an independent set with application to $(K_q; k)$-stable graphs

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Abstract

We introduce a natural generalization of an independent set of a graph and give a sharp lower bound on its size. The bound generalizes the widely known Caro and Wei result on the independence number of a graph. We use this result in the following problem. Given non-negative real numbers α, β the cost $c(G)$ of a graph G is defined by $c(G) = \alpha|V(G)| + \beta|E(G)|$. We estimate the minimum cost of a $(K_q; k)$-vertex stable graph, i.e. a graph which contains a clique K_q after removing any k of its vertices.

keywords: independent set; independence number; clique

1 Introduction

By a word graph we mean a simple graph in which multiple edges and loops are not allowed. Given a graph G, $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. Furthermore, $|G| := |V(G)|$ is the order of G and $||G|| := |E(G)|$ is the size of G. Let $N_G(v) = \{u : uv \in E(G)\}$ denote the set of neighbors of v in G, and let $d_G(v)$ (in short $d(v)$) denote the number of neighbors of v in G, i.e. $d_G(v) = |N_G(v)|$. The minimum degree of G is denoted by $\delta(G)$.

A set $I \subset V(G)$ is an independent set of G if a subgraph $G[I]$ of G induced by I is edgeless. A maximum independent set is a largest independent set for a given graph G and its size is denoted $\alpha(G)$. We start with the following generalization of the concept of an independent set. A set $I \subset V(G)$ is called a *p-independent set* of G if a subgraph of G induced by I has chromatic number less than or equal to p. In particular, a 1-independent set is an independent set. A *maximum p-independent set* is a largest p-independent set for a given graph G and its size is denoted $\alpha_p(G)$. The well known result about the size of an independent set states that

Theorem 1 *([2, 12]) Let G be a graph. Then*

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d_G(v) + 1}. \tag{1}$$

As a corollary of a more general result (see Theorem 6, Section 3) we obtain the following theorem.

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Theorem 2 *Let G be a graph and $p \geq 1$ an integer. If $\delta(G) \geq p-1$, then*

$$\alpha_p(G) \geq \sum_{v \in V(G)} \frac{p}{d_G(v)+1}. \tag{2}$$

Moreover, if $\delta(G) \geq p$, then the equality holds if and only if G is a disjoint union of cliques.

We apply this result in the following problem. Suppose that we want to build a construction having certain properties using elements from sets $S_1, ..., S_t$. Each element from a set S_i has a given cost c_i. Thus, the total cost (depending on the numbers and costs of used elements) of every construction can be computed. The purpose is to find a feasible construction with minimal total cost. We will consider this kind of problem in case where the feasible constructions can be modeled by graphs. Naturally, the graphs in question are supposed to have some properties that depends on the properties that are required for feasible constructions. In the paper we want that a feasible graph G contains a given subgraph H. In fact, we require more. Some elements may get damaged, hence, we want that even if some of them are spoiled, G still contains a copy of H.

Formally, let H be any graph and k a non-negative integer. A graph G is called $(H;k)$-*stable* if $G-S$ contains a subgraph isomorphic to H for every set $S \subset V \cup E$ with $|S| \leq k$. Given a cost $\alpha \geq 0$ of every vertex, and a cost $\beta \geq 0$ of every edge, the total cost $c(G)$ of G is defined by $c(G) = \alpha|G| + \beta||G||$. Then $\mathrm{stab}_{(\alpha,\beta)}(H;k) = \min\{c(G) : G \text{ is } (H;k) \text{ stable}\}$ denotes the minimum cost among the costs of all $(H;k)$-stable graphs.

Note that if $S \subset V$ and $\alpha = 0, \beta = 1$ then the above problem reduces to the problem of finding minimum $(H;k)$-vertex stable graphs, with minimum size (= minimum cost) denoted by $\mathrm{stab}(H;k)$. So far the exact value of $\mathrm{stab}(H;k)$ for any k is known in the case when $H = C_3$, C_4, K_4, $K_{1,m}$ [5], $H = K_5$ [8], and $H = K_q$ with k sufficiently large [13]. On the other hand, for small k the value $\mathrm{stab}(H;k)$ is known when $H = K_{m,n}$ and $k = 1$, see [6, 7], and when $H = K_n$ and $k \leq n/2+1$, see [9]. In all the above cases minimal vertex stable graphs are characterized. Furthermore, $\mathrm{stab}(C_n;1)$ is known for infinitely many n's and for remaining n's it has one of only two possible values, see [4]. An upper and a lower bound on $\mathrm{stab}(C_n;k)$ for sufficiently large n is also presented therein. There are also some general bounds on $\mathrm{stab}(H;k)$ that involve the minimal degree, connectivity and the order of H, see [3, 14].

In this paper we explore the methods from [13] in order to obtain generalizations. On the other hand, a 'clear' edge version (i.e. with $S \subset E$ and $\alpha = 0, \beta = 1$) has also been considered, see [10, 11].

Observe that if $\alpha = 0$ and H does not have isolated vertices, then after adding to or removing from a $(H;k)$-stable graph any number of isolated vertices we still have a $(H;k)$-stable graph with the same cost. So, in this particular case we will assume throughout the paper that no graph in question has isolated vertices.

We end this section by the following simple proposition.

Proposition 3 *Let G be a $(H;k)$-stable graph with minimum cost. If $\alpha \geq 0$ and $\beta > 0$ then each vertex as well as each edge of G is contained in some copy of H. In particular, $d_G(v) \geq \delta(H)$ for each vertex $v \in V(G)$.*

On the other hand, trivially, $\mathrm{stab}_{(\alpha,0)}(H;k) = \alpha(|H|+k)$ with $K_{|H|+k}$ being a minimum $(H;k)$-stable graph.

2 Two inequalities

Proposition 4 *Let $(x_i)_{i=1}^n$ be a sequence of non-negative numbers and let $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$. Then*

$$\sum_{i=1}^{n} \frac{1}{x_i + 1} \geq n \frac{1}{\bar{x} + 1}.$$

Proof. It is enough to show that the expression $\sum_{i=1}^n \frac{1}{x_i+1}$ is minimal if all x_i are equal to $\bar{x}$. Suppose on the contrary that $S := \sum_{i=1}^n \frac{1}{x_i+1}$ is minimal with $x_s \neq x_t$ for some $s, t \in [1, n]$. Without loss of generality we assume that $x_t = x_s + 2\epsilon$, where $\epsilon > 0$. Let $x_i' = x_i$ for $i \notin \{s, t\}$, and $x_t' = x_s' = x_s + \epsilon$. Let $S' = \sum_{i=1}^n \frac{1}{x_i'+1}$. Then

$$\begin{aligned} S' &= S + \frac{2}{x_s + 1 + \epsilon} - \frac{1}{x_s + 1 + 2\epsilon} - \frac{1}{x_s + 1} \\ &= S + \frac{2(x_s+1)(x_s+1+2\epsilon) - (x_s+1)(x_s+1+\epsilon) - (x_s+1+\epsilon)(x_s+1+2\epsilon)}{(x_s+1)(x_s+1+\epsilon)(x_s+1+2\epsilon)} \\ &= S - \frac{2\epsilon^2}{(x_s+1)(x_s+1+\epsilon)(x_s+1+2\epsilon)} < S, \end{aligned}$$

a contradiction with the minimality of S. ■

Proposition 5 *Let $(x_i)_{i=1}^n$ be a sequence of non-negative integers and let $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$. Then there exist m, $0 \leq m \leq n$, such that $\frac{1}{n}(m\lceil\bar{x}\rceil + (n-m)\lfloor\bar{x}\rfloor) = \bar{x}$ and*

$$\sum_{i=1}^{n} \frac{1}{x_i + 1} \geq m \frac{1}{\lceil\bar{x}\rceil + 1} + (n-m) \frac{1}{\lfloor\bar{x}\rfloor + 1}.$$

Proof. It is enough to show that the expression $\sum_{i=1}^n \frac{1}{x_i+1}$ is minimal if $|x_i - \bar{x}| < 1$ for all i. Suppose on the contrary that $S := \sum_{i=1}^n \frac{1}{x_i+1}$ is minimal with $|x_s - \bar{x}| \geq 1$ for some $s \in [1, n]$. If $x_s > \bar{x}$ ($x_s < \bar{x}$) then there exist x_t such that $x_t < \bar{x}$ ($x_t > \bar{x}$). Without loss of generality we assume that $x_s > x_t + 1$. Let $x_i' = x_i$ for $i \notin \{s, t\}$, and $x_s' = x_s - 1$, $x_t' = x_t + 1$. Clearly $(x_i)_{i=1}^n$ and $(x_i')_{i=1}^n$ have the same arithmetic mean. Let $S' := \sum_{i=1}^n \frac{1}{x_i'+1}$. Then

$$\begin{aligned} S' &= S - \frac{1}{x_s+1} - \frac{1}{x_t+1} + \frac{1}{x_s'+1} + \frac{1}{x_t'+1} \\ &= S - \frac{1}{x_s+1} - \frac{1}{x_t+1} + \frac{1}{x_s} + \frac{1}{x_t+2} \\ &= S - \frac{x_s(x_t+1)(x_t+2) + x_s(x_s+1)(x_t+2) - (x_s+1)(x_t+1)(x_t+2) - x_s(x_s+1)(x_t+1)}{x_s(x_s+1)(x_t+1)(x_t+2)} \\ &= S - \frac{x_s^2 + x_s - x_t^2 - 3x_t - 2}{x_s(x_s+1)(x_t+1)(x_t+2)} \\ &< S - \frac{(x_t+1)^2 + (x_t+1) - x_t^2 - 3x_t - 2}{x_s(x_s+1)(x_t+1)(x_t+2)} = S \end{aligned}$$

a contradiction with the minimality of S. ■

3 Large induced subgraphs with bounded coloring number

The coloring number $col(G)$ of a graph G is the least integer c for which there exists an ordering of the vertices of G in which each vertex has fewer than c neighbors that are earlier in the ordering.

Theorem 6 *Let G be a graph of order n and minimum degree δ. If $c \geq 1$ is an integer such that $\delta \geq c-1$, then G contains an induced subgraph H with coloring number $col(H) \leq c$ and order*

$$|H| \geq \sum_{v \in V(G)} \frac{c}{d_G(v)+1}. \tag{3}$$

Moreover, if $\delta \geq c$ then the equality holds if and only if G is a disjoint union of cliques.

Proof. Let σ be an ordering of the vertices of G. For $v \in V(G)$ let $\deg_\sigma^-(v)$ denote the number of neighbors of v that are earlier in σ. Let S_σ denote the set of all vertices v with $\deg_\sigma^-(v) \leq c-1$. In what follows, we use an argument similar to the one that was used by Alon and Spencer [1] in their proof of Caro [2] and Wei [12] result concerning independence number of graphs. We further assume that $c \geq 2$, because for $c = 1$ each set S_σ is an independent set and the under-mentioned facts are well known. Given a random ordering σ, the probability that a vertex v has at most i neighbors, $i \leq d_G(v)$, earlier in the ordering σ is equal

$$Pr(\deg_\sigma^-(v) \leq i) = \frac{\binom{n}{d_G(v)+1}(i+1)(d_G(v))!(n-d_G(v)-1)!}{n!} = \frac{i+1}{d_G(v)+1}.$$

Thus,

$$Pr(v \in S_\sigma) = \frac{c}{d_G(v)+1}.$$

Hence,

$$E\left(|S_\sigma|\right) = \sum_{v \in V(G)} \frac{c}{d_G(v)+1}.$$

Thus, there exists an ordering σ with the required number of vertices in S_σ, whence for a desired subgraph H we can take the subgraph induced by S_σ, $H = G[S_\sigma]$. Furthermore, the equality in (3) may hold only if $|S_\sigma|$ is the same for every ordering σ (if there is a σ with $|S_\sigma| < \sum_{v \in V(G)} \frac{c}{d_G(v)+1}$, then there is also a σ' with $|S_{\sigma'}| > \sum_{v \in V(G)} \frac{c}{d_G(v)+1}$ because the expectation is exactly that number). Now we will prove that if $\delta \geq c$, then this is possible only for the disjoint union of cliques. Let C be any component of G and let $v \in V(C)$. Consider the following ordering σ of vertices of C:

$$v_1, v_2, \ldots, v_{c-1}, v_c, v_{c+1}, v_{c+2}, \ldots, v_{|C|},$$

where $v_{c+1} = v$ and $v_1, v_2, \ldots, v_c$ are any neighbors of v. Observe that $v_{c+1} \notin S_\sigma$. Next consider an ordering σ'

$$v_{c+1}, v_1, v_2, \ldots, v_{c-1}, v_c, v_{c+2}, \ldots, v_{|C|}.$$

Note that since $|S_\sigma| = |S_{\sigma'}|$ and $v_{c+1} \in S_{\sigma'}$, $v_c \notin S_{\sigma'}$. Thus, $\deg_{\sigma'}^-(v_c) = c$. Analogously we obtain that $\deg_{\sigma''}^-(v_{c-1}) = c$ in an ordering $\sigma'' : v_c, v_{c+1}, v_1, v_2, \ldots, v_{c-1}, v_{c+2}, \ldots, v_{|C|}$, and so on. Therefore, vertices $v_1, v_2, \ldots, v_c, v_{c+1}$ induce a clique. Since v and its neighbors (at least 2) have been chosen arbitrarily, $\{v\} \cup N_G(v)$ induce a clique for each $v \in V(C)$. This implies that C is a clique. $\square$

Since the chromatic number of a graph is bounded by the coloring number, we obtain Theorem 2 as a corollary of Theorem 6.

4 $(K_q;k)$ stable graphs with minimum cost

Lemma 7 *Let $\alpha \geq 0$ and $\beta > 0$ be real numbers. If G is $(K_q;k)$-stable graph with minimum cost, then*

$$|G| - (q-1) \sum_{v \in V(G)} \frac{1}{d_G(v)+1} \geq k+1. \tag{4}$$

Moreover, if G is not a union of cliques then the inequality (4) is strong.

Proof. By Proposition 3, $\delta(G) \geq q-1$. Thus, by Theorem 2, there exists a set $I \subset V(G)$ such that $\chi(G[I]) \leq q-1$ and $|I| \geq \sum_{v \in V(G)} \frac{q-1}{d_G(v)+1}$ where the latter inequality is strong if G is not a union of cliques. Since $\chi(G[I]) \leq q-1$, $G[I]$ does not contain any K_q. Thus and since G is $(K_q;k)$-stable, $|G| - |I| \geq k+1$. $\square$

Theorem 8 *Let $\alpha \geq 0$, $\beta > 0$ be real numbers and $k \geq 0$, $q \geq 2$ be integers. Let $r = \left\lfloor \sqrt{(q-1)(q-2+\frac{2\alpha}{\beta})} \right\rfloor - q$.*

1. *If $\alpha(q-1) > \beta(2q + qr + (r^2+r-2)/2)$, then*

$$\mathrm{stab}_{(\alpha,\beta)}(K_q;k) \geq (k+1)\frac{2q+r}{q+1+r}\left(\alpha + \beta\left(q + \frac{r-1}{2}\right)\right), \tag{5}$$

with equality if and only if $k = a(q+1+r)-1$ for some positive integer a. Moreover, if G is $(K_q;k)$-stable with the cost given by right hand side of inequality (5), then G is a disjoint union of cliques K_{2q+r}.

2. *If $\alpha(q-1) = \beta(2q + qr + (r^2+r-2)/2)$, then*

$$\mathrm{stab}_{(\alpha,\beta)}(K_q;k) \geq (k+1)\frac{2q+r}{q+1+r}\left(\alpha + \beta\left(q + \frac{r-1}{2}\right)\right), \tag{6}$$

with equality if and only if $k = a(q+1+r) + b(q+r) - 1$ for some positive integers a, b. Moreover, if G is $(K_q;k)$-stable with the cost given by the right hand side of inequality (6) then G is a disjoint union of cliques K_{2q+r} and K_{2q-1+r}.

3. *If $\alpha(q-1) < \beta(2q + qr + (r^2+r-2)/2)$, then*

$$\mathrm{stab}_{(\alpha,\beta)}(K_q;k) \geq (k+1)\frac{2q-1+r}{q+r}\left(\alpha + \beta\left(q + \frac{r-2}{2}\right)\right), \tag{7}$$

with equality if and only if $k = a(q+r)-1$ for some positive integer a. Moreover, if G is $(K_q;k)$-stable with the cost given by right hand side of inequality (7) then G is a disjoint union of cliques K_{2q-1+r}.

Proof. Let

$$c_0 = (k+1)\cdot \min\left\{\frac{2q-1+r}{q+r}\left(\alpha+\beta\left(q+\frac{r-2}{2}\right)\right), \frac{2q+r}{q+1+r}\left(\alpha+\beta\left(q+\frac{r-1}{2}\right)\right)\right\}.$$

Let G be a $(K_q;k)$-stable graph with minimum cost. One can check that

$$\frac{2q+r}{q+1+r}\left(\alpha+\beta\left(q+\frac{r-1}{2}\right)\right) + \frac{2q-1+r}{q+r}\cdot\frac{\alpha(q-1)-\beta(2q+qr+(r^2+r-2)/2)}{(2q-1+r)(q+1+r)} = \frac{2q-1+r}{q+r}\left(\alpha+\beta\left(q+\frac{r-2}{2}\right)\right). \tag{8}$$

Hence, in order to prove lower bounds (5), (6) and (7), it suffices to show that $\mathrm{stab}_{(\alpha,\beta)}(K_q;k) \geq c_0$. By Lemma 7 and Proposition 4 we have that

$$|G| \geq (q-1)\sum_{v\in V(G)} \frac{1}{d_G(v)+1} + k + 1 \geq |G|\frac{q-1}{d_G+1} + k + 1, \text{ and so}$$

$$|G| \geq (k+1)\frac{d_G+1}{d_G-q+2}, \tag{9}$$

where $d_G = \frac{2||G||}{|G|} = \frac{1}{|G|}\sum_{v\in V(G)} d_G(v)$ is the average degree of G. Thus, by (9),

$$c(G) = \alpha|G| + \beta||G|| = \alpha|G| + \frac{\beta d_G}{2}|G| \geq \left(\alpha + \frac{\beta d_G}{2}\right)(k+1)\frac{d_G+1}{d_G-q+2}. \tag{10}$$

Recall that, by Proposition 3, $d_G \geq q-1$. Consider the following function

$$f(x) = \left(\alpha + \frac{\beta x}{2}\right)(k+1)\cdot\frac{x+1}{x-q+2},\ x \geq q-1.$$

Note that the derivative of f is equal to

$$\begin{aligned} f'(x) &= (k+1)\left(\frac{\beta}{2}\frac{x+1}{x-q+2} + (\alpha + \frac{\beta x}{2})\frac{1-q}{(x-q+2)^2}\right) \\ &= (k+1)\beta\frac{x^2 + x(4-2q) + 2\frac{\alpha}{\beta} + 2 - q}{(x-q+2)^2}. \end{aligned}$$

Thus

$$f'(x) \geq 0 \Longleftrightarrow x^2 + x(4-2q) + 2\frac{\alpha}{\beta} + 2 \geq 0.$$

Since $\Delta = 4(q-1)(q-2+2\alpha/\beta)$, we obtain that f is decreasing for $x \leq x_0$ and increasing for $x \geq x_0$ where $x_0 = q-2+\sqrt{(q-1)(q-2+\frac{2\alpha}{\beta})}$. Note that

$$\begin{aligned} 2q-2+r = q-2+\left\lfloor\sqrt{(q-1)(q-2+\frac{2\alpha}{\beta})}\right\rfloor \leq x_0 \\ \leq q-2+\left\lfloor\sqrt{(q-1)(q-2+\frac{2\alpha}{\beta})}\right\rfloor + 1 = 2q-1+r. \end{aligned} \tag{11}$$

Therefore, we can assume that $d_G \in [2q-2+r, 2q-1+r]$ because otherwise, by (10) and by the above mentioned property of f,

$$\begin{aligned} c(G) &> \min\{f(2q-2+r), f(2q-1+r)\} \\ &= \min\{(k+1)(\alpha + \frac{\beta(2q-2+r)}{2})\frac{2q-1+r}{q+r}, (k+1)(\alpha+\frac{\beta(2q-1+r)}{2})\frac{2q+r}{q+r+1}\} \\ &= (k+1)\cdot\min\left\{\frac{2q-1+r}{q+r}\left(\alpha+\beta(q+\frac{r-2}{2})\right), \frac{2q+r}{q+1+r}\left(\alpha+\beta(q+\frac{r-1}{2})\right)\right\} = c_0. \end{aligned}$$

Recall that

$$c(G) = \alpha|G| + \beta||G|| = (\alpha + \frac{\beta d_G}{2})|G|.$$

Thus, given an average degree d_G of G, $c(G)$ is minimal if $|G|$ is minimal. Since $d_G \in [2q-2+r, 2q-1+r]$, by Lemma 7 and Proposition 5, we have

$$\begin{aligned} |G| \geq m\frac{q-1}{2q-1+r} + (|G|-m)\frac{q-1}{2q+r} + k + 1, \text{ and} \\ \sum_{v\in V(G)} d_G(v) = m(2q-2+r) + (|G|-m)(2q-1+r), \end{aligned} \tag{12}$$

for some m, $0 \leq m \leq |G|$. Hence,

$$|G| \geq (k+1)\frac{2q+r}{q+1+r} + m\frac{q-1}{(2q-1+r)(q+1+r)}. \tag{13}$$

Furthermore, if equality holds, then G is a disjoint union of cliques. Thus, by (12) and (13),

$$\begin{aligned}
c(G) = \alpha|G| + \beta||G|| &\geq \alpha(k+1)\frac{2q+r}{q+1+r} + \alpha m\frac{q-1}{(2q-1+r)(q+1+r)} + \frac{\beta}{2}\sum_{v\in V(G)} d_G(v) \\
&= \alpha(k+1)\frac{2q+r}{q+1+r} + \alpha m\frac{q-1}{(2q-1+r)(q+1+r)} \\
&+ \frac{\beta}{2}\left(m(2q-2+r) + (|G|-m)(2q-1+r)\right) \qquad (14)\\
&= \alpha(k+1)\frac{2q+r}{q+1+r} + \alpha m\frac{q-1}{(2q-1+r)(q+1+r)} - \frac{\beta}{2}m + \frac{\beta}{2}|G|(2q-1+r) \\
&\geq \alpha(k+1)\frac{2q+r}{q+1+r} + \alpha m\frac{q-1}{(2q-1+r)(q+1+r)} - \frac{\beta}{2}m \\
&+ \frac{\beta}{2}(2q-1+r)(k+1)\frac{2q+r}{q+1+r} + \frac{\beta}{2}m\frac{q-1}{q+1+r} \\
&= (k+1)\frac{2q+r}{q+1+r}\left(\alpha + \beta(q+\frac{r-1}{2})\right) + m\frac{\alpha(q-1) - \beta(2q+qr+(r^2+r-2)/2)}{(2q-1+r)(q+1+r)}.
\end{aligned}$$

Thus, if $\alpha(q-1) \geq \beta(2q+qr+(r^2+r-2)/2)$, then $c(G) \geq c_0$. On the other hand if $\alpha(q-1) < \beta(2q+qr+(r^2+r-2)/2)$, then the minimum in (14) is obtained for $m = |G|$. In this case, by (12),

$$|G| \geq (k+1)\frac{2q-1+r}{q+r},$$

so

$$\begin{aligned}
c(G) = \alpha|G| + \beta||G|| = \alpha|G| + \frac{\beta}{2}|G|(2q-2+r) \\
\geq \left(\alpha + \beta\left(q + \frac{r-2}{2}\right)\right)(k+1)\frac{2q-1+r}{q+r}.
\end{aligned} \tag{15}$$

Therefore, $c(G) \geq c_0$, as well. Furthermore, if $\alpha(q-1) > \beta(2q+qr+(r^2+r-2)/2)$ and $c(G) = c_0$, then, by formula (14), $m = 0$. Hence, G is a disjoint union of cliques K_{2q+r}. Similarly, if $\alpha(q-1) = \beta(2q+qr+(r^2+r-2)/2)$ and $c(G) = c_0$, then G is the disjoint union of cliques K_{2q-1+r} and K_{2q+r} (m can be arbitrary). Finally, if $\alpha(q-1) < \beta(2q+qr+(r^2+r-2)/2)$ and $c(G) = c_0$, then $m = |G|$, so G is the disjoint union of cliques K_{2q-1+r}.

Therefore, it remains to show that indeed $c(G) = c_0$ in the above cases. Note that aK_{2q+r} is $(K_q; a(q+1+r)-1)$-stable, aK_{2q-1+r} is $(K_q; a(q+r)-1)$-stable, and $aK_{2q+r} + bK_{2q-1+r}$ is $(K_q; a(q+1+r) + b(q+r) - 1)$-stable. Suppose that $k = a(q+1+r) + b(q+r) - 1$ and

$\alpha(q-1) = \beta(2q+qr+(r^2+r-2)/2)$. Hence,

$$\begin{aligned} c(aK_{2q+r}+bK_{2q-1+r}) =& a\left(\alpha(2q+r)+\beta\frac{(2q+r)(2q+r-1)}{2}\right) \\ &+b\left(\alpha(2q+r-1)+\beta\frac{(2q+r-1)(2q+r-2)}{2}\right) \\ =& a\,(q+1+r)\,\frac{2q+r}{q+1+r}\,(\alpha+\beta(q+(r-1)/2)) \\ &+b\,(q+r)\,\frac{2q+r-1}{q+r}\,(\alpha+\beta(q+(r-2)/2)) \\ =&(k+1)\frac{2q+r}{q+1+r}\,(\alpha+\beta(q+(r-1)/2))\,, \end{aligned}$$

by (8), as required. Similar (and easier) computations can be made in the remaining cases. □

For two special cases ($\alpha=\beta=1$ and $\alpha=0, \beta=1$) we obtain the following corollaries.

Corollary 9 *Let $q \geq 2$. If $k \geq (q-1)(q-2)-1$, then*

$$\mathrm{stab}_{(1,1)}(K_q;k) = (2q-1)(k+1).$$

Moreover, if G is a $(K_q;k)$-stable with $c(G)=(2q-1)(k+1)$ then G is a disjoint union of cliques K_{2q-2} and K_{2q-1}.

Proof. Note that in this case $r=-1$ and the situation from Theorem 8 case 2 occurs. Thus, $\mathrm{stab}_{(1,1)}(K_q;k)=(2q-1)(k+1)$ for every $k=aq+b(q-1)-1$. On the other hand $(q-1)(q-2)-1$ is the Frobenious number for $\{q, q-1\}$, namely the largest integer that cannot be presented in the form $aq+b(q-1)$. □

Corollary 10 *([13]) Let $q \geq 2$. If $k \geq (q-3)(q-2)-1$, then*

$$\mathrm{stab}(K_q;k) = (2q-3)(k+1).$$

Moreover, if G is a $(K_q;k)$-stable with $||G|| = (2q-3)(k+1)$ then G is a disjoint union of cliques K_{2q-3} and K_{2q-2}.

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