Growth order for the size of smallest hamiltonian chain saturated uniform hypergraphs

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April 10, 2014

Abstract

We say that a hypergraph H is hamiltonian chain saturated if H does not contain a hamiltonian chain but by adding any new edge we create a hamiltonian chain in H. In this paper, for each $k \geq 3$, we establish the right order of magnitude n^{k-1} for the size of the smallest k-uniform hamiltonian chain saturated hypergraph. This solves an open problem of G. Y. Katona.

Keywords: saturated hypergraph, hamiltonian path, hamiltonian cycle, hamiltonian chain.

2000 Mathematics Subject Classification: 05C35.

1 Introduction

Let H be a k-uniform hypergraph on a vertex set $V(H)$ with $|V(H)| = n \geq k$. The set of the edges — k-element subsets of $V(H)$ — is denoted by $E(H)$. For simplicity of notation v_{n+x} with $x \geq 0$ denotes the same vertex as v_x. In [21] the authors defined the notion of a hamiltonian chain.

Definition 1 A cyclic ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set is called a *hamiltonian chain* and denoted $C_n^{(k)}$, if and only if $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} \in E(H)$ whenever $1 \leq i \leq n$. An ordering $(v_1, v_2, \ldots, v_l)$ of a subset of the vertex set is called an *open chain* and denoted $P_l^{(k)}$, if and only if $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} \in E(H)$ whenever $1 \leq i \leq l-k+1$. An open chain $P_n^{(k)}$ is an *open hamiltonian chain.*

For $v \in V(H)$, let $H - v$ be the hypergraph obtained by deleting v and all edges incident to v. We refer to this operation as *removing* v from H. Given a k-uniform hypergraph H and a k-element set $e \in H^c$, where H^c is the complement of H, we denote by $H + e$ the hypergraph obtained from H by adding e to its edge set.

*The author was partially supported by the Polish Ministry of Science and Higher Education.

Definition 2 We say that a hypergraph H is *hamiltonian chain saturated* if H does not contain a hamiltonian chain but for every $e \in H^c$ the hypergraph $H + e$ does contain a hamiltonian chain.

In the paper we study k-uniform hamiltonian chain saturated hypergraphs with as few as possible edges. Let $sat(n, C_n^{(k)})$ denote the minimum size (number of edges) among the sizes of k-uniform hamiltonian chain saturated hypergraphs on n vertices. The problem is solved for graphs, i.e. 2-uniform hypergraphs. Namely, $sat(n, C_n) = \lceil \frac{3n}{2} \rceil$ (apart from a few small values of n) which follows from [5, 9, 10]. Similar problem for hamiltonian paths is also solved in case of graphs, see [11] and [16]. Much less is known for $k \geq 3$. Namely, $sat(n, C_n^{(k)}) \geq \binom{n}{k}/(k(n-k)+1)$, see [12]. Hence the lower bound is of order n^{k-1}. In [20] the author conjectures that this is the right order of magnitude for $sat(n, C_n^{(k)})$ (a corresponding conjecture in case of hamiltonian path saturated hypergraphs was also posed in [12]). So far only hypergraphs with $O(n^{k-1/2})$ edges are known [13]. In this paper we prove that $sat(n, C_n^{(k)}) = \Theta(n^{k-1})$ for each $k \geq 3$. The proof is constructive. This solves the above mentioned open problem of G. Y. Katona. Our construction is based on the construction from [13]. However, it is much more detailed. Moreover, the key idea of using constructions of smallest hamiltonian cycle saturated graphs, which allowed us to obtain the right order of magnititude is entirely new.

In fact, the above problem belongs to the much wider theory of saturated graphs and hypergraphs. Given a hypergraph F, we say that the hypergraph H is F-saturated if H has no F as a subhypergraph, but does contain F after the addition of any new edge. The minimum number of edges in an F-saturated hypergraph on n vertices is denoted by $sat(n, F)$. There are many results on $sat(n, F)$ for graphs, see for example [15] for complete graphs, [3, 7, 8, 23] for complete s-partite graphs or [1, 2, 6, 17, 19, 22, 25] for cycles. In case $k \geq 3$, Bollobás [4] generalized Erdős, Hajnal and Moon's result [15] for complete k-uniform hypergraphs. Erdős, Füredi and Tuza [14] obtained $sat(n, F)$ for some particular hypergraphs F with few edges. Pikhurko [24] proved that $sat(n, F) = O(n^{k-1})$ for any fixed hypergraph F (generalizing previous result for graphs by Erdős, Füredi and Tuza [14]). Finally, note that asking for the maximum (instead of minimum) number $ex(n, F)$ of edges in an F saturated hypergraph on n vertices is the Turan problem. The problem of establishing $ex(n, C_n^{(k)})$ was also studied, see [18, 21, 26].

The paper is organized as follows. In Section 2 we recall the definition and usefull properties of Isaac's graphs. Then we present our construction. In Section 3 we prove that the construction gives hamiltonian chain saturated hypergraphs, provided that the parameters used in it satisfy certain conditions. As a corollary we obtain that $sat(n, C_n^{(k)}) = \Theta(n^{k-1})$.

2 Construction

We begin this section by recalling the definition and usefull properties of Isaac's graphs. Isaac's graphs J_s for odd $s \geq 3$ are defined as follows. Let $V(J_s) = \{v_i : 0 \leq i \leq 4s-1\}$ and $E(J_s) = E_0 \cup E_1 \cup E_2 \cup E_3$ where

$E_0 = \bigcup_{j=0}^{s-1} \{v_{4j}v_{4j+1}, v_{4j}v_{4j+2}, v_{4j}v_{4j+3}\}$,

$E_1 = \{v_{4j+1}v_{4j+7} : 0 \leq j \leq s-1\}$,

$E_2 = \{v_{4j+2}v_{4j+6} : 0 \leq j \leq s-1\}$ and

$E_3 = \{v_{4j+3}v_{4j+5} : 0 \leq j \leq s-1\}$.

Subscripts should be read modulo $4s$. Note that J_s is 3-regular.

Theorem 1 *([9]) Let $e = v_{4j}v_{4j+2}$ where $0 \le j < s-1$ and $s \ge 5$. For any two nonadjacent vertices u and v of J_s there exists a hamiltonian (u,v)-path in J_s containing edge e.*

For $w \in V(G)$ with neighbors v_1, v_2, v_3 and $w_1, ..., w_l \notin V(G)$ let $G^l(w)$ be a graph with $V(G^l(w)) = V(G-w) \cup \{w_1, ..., w_l\}$ and $E(G^l(w)) = E(G-w) \cup \left(\bigcup_{i<j} w_i w_j\right) \cup \{w_1v_1, w_2v_2, w_3v_3\}$. Note that $G^l(w)$ is hamiltonian if and only if G is hamiltonian.

Theorem 2 *([10]) Let $e = v_0v_2$ and $s \ge 9$. For any two nonadjacent vertices u and v of $J_s^3(v_{14})$ there exists a hamiltonian (u,v)-path in $J_s^3(v_{14})$ containing edge e.*

Given a vertex v, by $N_G(v)$ (in short $N(v)$) we denote the set of neighbors of v in G, i.e. $N_G(v) = \{u : uv \in E(G)\}$. Furthermore, given an edge $e = xy$ of G, we say that G has property $\mathcal{Q}(e)$ if $N(x) \cap N(y) = \emptyset$ and for every pair of non-adjacent vertices $u, v \in V(G)$, the graph $G + uv$ has a hamiltonian cycle containing e.

Lemma 3 *For each $n \ge 44$ there exists a graph G_n of order n and size $m \le \frac{3}{2}(n+39)$ with the property $\mathcal{Q}(e)$ for some $e \in E(G)$.*

Proof. Let $s \ge 9$ and $l = 0$ or $l \ge 3$. First we will prove that for every pair of non-adjacent vertices u, v of $J_s^l(v_{14})$, the graph $J_s^l(v_{14}) + uv$ has property $\mathcal{Q}(v_0v_2)$. If $l = 0$ or $l = 3$ then this is true by Theorems 1 or 2 respectively. Suppose that $l \ge 4$. Since vertices $w_1, ..., w_l$ induce a complete graph in $J_s^l(v_{14})$, we may assume that $v \notin \{w_1, ..., w_l\}$.

If $u \notin \{w_4, ..., w_l\}$, then, by Theorem 2, there is an appropriate hamiltonian cycle in $\left(J_s^l(v_{14}) + uv\right) - \{w_4, ..., w_l\} = J_s^3(v_{14}) + uv$. This cycle contains an edge $f = w_1w_2$ or $f = w_1w_3$. Thus by replacing f by a path $w_1, w_4, ..., w_l, w_2$ (or $w_1, w_4, ..., w_l, w_3$) we obtain a required hamiltonian cycle in $J_s^l(v_{14}) + uv$.

Suppose now that $u = w_r$ for some $r \in \{4, ..., l\}$. Let $w' \in \{w_1, w_2, w_3\}$ such that $w'v \notin E\left(J_s^l(v_{14})\right)$. Then, by Theorem 2, there is an appropriate hamiltonian cycle in $\left(J_s^l(v_{14}) + w'v\right) - \{w_4, ..., w_l\} = J_s^3(v_{14}) + w'v$. Thus by replacing $w'v$ by a path $w', w_4, ..., w_{r-1}, w_{r+1}, ..., w_l, w_r, v$ we obtain a required hamiltonian cycle in $J_s^l(v_{14}) + w_rv = J_s^l(v_{14}) + uv$.

Note that $|V(J_s^l(v_{14}))| = 4s + l - 1$ and $|E(J_s^l(v_{14}))| = 6s + \binom{l}{2}$ because J_s is 3-regular. Let $n = 4s + r$ where s is odd and $0 \le r \le 7$. Since $n \ge 44$, we have $s \ge 11$. If $r = 0$, then $G_n = J_s$ is a graph with property $\mathcal{Q}(v_0v_2)$. Furthermore, for $2 \le r \le 7$ we can have $G_n = J_s^{r+1}(v_{14})$. Finally, if $r = 1$ then we choose $G_n = J_{s-2}^{10}(v_{14})$ (note that $s - 2 \ge 9$). Observe that $|E(J_s^l(v_{14}))| = |V(J_s^l(v_{14}))| + \frac{(l-1)(l+3)}{2}$. Thus $|E(G_n)| \le \frac{3}{2}n + \frac{9 \cdot 13}{2} = \frac{3}{2}(n+39)$, because $l \le 10$.

□

Now we are in the position to define our construction.

Definition 3 Let q be a non-negative integer and $U_0, U_1, \ldots, U_q$ be pairwise disjoint sets of vertices such that $|U_0| \ge 2$ and $|U_i| \ge 44$ for $i = 1, 2, \ldots, q$. Let $G(U_i)$, $i = 1, ..., q$, be a graph isomorphic to $G_{|U_i|}$ with the vertex set $V(G(U_i)) = U_i$ and $x_i, y_i \in U_i$ being vertices that correspond, respectively, to the vertices x, y of $G_{|U_i|}$. Define the vertex set of the hypergraph H to be $V(H) = \bigcup_{i=0}^{q} U_i$. Let $u_1 \in U_{i_1}, ..., u_k \in U_{i_k}$ with $0 \le i_1 \le i_2 \le ... \le i_k \le q$. Then $\{u_1, ..., u_k\} =: e \in E(H)$ if and only if one of the following conditions is satisfied

1. $|e \cap U_0| = 0$, $|e \cap U_{i_1}| = 2$ and if $u, v \in e \cap U_{i_1}$ then uv is an edge of $G(U_{i_1})$

2. $|e \cap U_0| = 0$ and $|e \cap U_{i_1}| \geq 3$

3. $|e \cap U_0| = 1$, $|e \cap U_{i_2}| = 1$ and $e \cap \{x_{i_2}, y_{i_2}\} \neq \emptyset$

4. $|e \cap U_0| = 1$ and $|e \cap U_{i_2}| \geq 2$

5. $|e \cap U_0| \geq 2$

6. $|e \cap U_0| = 1$, $|e \cap U_1| = 1$ and $|e \cap U_2| \geq 1$ with $e \cap \{x_1, y_1, x_2, y_2\} = \emptyset$.

If $|U_0| = t$ and $|U_i| = p_i$, $i = 1, ..., q$, then the hypergraph obtained by this construction is denoted by $\mathcal{H}_k(t, p_1, ..., p_q)$.

Example 1 Let $H = \mathcal{H}_5(t, p_1, ..., p_q)$. Then

- a 5-tuple of the form $\{2, 2, 2, 4, 4\}$ (which means that the 5-tuple contains 3 vertices from the set U_2 and 2 vertices from the set U_4) is an edge of H (an edge of type 2)
- a 5-tuple of the form $\{0, 0, 2, 4, 4\}$ is an edge of H (an edge of type 5)
- a 5-tuple of the form $\{1, 2, 2, 4, 4\}$ is not an edge of H (it contains only one vertex from the set $U_{i_1} = U_1$)
- a 5-tuple of the form $\{0, 1, 1, 2, 4\}$ is an edge of H (an edge of type 4)
- a 5-tuple of the form $\{0, 2, 3, 3, 4\}$ is an edge of H (an edge of type 3) but only if a vertex in the position denoted by 2 is either x_2 or y_2.
- a 5-tuple of the form $\{1, 1, 2, 2, 4\}$ is an edge of H (an edge of type 1) but only if the two vertices from U_1 (in the positions denoted by 1) are connected in $G(U_1)$.
- a 5-tuple of the form $\{0, 1, 2, 2, 3\}$ is an edge of H if it has no vertex from $\{x_1, y_1, x_2, y_2\}$ (an edge of type 6) or it has a vertex from $\{x_1, y_1\}$ (an edge of type 3).

Definition 4 Let $k \geq 3$ and let t be an even integer if k is odd, or an arbitrary integer if k is even. For integers $\beta \geq \alpha \geq 22$, let $H_k(t, \alpha, \beta)$ denote a hypergraph $\mathcal{H}_k(t, p_1, ..., p_q)$ such that $q = \frac{t}{2}k + 1$, $p_1 = 2\beta + 1$, $p_i = 2\alpha + 1$ for $i = 2, ..., q - 1$ and $p_q = (\beta - \alpha)(k - 2) + \frac{t}{2}(k - 4) + 1 \geq 44$.

Proposition 4 *The number of vertices in $H_k(t, \alpha, \beta)$ is equal to $(t - 1)k\alpha + k\beta + t(k - 1) + 1$.*

Proof. Since $q = \frac{t}{2}k + 1$, we have

$$\begin{aligned} |V(\mathcal{H}_k(t, \alpha, \beta))| &= t + 2\beta + 1 + (2\alpha + 1)(q - 2) + (\beta - \alpha)(k - 2) + \frac{t}{2}(k - 4) + 1 \\ &= t + 2\beta + 1 + (2\alpha + 1)(\frac{t}{2}k - 1) + (\beta - \alpha)(k - 2) + \frac{t}{2}(k - 4) + 1 \\ &= t(k - 1) + k\beta + (t - 1)k\alpha + 1 \end{aligned}$$

□

Proposition 5 *If $\alpha = \Theta(n^{1/2})$, $\beta = \Theta(n^{1/2})$ and $t = \Theta(n^{1/2})$ then $|E(H_k(t, \alpha, \beta))| = \Theta(n^{k-1})$.*

Proof. Let E_i denote the set of edges of type i, $i = 1, ..., 6$, from Definition 3. Thus,

$$
\begin{aligned}
|E_1| &\le \sum_{i=1}^{q} \frac{3}{2}\left(|U_i| + 39\right) \cdot \binom{n}{k-2} = \Theta(n^{k-1}) \\
|E_2| &\le \sum_{i=1}^{q} \binom{|U_i|}{3} \cdot \binom{n}{k-3} = \Theta(n^{k-1}) \\
|E_3| &\le t \cdot 2q \cdot \binom{n}{k-2} = \Theta(n^{k-1}) \\
|E_4| &\le t \cdot \sum_{i=1}^{q} \binom{|U_i|}{2} \cdot \binom{n}{k-3} = \Theta(n^{k-1}) \\
|E_5| &\le \binom{t}{2}\binom{n}{k-2} = \Theta(n^{k-1}) \\
|E_6| &\le t \cdot |U_1| \cdot |U_2| \cdot \binom{n}{k-3} = \Theta(n^{k-3/2}),
\end{aligned}
$$

where the first inequality holds by Lemma 3. □

3 The main result

For $S \subset V(\mathcal{H}_k(t, p_1, ..., p_q))$ let $\min(S) = \min\{i : S \cap U_i \ne \emptyset\}$.

Lemma 6 *If $\alpha \le \beta$ and $44 \le (\beta - \alpha)(k-2) + \frac{t}{2}(k-4) + 1 \le k\alpha + (k-2)$, then $H_k(t, \alpha, \beta)$ does not have a hamiltonian chain.*

Proof. Let $H = H_k(t, \alpha, \beta)$. For an indirect proof let us suppose that H contains a hamiltonian chain $(v_1, v_2, ..., v_n)$. Let us partition this chain by removing the vertices from U_0. Thus the chain $(v_1, v_2, ..., v_n)$ falls to parts $A_1, ..., A_m$ with $m \le t$. Each part induces an open chain in H or consists of at most $k-1$ vertices. Note that for any two adjacent edges e and e' belonging to one part $\min(e) = \min(e')$. Indeed, since e and e' are of type 1 or 2 of Definition 3 (vertices from U_0 have been removed) $\left|e \cap U_{\min(e)}\right| \ge 2$ and $\left|e' \cap U_{\min(e')}\right| \ge 2$. Thus, $e' \cap U_{\min(e)} \ne \emptyset$, so $\min(e') \le \min(e)$, and vice versa. Hence, every edge in a part A_i has at least two vertices from the set U_j, where $j = \min(A_i)$. We say that the set U_j is the *dominating set* for this part.

First we will show that if the hamiltonian chain consists only from edges of types 1-5 of Definition 3, then the following formula holds:

$$
|A_i| \le \frac{|U_j| - 1}{2}k + (k-2) = \begin{cases} \alpha k + (k-2) & \text{if } j \ne 1, q \\ \beta k + (k-2) & \text{if } j = 1 \end{cases} \tag{1}
$$

where U_j is the set dominating A_i. Indeed, consider consecutive disjoint k-tuples consisting of consecutive vertices of A_i and one r-tuple, $0 \le r \le k-1$, at the end. Clearly, every such k-tuple is an edge in H, hence, contains at least two vertices from the dominating set. Let ϵ denote the number of vertices from the dominating set which are in the r-tuple. Then $|A_i| \le \left\lfloor \frac{|U_j| - \epsilon}{2} \right\rfloor k + r$. If $r \le k-2$ or $\epsilon \ge 2$ then (1) is true, because $|U_j|$ is odd.

So let us suppose that $r = k-1$, $\epsilon \le 1$ and $|A_i| = \frac{|U_j|-1}{2}k + (k-1)$ ((1) does not hold). Then every considered k-tuple contains exactly two vertices from U_j. In fact $\epsilon = 1$, because the last k vertices of A_i form an edge in H, too. Let us consider again consecutive disjoint k-tuples, but this time taken in the opposite direction. Thus, again, every new k-tuple contains exactly two vertices from U_j and the new r-tuple contains exactly one vertex from U_j. It is easily seen now that every two consecutive (in $A_i \cap U_j$) vertices from U_j are the only vertices from U_j in some edge of H. Hence, in $G(U_j)$ they are adjacent. Thus, $G(U_j)$ has a hamiltonian path. Moreover, since in the original chain both r-tuples are contained in an edge of type 3 from Definition 3 (recall that so far we do not use edges of type 6 of Definition 3), this hamiltonian path starts in x_j and ends in y_j. This, however, is a contradiction because $x_j y_j \in E(G(U_j))$ and $G(U_j)$ is not hamiltonian. Therefore, $|A_i| < \frac{|U_j|-1}{2}k + (k-1)$, so (1) holds.

By the assumption we have that $|U_q| \leq k\alpha + (k-2)$. Hence, and by (1), every part A_i contains at most $k\alpha + (k-2)$ vertices exept possibly one with dominating set U_1 which contains at most $k\beta + (k-2)$ vertices. Thus and since $m \leq t$ we have

$$\sum_{i=1}^{m} |A_i| \leq \sum_{i=1}^{m-1} (k\alpha + (k-2)) + k\beta + (k-2) \leq (m-1)(k\alpha + k - 2) + k\beta + k - 2$$
$$\leq (t-1)k\alpha + k\beta + t(k-2) \qquad (2)$$

Therefore, by (2) and Proposition 4,

$$n = |U_0| + \sum_{i=1}^{m} |A_i| \leq (t-1)k\alpha + k\beta + t(k-1) = n-1, \qquad (3)$$

a contradiction.

Now, we will prove that inequality (2) remains true even if the edges of type 6 are also used (although inequality (1) may not hold). This will complete the proof of the lemma because (2) leads to a contradiction. Let e' be an edge of type 6 of Definition 3 (we assume that at least one such edge is used in the chain), namely an edge of the form $\{0, 1, 2, ..., i_k\}$ with $x_1, y_1, x_2, y_2 \notin e'$. Let u be the only vertex from $U_0 \cap e'$. Let A_L, A_R be two adjacent parts, A_L 'on the left' from u and A_R 'on the right'. Clearly, U_1 is the dominating set for one of those parts, say A_L.

We again consider consecutive disjoint k-tuples and an r_L-tuple, $r_L \leq k-1$, of A_L (from the left to the right). Simirarily, we consider consecutive disjoint k-tuples and an r_R-tuple, $r_R \leq k-1$, of A_R, this time taken from the right to the left. Note that if $r_L, r_R \leq k-2$ then the number of vertices in A_L and A_R do not exceed the bound (1), so inequality (2) holds. Assume that $r_L = k-1$. Thus, the number of vertices of A_L exceeds (by one) the bound (1). It is possible only if the r_L-tuple contains exactly one vertex from U_1. Since e' contains already a vertex from $U_1 \setminus \{x_1, y_1\}$ the r_L-tuple plus u form an edge e'' that is also of type 6 of Definition 3. Note that if $r_R \leq k-3$ then the number of vertices in A_R is at least one smaller then the bound (1). Thus, the sum of vertices of A_L and A_R is bounded by the same number as before and (2) remains true. So, let $k-1 \geq r_R \geq k-2$. Let $w_1 \in e'' \cap U_1$ and w_2 be the vertex from $e'' \cap U_2$ which is the closest to u among all vertices from $e'' \cap U_2$. Suppose that the distance between w_1 and u is smaller than the distance between w_2 and u and consider the k-tuple which begins in w_1. Thus, this k-tuple intersects the r_R-tuple. Moreover, this intersection contains a vertex from U_2. Indeed, since e'' contains exactly one vertex from U_1, namely w_1, and $w_1 \notin \{x_1, y_1\}$, e'' is of type 6 in this case. On the other hand, if the distance between w_2 and u is smaller than the distance between w_1 and u then the intersection of the r_R-tuple and a k-tuple that begins in w_2 also contains a vertex from U_2. Indeed, since $w_2 \notin \{x_2, y_2\}$, e'' is of type 4 in this case. Thus, U_2 is the dominating set for A_R.

Furthermore, the r_L-tuple as well as the r_R-tuple contains at least one vertex from U_2. Hence,

$$|A_R| \le \left\lfloor \frac{|U_2|-2}{2} \right\rfloor k + r_2 = \alpha k - 1 \le \alpha k + (k-4). \tag{4}$$

Hence, the sum of the numbers of vertices in A_L and A_R is even smaller than before, so (2) holds.

Therefore, we may assume that $r_L \le k-2$. Then the number of vertices in A_L does not exceed (1). If the dominating set of A_R is different from U_1, then A_R does not have any vertex from U_1. Hence, in $\{u\} \cup A_R$ we can use only edges of types 1-5 of Definition 3. Thus, (1) holds for A_R and so (2) remains true. Finally, if U_1 is the dominating set for both A_L and A_R, then

$$\begin{aligned} |A_L| + |A_R| &\le \left\lfloor \frac{|A_L \cap U_1|}{2} \right\rfloor + (k-1) + \left\lfloor \frac{|A_R \cap U_1|}{2} \right\rfloor + (k-1) \\ &\le \beta k + 2k - 4, \end{aligned}$$

and so $|A_L| + |A_R|$ is by far less than $\alpha k + (k-2) + \beta k + (k-2)$, the sum of the bounds (1). Thus, (2) remains true in each case. □

Example 2 Consider the hypergraph $\mathcal{H}_3(2,1,1)$ with $U_0 = \{u_0, v_0\}$ and $G(U_i)$ being the path on the vertices u_i, x_i, y_i in order, $i = 1, ..., 3$. Note that $\alpha = 1$, $\beta = 1$, $q = 4$ and $p_4 = 0$, so the parameters do not satisfy all conditions required in Definition 4. However, in this particular case $G(U_i)$ has property $\mathcal{Q}(x_i y_i)$, which will allow us to illustrate the key ideas that occur in the proof of the main theorem (in fact, this example is not hamiltonian chain saturated). The reasoning from Lemma 6 can be repeated in order to prove that $\mathcal{H}_3(2,1,1)$ has no hamiltonian chain. Below we will show how to construct a hamiltonian chain in $\mathcal{H}_3(2,1,1) + e_0$ for three different non-edges e_0.

1. If $e_0 = \{x_1, x_2, x_3\}$, then the cyclic ordering

$$(u_0 y_1 u_1 \underbrace{x_1 x_3 x_2}_{e_0} u_2 y_2 v_0 y_3 u_3)$$

is a hamiltonian chain.

2. If $e_0 = \{u_1, y_1, x_3\}$, then the cyclic ordering

$$(u_0 \underbrace{y_1 x_3 u_1}_{e_0} x_1 u_3 v_0 y_2 u_2 x_2 y_3)$$

is a hamiltonian chain.

3. If $e_0 = \{v_0, u_1, x_3\}$, then the cyclic ordering

$$(u_0 y_1 y_3 x_1 \underbrace{u_1 x_3 v_0}_{e_0} y_2 u_2 x_2 u_3)$$

is a hamiltonian chain.

Theorem 7 *Let $k \ge 3$. If t, α, β satisfy the following conditions*

$$\begin{aligned} &1.\ t \ge 2k \text{ is an even integer if } k \text{ is odd, or an arbitrary integer if } k \text{ is even,} \\ &2.\ \alpha \ge 2kt \\ &3.\ 2\alpha + t + 4 \le \beta \le 2\alpha + 3t + 5, \end{aligned} \tag{5}$$

then $\mathcal{H}_k(t, \alpha, \beta)$ is hamiltonian chain saturated. Moreover, if $t = \Theta(n^{1/2})$, $\alpha = \Theta(n^{1/2})$ and $\beta = \Theta(n^{1/2})$, then $|E(\mathcal{H}_k(t, \alpha, \beta))| = \Theta(n^{k-1})$.

Proof. Let $H = H_k(t, \alpha, \beta)$. It is easy to check that the upper bound on β from (5) implies that $|U_q| = (\beta - \alpha)(k-2) + \frac{t}{2}(k-4) + 1 \leq k\alpha + (k-2)$, if $\alpha \geq 2kt$. Hence, by Lemma 6, H does not have any hamiltonian chain. Recall first that, for each $i = 1, ..., q$, $G(U_i) + uv$, with $uv \notin E(G(U_i))$, has a hamiltonian cycle containing the edge x_iy_i. In particular, for each vertex $u_i \in V(G(U_i))$ with $u_iy_i \notin E(G(U_i))$, $G(U_i) + u_iy_i$, has a hamiltonian cycle containing the edge x_iy_i (and of course the edge u_iy_i). Thus, $G(U_i) - y_i$ has a hamiltonian path, say $P_i(u_i)$, with endpoints x_i and u_i. Analogously, for each vertex $v_i \in V(G(U_i))$ with $v_iy_i \notin E(G(U_i))$ $G(U_i) - x_i$ has a hamiltonian path, say $P_i'(v_i)$, with endpoints y_i and v_i. Note that using edges of H we are able to construct t open chains $\mathcal{C}_i$, $i = 1, ..., t$, of the following form

$$y_i \underbrace{*...*}_{k-3} u_i i \underbrace{*...*}_{k-2} ii \underbrace{*...*}_{k-2} ii \underbrace{*...*}_{k-2} ii \underbrace{*...*}_{k-2} \cdots ii \underbrace{*...*}_{k-2} ii \underbrace{*...*}_{k-2} ix_i \underbrace{*...*}_{k-2} \tag{6}$$

which means that $\mathcal{C}_i$ contains

- vertices y_i, x_i and u_i,
- all remaining vertices from U_i – in the positions denoted by i in order given by $P_i(u_i)$.
- $\frac{|U_i|-1}{2}(k-2) + (k-3)$ arbitrary vertices from sets U_j, $j \geq t+1$ – in the positions denoted by $*$,

Note that $\mathcal{C}_i$ is indeed an open chain: first (from the left) edge is of type 2 of Definition 3 and each remaining is of type 1 of Definition 3 (because vertices $(V(\mathcal{C}_i) \setminus \{y_i\}) \cap U_i$ are arranged in order given by a path $P_i(u_i)$). Note also, that after deleting any vertices from positions $*$ we still have an open chain. Moreover,

$$|\mathcal{C}_i| = \frac{|U_i| - 1}{2}k + (k-2) = \begin{cases} \alpha k + (k-2) & \text{if } i \neq 1 \\ \beta k + (k-2) & \text{if } i = 1 \end{cases} \tag{7}$$

The concatenation of these chains, obtained by placing a vertex from U_0 between $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$, $i = 1, ..., t-1$, and between $\mathcal{C}_t$ and $\mathcal{C}_1$, is a (closed) chain. Indeed, since $\mathcal{C}_i$ are open chains it sufices to check that every k-tuple (consisting of consecutive vertices) that contains a vertex from U_0 is an edge of H. Note that every such k tuple e has the form $x_i \underbrace{*...*}_{k-2} 0$ or $\underbrace{*...*}_{k'} 0y_i \underbrace{*...*}_{k''}$, with $k' + k'' = k-2$ and $i \leq t$ (where 0 represents a vertex from U_0). Moreover, in the positions $*$ there are vertices from sets U_j with $j \geq t+1$. Hence, $\min(e \setminus U_0) = i$. Therefore, e is of type 3 of Definition 3. Furthermore,

$$|U_0| + \sum_{i=1}^{t} |\mathcal{C}_i| = t + (k\beta + k - 2) + (t-1)(k\alpha + k - 2) = n - 1, \tag{8}$$

by Proposition 4. Hence, we are able to construct a (closed) chain which has $n-1$ vertices. Briefly speaking, we will show that by adding any new edge we will be able to modify one open chain $\mathcal{C}_i$ in such a way that the resulting open chain will have at least one vertex more.

Let e_0 be a new edge. Let e_0 have the form $\{i_1, i_2, ..., i_k\}$, $0 \leq i_1 \leq i_2 \leq ... \leq i_k$, which means that $e_0 = \{u_1, u_2, ..., u_k\}$ with $u_1 \in U_{i_1}, u_2 \in U_{i_2}, ..., u_k \in U_{i_k}$. Let $I = \{i \geq 1 : e_0 \cap U_i \neq \emptyset\}$. Let $j_1 < j_2 < ... < j_{q-|I|}$ be consecutive elements of the set $\{1, 2, ..., q\} \setminus I$. Furthermore, let $J = \{j_1, ..., j_{t-1}\}$ and $R = \{j_t, ..., j_{q-|I|}\}$. Let $\mathcal{C}_{j_i}$, $i = 2, ..., t-2$, have the form

$$y_{j_i} \underbrace{*...*}_{k-3} u_{j_i} j_i \underbrace{*...*}_{k-2} j_i j_i \underbrace{*...*}_{k-2} j_i j_i \underbrace{*...*}_{k-2} \cdots j_i j_i \underbrace{*...*}_{k-2} j_i j_i \underbrace{*...*}_{k-2} j_i x_{j_i} \underbrace{*...*}_{k-2}$$

which means that $\mathcal{C}_{j_i}$ contains

- vertices y_{j_i}, u_{j_i} and x_{j_i},
- all remaining vertices from U_{j_i} – in the positions denoted by j_i in order given by some $P_{j_i}(u_{j_i})$,
- some other vertices in positions denoted by $*$ (this will be decided later).

Furthermore, let $\mathcal{C}_{j_{t-1}}$ have the form

$$y_{j_{t-1}} \underbrace{*...*}_{k-3} u_{j_{t-1}} j_{t-1} \underbrace{*...*}_{k-2} j_{t-1} j_{t-1} \underbrace{*...*}_{k-2} j_{t-1} j_{t-1} \underbrace{*...*}_{k-2} \cdots j_{t-1} j_{t-1} \underbrace{*...*}_{k-2} j_{t-1} j_{t-1} \underbrace{*...*}_{k-2} j_{t-1} x_{j_{t-1}} \underbrace{q...q}_{k-2}.$$

Note that $\mathcal{C}_{j_{t-1}}$ differs slightly from the other open chains $\mathcal{C}_{j_i}$ because it is supossed to have vertices from $U_q \setminus e_0$ on the last positions. This will be important later, when we will concatenate constructed open chains. Since e_0 is not an edge of H we have three cases

C1. $|e_0 \cap U_0| = 0$ and $|e_0 \cap U_{i_1}| = 1$.

C2. $|e_0 \cap U_0| = 0$, $e_0 \cap U_{i_1} = \{u, v\}$ with $uv \notin E(G(U_{i_1}))$.

C3. $|e_0 \cap U_0| = 1$, $e_0 \cap U_{i_2} = \{u\}$ with $u \notin \{x_{i_2}, y_{i_2}\}$.

Consider C1: Assume first that $\{i_1, i_2\} \neq \{1, 2\}$. Let $\{u\} = e_0 \cap U_{i_1}$. Suppose that $u \notin \{x_{i_1}, y_{i_1}\}$. Since $N(x_{i_1}) \cap N(y_{i_1}) = \emptyset$ in $G(U_{i_1})$, we may assume that u is not a neighbour of y_{i_1} in $G(U_{i_1})$ (the case when u is not a neighbour of x_{i_1} in $G(U_{i_1})$ is analogous). Let $\mathcal{C}_0^+$ be an open chain of the form

$$y_{i_1} \underbrace{*...*}_{k-3} x_{i_1} i_1 \underbrace{*...*}_{k-2} i_1 i_1 \underbrace{*...*}_{k-2} \cdots i_1 i_1 \underbrace{*...*}_{k-2} i_1 \underbrace{u i_3 ... i_k i_2}_{e_0} i_2 i_2 i_2 ... i_2 0$$

which means that $\mathcal{C}_0^+$ contains

- vertices x_{i_1}, y_{i_1} and u,
- all remaining vertices from U_{i_1} – in the positions denoted by i_1 in order given by $P_{i_1}(u)$
- all vertices from U_{i_2} – in the positions denoted by i_2 (or, if $i_2 = q$, all vertices from $U_q \setminus V(\mathcal{C}_{j_{t-1}})$),
- some other vertices in positions denoted by $*$ (this will be decided later),
- a vertex from U_0 – in the position denoted by 0.

If $u = x_{i_1}$ (or $u = y_{i_1}$) then let $\mathcal{C}_0^+$ has the form

$$y_{i_1} \underbrace{*...*}_{k-3} i_1 i_1 \underbrace{*...*}_{k-2} i_1 i_1 \underbrace{*...*}_{k-2} \cdots i_1 i_1 \underbrace{*...*}_{k-2} i_1 \underbrace{x_{i_1} i_3 ... i_k i_2}_{e_0} i_2 i_2 i_2 ... i_2 0$$

(or analogous one), where vertices from $U_{i_1} \setminus \{y_{i_1}\}$ are arranged in order given by some path $P_{i_1}(v)$ for an arbitrary vertex $v \in U_{i_1}$ which is not a neighbour of y_{i_1} in $G(U_{i_1})$. Note that the sequence $C_0^+ \setminus \{0\}$ is (by far) longer than an analogous sequence (6).

Assume now that $\{i_1, i_2\} = \{1, 2\}$. Let C_0^+ be an open chain of the form

$$y_1 \underbrace{*...*}_{k-3} 11 \underbrace{*...*}_{k-2} 11 \underbrace{*...*}_{k-2} 11 \underbrace{*...*}_{k-2} \cdots \underbrace{*...*}_{k-2} 1 \underbrace{1 i_3 ... i_k 2}_{e_0} 22 ... x_2 0$$

if $x_2 \notin e_0$ (or analogous one if $y_2 \notin e_0$, i.e. with x_2 replaced by y_2). The fact that the vertex x_2 is present on the penultimate position in C_0^+ will be very important later, when we will concatenate open chains.
If $x_2, y_2 \in e_0$, then let C_0^+ have the form

$$y_1 \underbrace{*...*}_{k-3} 11 \underbrace{*...*}_{k-2} 11 \underbrace{*...*}_{k-2} 11 \underbrace{*...*}_{k-2} \cdots 11 \underbrace{*...*}_{k-2} 1 \underbrace{1 i_4 ... i_k y_2 x_2}_{e_0} 0 \tag{9}$$

Note that the latter open chain differs significantly from the previous ones. However, also in this case the sequence $C_0^+ \setminus \{0\}$ is longer than an analogous sequence (6), but this time only by one vertex.

Consider now C2. Thus, $e_0 = \{u, v, i_3, .., i_k\}$ where $u, v \in U_{i_1}$ and $uv \notin E(G(U_{i_1}))$. Since $G(U_{i_1}) \cup uv$ has a hamiltonian cycle containing the edge $x_{i_1} y_{i_1}$, there are two paths (possibly one of them trivial), one from x_{i_1} to u (or v), say $P_{i_1}(x_{i_1}, u)$, and the other from y_{i_1} to v (resp. u), say $P_{i_1}(y_{i_1}, v)$, which together cover all vertices of $G(U_{i_1})$. Without loss of generality we may assume that $P_{i_1}(x_{i_1}, u)$ is a path from x_{i_1} to u and has even order while $P_{i_1}(y_{i_1}, v)$ is a path from y_{i_1} to v and has odd order. In this case let $\mathcal{C}_0^+$ have the form

$$y_{i_1} \underbrace{*...*}_{k-2} i_1 i_1 \underbrace{*...*}_{k-2} \cdots i_1 i_1 \underbrace{*...*}_{k-2} i_1 \underbrace{v i_3 ... i_k u}_{e_0} i_1 \underbrace{*...*}_{k-2} i_1 i_1 \underbrace{*...*}_{k-2} \cdots i_1 i_1 \underbrace{*...*}_{k-2} i_1 x_{i_1} \underbrace{q...q}_{k-2} 0$$

which means that $\mathcal{C}_0^+$ contains

- vertices y_{i_1}, x_{i_1}, u, v,
- all remaining vertices from U_{i_1} – in the positions denoted by i_1 in order given by $P_{i_1}(x_{i_1}, u)$ and $P_{i_1}(y_{i_1}, v)$,
- some other vertices in positions denoted by $*$ (this will be decided later),
- some vertices from U_q – in the positions denoted by q,
- a vertex from U_0 – in the position denoted by 0.

Note that the sequence $C_0^+ \setminus \{0\}$ is longer by one vertex than an analogous sequence (6).

Consider C3. Assume first that $\{i_2, i_3\} \neq \{1, 2\}$. Hence, $e_0 = \{0, u, i_3, ..., i_k\}$ where $u \in U_{i_2} \setminus \{x_{i_2}, y_{i_2}\}$. Since $N(x_{i_2}) \cap N(y_{i_2}) = \emptyset$ in $G(U_{i_2})$, we may assume that $u \notin N(y_{i_2})$ (the case when $u \notin N(x_{i_2})$ is analogous). Then there is a hamiltonian path from u to y_{i_2} in $G(U_{i_2})$. In this case let $\mathcal{C}_0^+$ have the form

$$y_{i_2} \underbrace{*...*}_{k-2} i_2 i_2 \underbrace{*...*}_{k-2} \cdots i_2 i_2 \underbrace{*...*}_{k-2} i_2 \underbrace{u i_3 ... i_k 0}_{e_0},$$

where vertices from U_{i_2} are arranged in order given by the hamiltonian path from u to y_{i_2} in $G(U_{i_2})$. Note that also in this case the sequence $C_0^+ \setminus \{0\}$ is longer (by one vertex) than an analogous sequence (6).

If $\{i_2, i_3\} = \{1, 2\}$, then $e_0 = \{0, u, v, i_4, ..., i_k\}$ with $u \in U_1 \setminus \{x_1, y_1\}$ and $v \in \{x_2, y_2\}$. Without loss of generality we assume that $v = x_2$. In this case $\mathcal{C}_0^+$ has the form

$$y_1 \underbrace{*...*}_{k-2} 11 \underbrace{*...*}_{k-2} \cdots 11 \underbrace{*...*}_{k-2} 1 \underbrace{u i_4 ... i_k x_2 0}_{e_0}.$$

Similarily as in (9), it is important to have a vertex x_2 on the penultimate position of $\mathcal{C}_0^+$.

Finally, let $\mathcal{C}_{j_1}$ be an open chain of the form

$$y_{j_1}\underbrace{*...*}_{k-3}u_{j_1}j_1\underbrace{*...*}_{k-2}j_1j_1\underbrace{*...*}_{k-2}\ \cdots\ j_1j_1\underbrace{*...*}_{k-2}j_1x_{j_1}\underbrace{q...q}_{k-2}$$

where vertices from $U_{j_1}\setminus\{y_{j_1}\}$ are arranged in order given by a path $P_{j_1}(u_{j_1})$.

Note that sequences $\mathcal{C}_0^+$, $\mathcal{C}_{j_1}$, ..., $\mathcal{C}_{j_{t-1}}$ contain all vertices from sets U_i with $i \in J$. Now we will insert vertices in positions denoted by $*$. Note that in order to assure that the resulting sequence is an open chain in H, it sufices to insert in $\mathcal{C}_{j_i}$ vertices from sets U_i with $i > j_i$, because then $U_{j_i} = \min(V(\mathcal{C}_{j_i})$. Similarly, in $\mathcal{C}_0^+$ it sufices to insert vertices from sets U_i with $i > i_1$ in cases C1 and C2, or vertices from sets U_i with $i > i_2$ in case C3. Firstly, we want to insert all vertices from $\left(\bigcup_{i\in I} U_i\right)\setminus U_q$ in $\mathcal{C}_0^+\cup\mathcal{C}_{j_1}$. Some of them are already inserted. The remaining ones will be inserted in $\mathcal{C}_{j_1}$, if $j_1 = 1$, or in $\mathcal{C}_0^+$ if $j_1 \neq 1$ (clearly, if $j_1 \neq 1$, then $i_1 = 1$ in cases C1 and C2 or $i_2 = 1$ in case C3). In case C1 we have already inserted all vertices from $U_{i_1}\cup U_{i_2}$ in $\mathcal{C}_0^+$ if $\mathcal{C}_0^+$ is different from (9). Thus, since $|I| \le k$ it remains to insert at most $(k-2)(2\alpha+1)$ vertices in positions denoted by $*$. On the other hand, if $\mathcal{C}_0^+$ has the from (9), then it contains all vertices from $U_{i_1} = U_1$. Moreover, $|I| \le k-1$ because e_0 contains two vertices from U_2. Hence, it remains to insert at most $(k-2)(2\alpha+1)$ vertices in positions denoted by $*$. Similarly, since in cases C2 and C3 we have $|I| \le k-1$ and we have already inserted all vertices from U_{i_1} (or U_{i_2}) in $\mathcal{C}_0^+$, it also remains to insert at most $(k-2)(2\alpha+1)$ vertices in positions denoted by $*$. Therefore, if $j_1 \neq 1$, then we can insert all remaining vertices from $\left(\bigcup_{i\in I} U_i\right)\setminus U_q$ in $\mathcal{C}_0^+$ because it has at least

$$\frac{(2\beta+1)-3}{2}(k-2)+k-3 \ge (2\alpha+1)(k-2) \tag{10}$$

($\beta \ge 2\alpha+2$ by the assumption) positions $*$. Similarly, if $j_1 = 1$, then we can insert all remaining vertices from $\left(\bigcup_{i\in I} U_i\right)\setminus U_q$ in $\mathcal{C}_{j_1}$. At this stage the sequences contain already all vertices from sets U_i, $i \in (I\cup J)\setminus\{q\}$. Consider now vertices from U_q. Our next goal is to fill up all positions $*$ in $\mathcal{C}_0^+$ by different vertices from U_q, but only if $j_1 = 1$. To achieve this we need sufficiently large number of vertices in U_q. We have already used $k-2$ vertices from U_q in $\mathcal{C}_{j_1}$, $k-2$ vertices from U_q in $\mathcal{C}_{j_{t-1}}$ and at most $k-1$ vertices from U_q in e_0. On the other hand, if $j_1 = 1$, we have at most $\frac{(2\alpha+1)-3}{2}(k-2)+k-2 = \alpha(k-2)$ positions $*$ in $\mathcal{C}_0^+$. Therefore we need that

$$(\beta-\alpha)(k-2)+\frac{t}{2}(k-4)+1 = |U_q| \ge \alpha(k-2)+2(k-2)+(k-1).$$

This is satisfied by the assumption (5) on α and β. However, if $i_2 = q$ then in case C1 we have already used all vertices from U_q in the positions denoted by i_2 in $\mathcal{C}_0^+$. In such situation we simply delete all positions $*$ without spoiling the open chain. However, we have to be sure that the length of the resulting sequence (minus a vertex from U_0) exceeds (7). This is possible when

$$|U_{i_1}|+|U_q|-2(k-2)-(k-1) > \alpha k+(k-2)$$

(some vertices from U_q are already used in different places), which is satisfied by the assumption (5) on α and β.

Now, since $j_1 < ... < j_{t-1} < j_t < ... < j_{q-|I|}$ we can arbitrarily fill up positions $*$ in all $\mathcal{C}_{j_1}$, ..., $\mathcal{C}_{j_{t-1}}$ (and also in $\mathcal{C}_0^+$, if $i_1 = 1$ in cases C1 and C2, or if $i_2 = 1$ in case C3) by different vertices from sets U_i with $i \in R$ and by not previously used vertices from U_q. Recall also, that after deleting any number of vertices from positions $*$ we still have an open chain. Hence, we can fill up these positions until the moment that we do not have any available vertices. If this happens then it means that constructed open chains contain all vertices from $V(H) \setminus U_0$. Otherwise, since $\mathcal{C}_0^+ \setminus \{0\}$ is in each case C1, C2 and C3 longer than an analogous open chain (6), we have

$$|\mathcal{C}_0^+| - 1 + \sum_{i=1}^{t-1} |\mathcal{C}_{j_i}| > n - 1 - |U_0|,$$

see formula (8). Thus in each situation

$$|\mathcal{C}_0^+| + \sum_{i=1}^{t-1} |\mathcal{C}_{j_i}| \geq n - |U_0| + 1, \tag{11}$$

Consider now the following cyclic ordering $\mathcal{C}$:

$$\left(\mathcal{C}_0^+ \mathcal{C}_{j_1} 0 \mathcal{C}_{j_2} 0 \cdots \mathcal{C}_{j_{t-1}} 0\right).$$

Since $|U_0| = t$, by formula (11) we have

$$|\mathcal{C}| \geq n.$$

Moreover, $\mathcal{C}$ is a (closed) chain. Indeed, $\mathcal{C}_0^+$ and all $\mathcal{C}_{j_i}$ are open chains. Hence, it suffices to check that each k-tuple (consisting of consecutive vertices) of $\mathcal{C}$ that contains a vertex from U_0 is an edge in H. Note that each such k-tuple e contains either vertex x_{j_i} or $y_{j_{i+1}}$ (eventually either $x_{j_{t-1}}$ or y_{i_1}). Furthermore, if $j_1 \leq 2$, then either $j_i = \min(e \setminus U_0)$ or $j_{i+1} = \min(e \setminus U_0)$ (either $j_{t-1} = \min(e \setminus U_0)$ or $i_1 = \min(e \setminus U_0)$, respectively). Thus, every such k-tuple is an edge of type 3 or 4 from Definition 3. Note however, that we have to be careful in the following concatenations $\mathcal{C}_{j_{t-1}} 0 \mathcal{C}_0^+$, $\mathcal{C}_0^+ \mathcal{C}_{j_1}$ and $\mathcal{C}_{j_1} 0 \mathcal{C}_{j_2}$. In order to assure that e is an edge (of type 3 or 4) it is required that $\min(e \setminus U_0) = i_1$ (or i_2 in case C3) in the first concatenation, $\min(e \setminus U_0) = j_1$ in the second and $\min(e \setminus U_0) = j_2$ in the third. This is the reason why we require vertices from U_q in the last positions in $\mathcal{C}_{j_1}$. If $j_1 \geq 3$ then the presence of x_2 in the penultimate position of $\mathcal{C}_0^+$ implies that each k tuple e having non-empty intersections with both $\mathcal{C}_0^+$ and $\mathcal{C}_{j_i}$ is an edge of H. Thus, $\mathcal{C}$ is a hamiltonian chain.

Finally, if $t = \Theta(n^{1/2})$, $\alpha = \Theta(n^{1/2})$ and $\beta = \Theta(n^{1/2})$, then $|E(\mathcal{H}_k(t, \alpha, \beta))| = \Theta(n^{k-1})$, by Proposition 5. □

Corollary 8 *For every $k \geq 3$ we have*

$$sat(n, C_n^{(k)}) = \Theta(n^{k-1}).$$

Proof. Let $n = |V(H_k(t, \alpha, \beta))|$. By Proposition 4, $n = (t-1)k\alpha + k\beta + t(k-1) + 1$. By Theorem 7, it is enough to prove that each sufficiently large integer n can be represented in such form with α, β, t being integers that satisfy (5) and $\alpha, \beta, t = \Theta(n^{1/2})$. Suppose first that k is even. Let $t = \left\lfloor \frac{2\sqrt{n}}{3k} \right\rfloor + \epsilon$ where $0 \leq \epsilon \leq k-1$ is chosen in such a way that $t + n - 1 \equiv 0 \mod k$. Thus $n' := n - 1 - t(k-1) \equiv 0 \mod k$. Let $n'' = n'/k$. Clearly, $n'' = x(t-1) + r$ where x and r are integers and $0 \leq r \leq t-2$. Let $y = \left\lceil \frac{2x+3}{t+1} \right\rceil + 1$. We set $\alpha = x - y$, $\beta = r + y(t-1)$. Thus,

$y(t-1) \le \beta \le y(t-1)+t-2$. Now,

$$\begin{aligned}\beta \ge & y(t-1) \ge \left(\frac{2x+3}{t+1}+1\right)(t-1) = 2x+3-2\frac{2x+3}{t+1}+t-1\\ \ge & 2x+3-(2y-2)+t-1 = 2\alpha+t+4, \text{ and}\\ \beta \le & y(t-1)+t-2 \le \left(\frac{2x+3}{t+1}+2\right)(t-1)+t-2 = 2x+3-2\frac{2x+3}{t+1}+3t-4\\ = & 2x+3-2\left(\frac{2x+3}{t+1}+2\right)+3t \le 2x+3-2y+3t = 2\alpha+3t+3.\end{aligned}$$

Furthermore, $\alpha = \frac{3}{2}\sqrt{n}+O(1)$. Thus, for suficiently large n, $\alpha \ge 2kt = \frac{4}{3}\sqrt{n}+O(1)$. Therefore, all conditions (5) are satisfied for sufficiently large n. Suppose now that k is odd. Hence we have to choose t even. Therefore, if $\left\lfloor \frac{2\sqrt{n}}{3k} \right\rfloor + \epsilon$ is odd we can take $t = k + \left\lfloor \frac{2\sqrt{n}}{3k} \right\rfloor + \epsilon$. All previous calculations remain valid. □

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