On $(K_q; k)$-stable graphs

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Abstract

A graph G is called $(H;k)$-*vertex stable* if G contains a subgraph isomorphic to H even after removing any k of its vertices. By $\text{stab}(H;k)$ we denote the minimum size among the sizes of all $(H;k)$-vertex stable graphs. Given an integer $q \geq 2$, we prove that, apart of some small values of k, $\text{stab}(K_q;k) = (2q-3)(k+1)$. This confirms in the affirmative the conjecture of Dudek et al. [(H,k) stable graphs with minimum size, Discuss. Math. Graph Theory 28(1) (2008) 137–149]. Furthermore, we characterize the extremal graphs.

1 Introduction

By the word graph we mean a simple graph without loops and multiple edges. Given a graph G, $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. Furthermore, $|G| := |V(G)|$ is the order of G and $||G|| := |E(G)|$ is the size of G.

The following problem has attracted some attention recently. Let H be any graph and k a non-negative integer. A graph G is called $(H;k)$-*vertex stable* (in short $(H;k)$-*stable*) if G contains a subgraph isomorphic to H even after removing any k of its vertices. Then $\text{stab}(H;k)$ denotes the minimum size among the sizes of all $(H;k)$-vertex stable graphs. A $(H;k)$-stable graph with minimum size shall be called a *minimum* $(H;k)$-stable graph. Note that if H does not have isolated vertices then after adding to or removing from a $(H;k)$-vertex stable graph any number of isolated vertices we still have a $(H;k)$-vertex stable graph with the same size. Therefore, in the sequel we assume that no graph in question has isolated vertices.

The notion of $(H;k)$-vertex stable graphs was introduced in [4] (an edge version of this notion was also considered, see [9, 10]). So far the above problem has been mainly investigated for specified graphs including cycles [3, 4], complete bipartite graphs [5, 6], and above all, complete graphs [4, 7, 8]. In [4] it was proved that $\text{stab}(K_3;k) = 3(k+1)$ and $\text{stab}(K_4;k) = 5(k+1)$, and the authors conjectured that $\text{stab}(K_q;k) = (2q-3)(k+1)$ for $k \geq k(q)$ for some sufficiently large integer $k(q)$. In [7] the authors gave the value of $\text{stab}(K_5;k)$ for all k and characterized minimum $(K_q;k)$-stable graphs for $q = 3,4,5$ and all k. In particular they confirmed in the affirmative the above mentioned conjecture in case $q = 5$ with $k(5) = 5$. In this paper we present a lower bound on $\text{stab}(K_q;k)$ for all $k \geq 0$ and $q \geq 2$. As a result, we confirm the above mentioned conjecture for all remaining q's with $k(q) = (q-3)(q-2)-1$. Furthermore, we characterize the minimum graphs. We also derive the value of $\text{stab}(K_6;k)$ for all k.

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2 General conditions

Recall the following simple observation.

Proposition 1 ([4]) *Let δ_H be the minimum degree of a graph H. Then in any minimum $(H;k)$-stable graph G, $d_G(v) \geq \delta_H$ for each vertex $v \in G$.*

The following theorem may be seen as a necessary condition for a graph G to be a minimum $(H;k)$-stable graph.

Theorem 2 *If G is a minimum $(H;k)$-stable graph then*

$$|G| - \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1} \geq k+1. \tag{1}$$

Moreover, if G is not a union of cliques then the inequality (1) is strong.

Proof. By Proposition 1 we assume that minimum degree of G is at least δ_H. Let σ be an ordering of the vertices of G. For $v \in V(G)$ let $\deg_\sigma^-(v)$ denote the number of neighbors of v that are on the left from v in ordering σ. Let S_σ denote the set of all vertices v with $\deg_\sigma^-(v) \leq \delta_H - 1$. Note that by removing from G all vertices from $V(G) \setminus S_\sigma$ we spoil all copies of H. Indeed, we can consecutively (from the right to the left) eliminate all vertices from S_σ because at each time the analized vertex has degree $\leq \delta_H - 1$ (and therefore cannot be in any copy of H contained in a graph induced by it and the vertices from $V(G) \setminus S_\sigma$ that are earlier in the ordering σ). Thus, since G is $(H;k)$-stable, $|G| - |S_\sigma| \geq k+1$ for each ordering σ.

Therefore, it suffices to find an ordering σ with $|S_\sigma| \geq \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1}$. We will achieve this by an argument similar to the one that was used by Alon and Spencer [1] in their proof of Caro [2] and Wei [11] result concerning independence number of graphs. We further assume that $\delta_H \geq 2$, because for $\delta_H = 1$ each set S_σ is an independent set and these facts are well known Caro and Wei theorem. Given a random ordering σ, the probability that a vertex v has at most i, $i \leq d_G(v)$, neighbors on its left side in the ordering σ is equal to

$$Pr(\deg_\sigma^-(v) \leq i) = \frac{\binom{n}{d_G(v)+1}(i+1)(d_G(v))!(n-d_G(v)-1)!}{n!} = \frac{i+1}{d_G(v)+1}.$$

Thus,

$$Pr(v \in S_\sigma) = \frac{\delta_H}{d_G(v)+1}.$$

Hence,

$$E\left(|S_\sigma|\right) = \sum_{v \in V(G)} \frac{\delta_H}{d_G(v)+1}.$$

Thus, there exists an ordering σ with the required number of vertices in S_σ. Furthermore, the equality in (1) may hold only if $|S_\sigma|$ is the same for every ordering σ (if there is a σ with $|S_\sigma| < \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1}$, then there is also a σ' with $|S_{\sigma'}| > \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1}$ because the expectation is exactly that number). Now we will prove that if G is minimum $(H;k)$-stable, then this is possible only for the disjoint union of cliques.

Let C be any component of G and let $v \in V(C)$. Let $\delta = \delta_H$. Consider the following ordering σ of vertices of C:

$$v_1, v_2, ..., v_\delta, v_{\delta+1}, v_{\delta+2}, ..., v_{|C|},$$

where $v_{\delta+1} = v$ and $v_1, v_2, ..., v_\delta$ are any neighbours of v (recall that each vertex of G has at least δ neighbors). Next consider an ordering σ'

$$v_{\delta+1}, v_1, v_2, ..., v_\delta, v_{\delta+2}, ..., v_{|C|}.$$

Note that since $|S_\sigma| = |S_{\sigma'}|$ and $v_{\delta+1} \in S_{\sigma'}$, $v_\delta \notin S_{\sigma'}$. Thus, $\deg^-_{\sigma'}(v_\delta) = \delta$. Analogously we obtain that $\deg^-_{\sigma''}(v_{\delta-1}) = \delta$ in an ordering $\sigma'' : v_\delta, v_{\delta+1}, v_1, v_2, ..., v_{\delta-1}, v_{\delta+2}, ..., v_{|C|}$, and so on. Therefore, vertices $v_1, v_2, ..., v_\delta, v_{\delta+1}$ induce a clique. Since v and its neighbours have been chosen arbitrarily, $\{v\} \cup N_G(v)$ induce a clique for each $v \in V(C)$. This implies that C is a clique. □

Corollary 3 *Let H be any graph and let δ_H denote the minimum degree of H. Then*

$$\mathrm{stab}(H;k) \geq (k+1)\left(\delta_H + \sqrt{\delta_H(\delta_H - 1)} - 1/2\right).$$

Proof. Clearly, it suffices to prove the bound on the size for minimum $(H;k)$-stable graphs. Let G be such a graph. By Theorem 2 we have that

$$|G| \geq \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1} + k + 1 \geq |G| \frac{\delta_H}{d_G + 1} + k + 1, \tag{2}$$

where $d_G = \frac{2||G||}{|G|}$ is the average degree of G. Note that the latter inequality follows from the fact that the expression $\sum_{j=1}^{l} \frac{1}{x_j}$ with $\sum_{j=1}^{l} x_j = const$ (the constant being equal to $2||G|| + |G|$ in our case) and $x_j > 0$, is minimal if all the x_j are equal. Indeed, suppose on the contrary that $S := \sum_{j=1}^{k} \frac{1}{x_j}$ is minimal with $x_s \neq x_t$ for some $s, t \in [1, l]$. Without loss of generality we assume that $x_t = x_s + 2\epsilon$, where $\epsilon > 0$. Let $x'_j = x_j$ for $j \notin \{s, t\}$, and $x'_t = x'_s = x_s + \epsilon$. Clearly $\sum_{j=1}^{l} x'_j = \sum_{j=1}^{l} x_j$. Let $S' = \sum_{j=1}^{l} \frac{1}{x'_j}$. Then

$$\begin{aligned} S' &= S + \frac{2}{x_s+\epsilon} - \frac{1}{x_s+2\epsilon} - \frac{1}{x_s} = S + \frac{2x_s(x_s+2\epsilon) - x_s(x_s+\epsilon) - (x_s+\epsilon)(x_s+2\epsilon)}{x_s(x_s+\epsilon)(x_s+2\epsilon)} \\ &= S - \frac{2\epsilon^2}{x_s(x_s+\epsilon)(x_s+2\epsilon)} < S, \end{aligned}$$

a contradiction with the minimality of S.

Thus, by (2),

$$||G|| = \frac{d_G}{2}|G| \geq \frac{k+1}{2} \cdot \frac{d_G(d_G+1)}{d_G + 1 - \delta_H}.$$

By examining the derivative of the function $f(x) = \frac{x(x+1)}{x+1-\delta_H}$ we obtain that f has minimum in $x_0 = \delta_H + \sqrt{\delta_H(\delta_H-1)} - 1$. Hence, $||G|| \geq \frac{k+1}{2} f(x_0) = (k+1)\left(\delta_H + \sqrt{\delta_H(\delta_H-1)} - 1/2\right)$. □

3 Complete graphs

Theorem 4 *Let G be a $(K_q;k)$-stable graph, $q \geq 2$ and $k \geq 0$. Then*

$$||G|| \geq (2q-3)(k+1), \tag{3}$$

with equality if and only if G is a disjoint union of cliques K_{2q-3} and K_{2q-2}.

Proof. We may assume that G is a minimum $(K_q;k)$-stable graph. Similarily as in the proof of Corollary 3 we have $||G|| \geq \frac{k+1}{2} \cdot \frac{d_G(d_G+1)}{d_G+1-(q-1)}$. By examining the derivative of the function $f(x) = \frac{x(x+1)}{x+1-(q-1)}$ we obtain that $f(x)$ is decreasing for $x \leq x_0$ and increasing for $x \geq x_0$ where $x_0 = q-1+\sqrt{(q-1)(q-2)}-1$, $2q-4 \leq x_0 \leq 2q-3$. On the other hand, $f(2q-4) = f(2q-3) = 2(2q-3)$. Therefore, the lower bound (3) can be achieved only if $d_G \in [2q-4, 2q-3]$. Then the sum $\sum_{v \in V(G)} \frac{1}{d_G(v)+1}$ is minimal if degrees of vertices of G differ as little as possible from d_G. Thus, we may assume that $d_G(v) \in \{2q-4, 2q-3\}$ for every $v \in V(G)$. Let m denote the number of vertices of G with degree equal to $2q-3$. Hence,

$$\sum_{v \in V(G)} \frac{1}{d_G(v)+1} \geq m\frac{1}{2q-2} + (|G|-m)\frac{1}{2q-3} = \frac{2(q-1)|G|-m}{2(q-1)(2q-3)}, \tag{4}$$

with equality if and only if $d_G(v) \in \{2q-4, 2q-3\}$ for every $v \in V(G)$. Therefore, by Theorem 2 we have

$$\begin{aligned} |G| - \frac{2(q-1)|G|-m}{2(2q-3)} &\geq k+1, \text{ and so} \\ |G| \geq (k+1)\frac{2q-3}{q-2} &- \frac{m}{2(q-2)}, \end{aligned} \tag{5}$$

with equality if and only if G is a disjoint union of cliques. Thus,

$$\begin{aligned} ||G|| &\geq \frac{m(2q-3) + (|G|-m)(2q-4)}{2} \\ &\geq \frac{m + (k+1)\frac{2q-3}{q-2}(2q-4) - \frac{m}{2q-4}(2q-4)}{2} \\ &= (k+1)(2q-3) \end{aligned} \tag{6}$$

with equality if and only if G is the disjoint union of cliques K_{2q-3} and K_{2q-2}. □

Theorem 5 *Let $q \geq 2$, $k \geq 0$ be non-negative integers. Then*

$$\operatorname{stab}(K_q;k) \geq (2q-3)(k+1),$$

with equality if and only if $k = a(q-2) + b(q-1) - 1$ for some non-negative integers a, b. In particular,

$$\operatorname{stab}(K_q;k) = (2q-3)(k+1) \text{ for } k \geq (q-3)(q-2)-1.$$

Furthermore, if G is a $(K_q;k)$-stable with $||G|| = (2q-3)(k+1)$ then G is a disjoint union of cliques K_{2q-3} and K_{2q-2}.

Proof. It is easy to see that $G = aK_{2q-3} + bK_{2q-2}$ is $(K_q; a(q-2)+b(q-1)-1)$-stable. On the other hand $(q-3)(q-2)-1$ is the Frobenious number for $\{q-2, q-1\}$, namely the largest integer that canot be presented in the form $a(q-2)+b(q-1)$. Lower bounds follows from Theorem 4. □

4 Concluding Remarks

Apart of some small values of k, we have determined the exact value of $\operatorname{stab}(K_q;k)$ for all q, together with minimum graphs. In [8] it is proved that for $q \geq 6$ and $k \leq q/2+1$ the only $(K_q;k)$-stable graph with minimum size is isomorphic to K_{q+k}. Thus, $\operatorname{stab}(K_6;k)$ is known for all

k except $k \in \{5, 6, 10\}$. In these cases Theorem 5 implies that $\mathrm{stab}(K_6; k) \geq 9(k+1)+1$. Since the graphs K_{11}, $K_8 \cup K_9$ and $K_{10} \cup K_{11}$ are, respectively, $(K_6; 5)$, $(K_6; 6)$ and $(K_6; 10)$-stable we have: $\mathrm{stab}(K_6; 5) = 55$, $\mathrm{stab}(K_6; 6) = 64$ and $\mathrm{stab}(K_6; 10) = 100$. Therefore, the value of $\mathrm{stab}(K_6; k)$ is known for all k. Similarily, $\mathrm{stab}(K_7; k)$ is known for all k except $k \in \{6, 7, 8, 12, 13, 18\}$. However, for $k \in \{6, 8, 12, 13, 18\}$, $\mathrm{stab}(K_7; k)$ can be computed in an analogous way as previously. Hence, the first unknown value is $\mathrm{stab}(K_7; 7)$.

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