Sparse graphs of girth at least five are packable

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Abstract

The following statement was conjectured by Faudree, Rousseau, Schelp and Schuster in 1981: every non-star graph G with girth $g \geq 5$ is packable.

The conjecture was proved by Faudree et al. with the additional condition that the size $||G|| \leq \frac{6}{5}n - 2$, where n is the order of G. In this paper, for each integer $k \geq 3$, we prove that every non-star graph with girth $g \geq 5$ such that $||G|| \leq \frac{2k-1}{k}n - \alpha_k(n)$ is packable, where $\alpha_k(n)$ is $o(n)$ for every k. This implies that the conjecture is true for sufficiently large planar graphs.

1 Introduction

We deal with finite, simple graphs without loops and multiple edges. We use standard graph theory notation. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of G is denoted by $|G|$ and the size is denoted by $||G||$. By $N_G(x)$ we denote the set of vertices adjacent to x in G. For a vertex set X, the set $N_G(X)$ denotes the external neighbourhood of X in G, i.e.

$$N_G(X) = \{y \in V(G) \setminus X : y \text{ is adjacent with some } x \in X\}.$$

We say that G is *packable in its complement* (G is packable, in short) if there is a permutation σ on $V(G)$ such that if xy is an edge in G, then $\sigma(x)\sigma(y)$ is not an edge in G. Thus, G is packable if and only if G is a subgraph of its complement. If $\sigma(x) \neq x$ for every vertex $x \in V(G)$, then we say that G is *fixed-point-free* packable. In the rest of this paper, in the notation of permutations we omit fixed points.

One of the classical results in the theory of packing graphs is the following theorem, proved independently in [1, 3, 9].

Theorem 1 ([1, 3, 9]) *Let G be a graph of order n and size $||G|| \leq n - 2$. Then G is packable.*

This theorem cannot be improved by raising the size of G since a star on n vertices is not packable. In [5] all non-packable graphs with order n and size n are presented. Each of the non-packable graphs has a cycle of length 3 or 4. These results motivate the following conjecture:

Conjecture 2 ([5]) *Every non-star graph G with girth $g \geq 5$ is packable.*

Woźniak [11] proved that every non-star graph G with girth $g \geq 8$ is packable. His result was improved by Brandt [2] who showed that every non-star graph G with girth $g \geq 7$ is packable. Another, relatively short proof of Brandt's result was given in [6]. Recently, the present authors proved [7] the following theorem.

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Theorem 3 ([7]) *Let G be a non-star graph with girth $g \geq 6$. Then G is packable.*

Note that this theorem implies that Conjecture 2 is true for bipartite graphs. Other results concerning Conjecture 2 were obtained by adding extra conditions on the size of a graph.

Theorem 4 ([5]) *Let G be a non-star graph with girth $g \geq 5$, order n and size $||G|| \leq \frac{6}{5}n - 2$. Then G is packable.*

In this paper we prove the following statement.

Theorem 5 *Let $k \geq 3$ be an integer. If G is a non-star graph with girth $g \geq 5$, order n and size $||G|| \leq \frac{2k-1}{k}n - 4\frac{k-1}{k}(2\sqrt{n}+1) - 2k(4k-5)$, then G is packable.*

By taking $k = 3$ we note that our new upper bound for the size of G is greater than the bound in Theorem 4 for $n > 285$. As a corollary of Theorem 5 we obtain that Conjecture 2 is true for large planar graphs.

Corollary 6 *Let G be a planar graph of order $n \geq 3850$. If G is a non-star graph with girth $g \geq 5$, then G is packable.*

Proof. Let f denote the number of faces of G. Since $g \geq 5$, every face of G has at least 5 edges. On the other hand every edge belongs to two faces. Hence, $2||G|| \geq 5 \cdot f$. Thus, by Euler's formula $||G|| + 2 = n + f$, we have $||G|| \leq \frac{5}{3}(n-2)$. Note that for $n \geq 3850$ we have

$$\frac{5}{3}(n-2) \leq \frac{11}{6}n - \frac{20}{6}(2\sqrt{n}+1) - 228, \tag{1}$$

where the RHS of (1) is our new bound on the size of G taken for $k = 6$. Indeed, the above inequality is equivalent to the following one

$$n - 40\sqrt{n} - 1368 \geq 0,$$

which is satisfied for $\sqrt{n} \geq 62.0476$ (and so for $n \geq 3849.91$). Hence G is packable by Theorem 5. □

We recall further classical results of packing theory which will be used in the proof of Theorem 5.

Theorem 7 ([9]) *Let G_1 and G_2 be graphs of order n each, and maximum degrees $\Delta(G_1)$ and $\Delta(G_2)$, respectively. If $2\Delta(G_1)\Delta(G_2) < n$, then the complete graph K_n contains edge-disjoint copies of G_1 and G_2.*

Theorem 8 ([10]) *Let G be a graph of order n and size $||G|| \leq n-2$. Then G is fixed-point-free packable.*

The paper is organized as follows. In the next section we prove some preliminary lemmas. They will be needed in the main part of the proof of Theorem 5 presented in the third section. The general idea of the proof of this part has also been succesfully applied in [8].

2 Lemmas

We use the following result from [7].

Lemma 9 ([7]) *Let G be a graph and $k \geq 1$, $l \geq 1$ be any positive integers. If there is a set $U = \{v_1, ...v_k, v_{k+1}, ..., v_{k+l}\} \subset V(G)$ of $k+l$ independent vertices of G such that*

1. *vertices $v_1, ..., v_k$ have degrees at most l and vertices $v_{k+1}, ..., v_{k+l}$ have degrees at most k;*
2. *vertices of U have mutually disjoint neighborhoods, i.e. $N(v_i) \cap N(v_j) = \emptyset$ for $i \neq j$;*

3. there is a packing σ' of $G-U$

then there exists a packing σ of G such that $\sigma|_{G-U}=\sigma'$.

In fact, the conclusion in the above lemma is slightly stronger than that in [7]. However, it can be derived directly from the proof of Lemma 3 in [7], without changing any line of the proof.

For convenience, let $\alpha_k(n)=4\frac{k-1}{k}(2\sqrt{n}+1)+2k(4k-5)$. In many places in the proofs we will use the following observation.

Proposition 10 *Let $k\geq 3$. Let G be a graph of order n and size $||G||\leq \frac{2k-1}{k}n-\alpha_k(n)$. If G' is a graph that arises from G by deleting m vertices and at least $2m$ edges, then $||G'||\leq \frac{2k-1}{k}n'-\alpha_k(n')$, where n' is the order of G'.*

Proof. Note that $\alpha_k(n)$ is increasing with respect to n. Thus,

$$\begin{aligned}||G'|| &\leq \frac{2k-1}{k}n-\alpha_k(n)-2m=\frac{2k-1}{k}(n-m)-\alpha_k(n)-\frac{m}{k}\\ &< \frac{2k-1}{k}(n-m)-\alpha_k(n-m)=\frac{2k-1}{k}(n')-\alpha_k(n').\end{aligned}$$

□

Lemma 11 *Let $k\geq 3$. Let G be a non-star graph of order n, girth $g\geq 5$ and size $||G||\leq \frac{2k-1}{k}n-\alpha_k(n)$. If $n\leq 10k^2$, then G is packable.*

Proof. If $||G||\leq \frac{6}{5}n-2$, then G is packable by Theorem 4. Note that

$$\begin{aligned}&\frac{2k-1}{k}n-4\frac{k-1}{k}(2\sqrt{n}+1)-2k(4k-5)\leq \frac{6}{5}n-2 \Longleftrightarrow\\ &\frac{4k-5}{5k}n-2k(4k-5)-4\frac{k-1}{k}(2\sqrt{n}+1)+2\leq 0 \qquad (2)\end{aligned}$$

Since $2-4\frac{k-1}{k}(2\sqrt{n}+1)\leq 0$, (2) holds if $n\leq 10k^2$. Thus, if $n\leq 10k^2$, then G is packable by Theorem 4. □

Let T_1, T_2 be vertex-disjoint trees (we include isolated vertices as trivial trees). Let x be a vertex belonging neither to the vertex set of T_1 nor T_2 and let B be any non-empty set of edges containing x. Then a graph $H=(V,E)$ is called *a starry tree* if $V=V(T_1)\cup V(T_2)\cup\{x\}$ and $E=E(T_1)\cup E(T_2)\cup B$. A vertex x we call *a middle vertex* of H. Note that a starry tree need not be connected.

Lemma 12 *Let H be a non-star starry tree of girth $g\geq 5$. Then there is a packing of H such that the middle vertex x of H is the image of one of its neighbors.*

Proof. Let $P_5(3)$ be a starry tree with vertex set $\{1,2,3,4,5\}$ and edge set $\{i(i+1); i=1,...,4\}$, and with the middle vertex $x=3$. First, note that $(1435)(2)$ is a packing of $P_5(3)$ as required.

In what follows, we will prove a slightly stronger statement than the one formulated in the lemma. Namely, we will prove that if H is a non-star and different from $P_5(3)$ starry tree of girth $g\geq 5$, then there is a fixed-point-free packing of H such that the middle vertex x of H is the image of one of its neighbors. The proof of this statement is by induction on $|T_1|+|T_2|$.

If $|T_1|+|T_2|=2$ then the claim obviously holds. Assume that $|T_1|+|T_2|\geq 3$. By a leaf of a tree we mean a vertex with degree equal to 1 (in particular, the vertex of a one-vertex tree is not a leaf). Observe that there is at least one leaf in T_1 or in T_2. We distinguish two cases:

Case 1. The middle vertex x is adjacent to all leaves in T_1 and T_2.

Case 2. There exists a leaf l in T_1 or T_2 such that the middle vertex x is not adjacent to l.

Consider Case 1. Without loss of generality we can assume that $|T_2|\geq |T_1|$. Thus, by the girth assumption, $|V(T_2)|\geq 4$. Furthermore, again by the girth assumption, every vertex in $V(T_2)$

is a neighbor of at most one leaf of T_2 (the same holds for T_1 as well). Let L be the set of all leaves of T_2. Hence, $|N_{T_2}(L)| = |L|$. If every vertex $v \in N_{T_2}(L)$ has degree 3 or more in T_2, then

$$\begin{aligned} 2||T_2|| &= \sum_{v \in V(T_2)} \deg v \geq |L| + 3|N_{T_2}(L)| + 2(n - |L| - |N_{T_2}(L)|) \\ &= |L| + 3|L| + 2(n - 2|L|) = 2n, \end{aligned}$$

which is not possible. Hence, there exists a leaf l'' in T_2 such that l'' is a neighbor of a vertex z with degree 2 in T_2. Let z' be the neighbor of z other than l''. Note that neither z nor z' is connected with x since $g \geq 5$. Let $H' = H - \{l'', z, z', x\}$. Observe that by Theorem 8, H' is fixed-point-free packable since $||H'|| \leq |H'| - 2$. Let σ' be such a packing of H'. Then $(l'', x, z, z')\sigma'$ is a packing as required of H .

Consider Case 2. Without loss of generality we assume that $l \in V(T_1)$. Let l' denote the neighbour of l in T_1. Consider a graph $H' = H - \{l\}$. Suppose that H' is a star. Hence, $V(H) = \{x, l, l', u\}$ with $\{l, l'\} = V(T_1)$ and $\{u\} = V(T_2)$. Since $g \geq 5$, xl is not an edge of H. Thus (x, l, l', u) is a packing as required of H. So we may assume that H' is not a star. On the other hand, if $H' = P_5(3)$ then by the girth assumption H is a path of length five. Hence, $(l23541)$ is a packing as required of H.

Therefore, we may assume that H' is not a star and H' is different from $P_5(3)$. Thus, by the induction hypothesis, there exists a fixed-point-free packing σ' of H' such that x is the image of one of its neighbors. If $l'\sigma'(l')$ is not an edge of H, then σ_1 such that $\sigma_1(\sigma'^{-1}(l')) = l$, $\sigma_1(l) = l'$ and $\sigma_1(v) = \sigma'(v)$ for remaining vertices is a packing as required of H. On the other hand, if $l'\sigma'(l')$ is an edge of H, (and so $\sigma'^{-1}(l')l'$ is not an edge in H since σ' is a packing of H'), then σ_2 such that $\sigma_2(l') = l$, $\sigma_2(l) = \sigma'(l')$ and $\sigma_2(v) = \sigma'(v)$ for remaining vertices is a packing as required of H, unless $\sigma'(l') = x$, in which case our additional assumption is not satisfied.

So we may assume that $\sigma'(l') = x$ and that $l'x$ is an edge in H. In particular, $\sigma'(x) \neq l'$. Observe that $N_{T_1}(l') \setminus \{l\} \neq \emptyset$ or T_1 is the edge ll'. Assume first that there is a vertex $z \in N_{T_1}(l') \setminus \{l\}$. Note that, by the girth assumption, z is not adjacent to x. Moreover, $\sigma'^{-1}(z)\sigma'^{-1}(l')$ is not an edge in H since σ' is a packing of H'. Let σ_3 be such that $\sigma_3(\sigma'^{-1}(z)) = l$, $\sigma_3(l) = z$ and $\sigma_3(v) = \sigma'(v)$ for remaining vertices. Then σ_3 is a packing as required of H. Therefore we must have that T_1 is the edge ll'. Moreover, without loss of generality we can assume that one of the following holds:

- $|T_2| = 1$
- $|T_2| = 2$ (and $H \neq P_5(3)$)
- all of the leaves in T_2 are adjacent to x.

Otherwise, we can replace T_1 by T_2 and using the above arguments we obtain a required packing of H. For $|T_2| = 1$ or $|T_2| = 2$ the existence of a packing as required is obvious. Suppose then that all of leaves in T_2 are adjacent to x. Then we proceed as in Case 1. □

Lemma 13 *Let $k \geq 3$. Let G be a graph with minimum order n such that G is a non-star, non-packable graph with girth $g \geq 5$ and size $||G|| \leq \frac{2k-1}{k}n - \alpha_k(n)$. Then G has no isolated vertices.*

Proof. Suppose for a contradiction, that y is an isolated vertex of G. By Proposition 11, $n > 10k^2 \geq 90$. Hence, $\Delta(G) \geq 7$. Indeed, otherwise $2\Delta^2(G) \leq 72 < n$ and G is packable by Theorem 7. Let $x \in G$ with $\deg x \geq 7$. Note that since $g \geq 5$, the graph $G' = G - \{x, y\}$ is not a star (otherwise x would be a vertex of some cycle of order 3 or 4). Furthermore, as we delete two vertices and at least 7 edges, $||G'|| \leq \frac{2k-1}{k}|G'| - \alpha_k(|G'|)$, by Proposition 10. Thus, by the minimality assumption there is a packing σ' of G'. Then $(xy)\sigma'$ is a packing of G, a contradiction.

□

Now we are going to show a similar result for vertices of degree 1, namely that if the number of vertices of degree 1 is large, then G is packable. However, first we need the following two preparatory lemmas.

Lemma 14 *Let G be a graph with girth $g \geq 5$ such that $S = \{v \in V(G) : \deg v \geq 3\}$ is a set of independent vertices in G. If G is not a star, then G is packable.*

Proof. Let G be a counterexample of the above lemma with minimal order n. By Theorem 3, we may assume that G contains a cycle of length 5. Since vertices of degree greater than or equal to 3 are independent, this cycle contains two adjacent vertices x, y with degree 2. Let x', y' be the neighbours of x, y different from y and x, respectively. Let $G' = G - \{x, y, x', y'\}$.

If G' is not a star then it is packable by the minimality assumption. Let σ' be a packing of G'. Then $(x, x', y, y')\sigma'$ is a packing of G.

If G' is a star, then $|S| = 1$. Thus $\deg x' = 2$ or $\deg y' = 2$. Without loss of generality we may assume that $\deg x' = 2$. Let $s \in S$. Hence, $G'' = G - \{s, x', x, y\}$ is different from a star. Thus, there is a packing σ'' of G''. Then $(x, y, x', s)\sigma''$ is a packing of G.

□

Lemma 15 *Let G be a graph of order $n \geq 1$. Let $U, W \subset V(G)$ be disjoint sets of independent vertices of G such that*

1. *vertices of W are isolated in G;*
2. *m vertices of U have degree at most 1;*
3. *vertices of U have mutually disjoint neighborhoods, i.e. $N(v_i) \cap N(v_j) = \emptyset$ for $i \neq j$;*
4. *there exists a packing σ' of $G - (U \cup W)$.*

If $|W| \geq \min\{\lfloor 2\sqrt{n} \rfloor, |U| - m + 1\}$ then G is packable. Moreover, there exists a packing σ of G such that $\sigma|_{G-(U\cup W)} = \sigma'$.

Proof. Let $G' := G - (U \cup W)$. Let $W = \{w_1, ..., w_t\}$ and $U = \{u_1, ..., u_s\}$ with $\deg u_1 \leq \deg u_2 \leq ... \leq \deg u_s$. If $|W| \geq |U| - m + 1$, then we have two cases:

Case 1. $m \leq 1$ (hence $t \geq s$): in this case $\sigma'' := (w_1, u_1)(w_2, u_2)...(w_s, u_s)\sigma'$ is a packing of G with the required property.

Case 2. $m \geq 2$ (so $t \geq s - m + 1$): in this case $\sigma'' := (w_1, u_{m+1})(w_2, u_{m+2})...(w_{s-m}, u_s)\sigma'$ is a packing of $G - \{u_1, ..., u_m\}$. Moreover, $\deg u_i \leq 1$ for $i = 1, ..., m$. Hence, this packing can be extended to a packing of G with the required property, by Lemma 9.

So we may assume that $|U| - m + 1 > \lfloor 2\sqrt{n} \rfloor$. Hence $|W| \geq \lfloor 2\sqrt{n} \rfloor$. If $\deg u_s < \lfloor \frac{s}{2} \rfloor + 1$, then we can extend σ', using Lemma 9 (with $k = \lfloor \frac{s}{2} \rfloor$ and $l = \lceil \frac{s}{2} \rceil$), to a packing of $G[V(G') \cup U]$ $(= G - W)$. Clearly, in this way we obtain also a packing of G because W consists of isolated vertices. Thus we may assume that

$$\deg u_s \geq \left\lfloor \frac{s}{2} \right\rfloor + 1.$$

Consider now $U_1 := U - u_s$. In the same way as before, if σ' cannot be extended, using Lemma 9, to a packing of $G[V(G') \cup U_1]$, then

$$\deg u_{s-1} \geq \left\lfloor \frac{s-1}{2} \right\rfloor + 1$$

and so on. Let $U_0 = U$ and let $U_l = U \setminus \{u_s, ..., u_{s-l+1}\}$ for $1 \leq l \leq s$. Let p, $0 \leq p \leq s-1$, be the smallest integer such that $\deg u_{s-p-1} \leq \left\lfloor \frac{s-p-1}{2} \right\rfloor$, or $p = s-1$ if $\deg u_{s-i} \geq \leq \left\lfloor \frac{s-i}{2} \right\rfloor + 1$ for every

$i = 1, ..., s$. Note that, by Lemma 9, σ' can be extended to a packing σ_{p+1} of $G[V(G') \cup U_{p+1}]$. We will show that $p \leq t-1$. Indeed,

$$\sum_{i=1}^{s} \deg u_i \geq \sum_{j=0}^{p} \left(\left\lfloor \frac{s-j}{2} \right\rfloor + 1 \right) \geq \sum_{j=0}^{p} \left(\frac{s+1-j}{2} \right) = (p+1)\frac{2s+2-p}{4}$$
$$\geq \frac{(p+1)p}{4}.$$

Because vertices in U have disjoint neighborhoods and $|U| \geq \lfloor 2\sqrt{n} \rfloor$,

$$n \geq |U| + |W| + \sum_{i=1}^{s} \deg u_i \geq 4\sqrt{n} - 2 + \frac{(p+1)p}{4}.$$

The last component of the above inequality is increasing for $p \geq 0$. Hence, if $p \geq 2\sqrt{n} - 1$, then

$$n \geq n + \frac{7}{2}\sqrt{n} - 2,$$

a contradiction. Thus, $p < 2\sqrt{n} - 1$ hence $p \leq t-1$. Hence, $\sigma = (w_1, u_{s-p})(w_2, u_{s-p+1})...(w_{p+1}, u_s)\sigma_{p+1}$ is a packing of G. □

Now we are in the position to show that if the number of vertices of degree 1 is large enough, then G is packable.

Lemma 16 *Let $k \geq 3$. Let G be a graph with minimum order n such that G is a non-star, non-packable graph with girth $g \geq 5$ and size $||G|| \leq \frac{2k-1}{k} n - \alpha_k(n)$. If all the vertices of G of degree 1 have a common neighbor then G has at most $4\sqrt{n} + 1$ vertices of degree 1.*

Proof. By Lemma 13, G has no isolated vertices. Let V_1 denote the set of all vertices of G with degree 1 and suppose that $|V_1| \geq 4\sqrt{n} + 2$. Let x be the common neighbor of all vertices from V_1. Furthermore, let y be one of the neighbors of x outside V_1 or if $N(x) = V_1$ let $y \in V(G) \setminus (V_1 \cup \{x\})$. We define $G' := G - x - N(x) - y - N(y)$.
Let U_1 denote the set of all different from y neighbors of x with degree at least 2. Furthermore let $z \neq x$ be one of the neighbors of y. Let U_2 be the set of all neighbors of y excluding x and z. Note that $U_1 \cap U_2 = \emptyset$ (because of girth assumptions if y is a neighbor of x, or because $U_1 = \emptyset$, otherwise). Finally let a and a' be certain vertices from V_1.
We choose disjoint subsets, W_1 and W_2, of $V_1 \setminus \{a, a'\}$ such that $|W_i| = \min\{\lfloor 2\sqrt{n} \rfloor, |U_i| - m_i + 1\}$, where m_i denotes the number of vertices of degree 2 in U_i. In particular,

$$|W_i| \leq |U_i| - m_i + 1.$$

Thus

$$\sum_{v \in U_i \cup W_i} \deg v \geq |W_i| + 2m_i + 3(|U_i| - m_i) = 3|U_i| + |W_i| - m_i$$
$$\geq 2|U_i| - m_i + |W_i| + |W_i| + m_i - 1 = 2|U_i \cup W_i| - 1. \tag{3}$$

Let $V_1' = V_1 \setminus (W_1 \cup W_2 \cup \{a, a'\})$. Denote vertices in V_1' by $\{u_1, ..., u_p\}$, see Figure 1. Let us choose a subset $A = \{v_1, ..., v_{|A|}\} \subset V(G')$ in the following way. Let $G_1' = G'$. Let $v_1 \in V(G_1')$ be a vertex such that v_1 is not a neighbor of z in G and v_1 has degree at least 3 in G_1'. In the $(i+1)$-th step we define $G_{i+1}' = G_i' - v_i$ and we choose such a vertex $v_{i+1} \in V(G_{i+1}')$ which is not a neighbor of z in G and which has degree at least 3 in G_{i+1}'. We continue this procedure until the time when $|A| = p$ or it is not possible to choose a successive vertex v_{i+1}. Let $G'' := G_{|A|+1}'$. Now we have two cases:

Case 1. $|A| < p$. In this case G'' contains all the neighbors of z in G' ($\deg z \geq 2$) which are independent because of girth assumption. Furthermore, all remaining vertices of G'' have degrees at most 2 in G''. Hence, G'' is packable by Lemma 14 or G'' is a star of order 2 or more (if G'' is a star of order 1, then it is trivially packable).

Case 2. $|A| = p$. Let $n'' = |G''|$. Hence $n'' = n - |U_1 \cup W_1| - |U_2 \cup W_2| - 2p - 5$. Note that because girth is greater than or equal to 5 the set $U_1 \cup U_2 \cup \{z\}$ is independent in G. Thus,

$$\begin{aligned} ||G''|| &\leq ||G|| - \left(\sum_{v \in U_1 \cup W_1 \cup U_2 \cup W_2} \deg v + \sum_{v \in A} 3 + \sum_{v \in V_1'} 1 + \deg a + \deg a' + \deg z \right) \\ &\leq ||G|| - (2|U_1 \cup W_1| - 1 + 2|U_2 \cup W_2| - 1 + 3p + p + 4), \text{ by (3)} \\ &= ||G|| + 2(n'' - n) + 8 \leq \frac{2k-1}{k} n - \alpha_k(n) + 2n'' - 2n + 8. \end{aligned}$$

Recall that, by Proposition 11, $n > 10k^2$. Hence $\frac{1}{k}(n - n'') \geq 8$ since $n \geq n'' + 4\sqrt{n} + 2$. Therefore

$$||G''|| \leq \frac{-1}{k} n'' - 8 - \alpha_k(n) + 2n'' + 8 \leq \frac{2k-1}{k} n'' - \alpha_k(n'')$$

for, clearly, $\alpha_k(n) \geq \alpha_k(n'')$. Therefore G'' is a star of order at least 2 or G'' is packable by the minimality assumption (if G'' is a star of order 1, then it is trivially packable).

In what follows we deal with both cases simultaneously. Suppose first that G'' is not a non-trivial star whence it is packable. Let σ'' denote a packing of G''. Then $\sigma' := (v_1, u_1)...(v_{|A|}, u_{|A|})\sigma''$ is a packing of $G' + V_1'$ (recall, that in the notation of a permutation we omit fixed points). Now, by Lemma 15, σ' can be extended to a packing σ_2' of $G' + V_1' + U_2 + W_2$ (the girth assumption implies that vertices from U_2 have disjoint neighborhoods). Consequently, by Lemma 15, σ_2' can be extended to a packing σ_1' of $G' + V_1' + U_2 + W_2 + U_1 + W_1$ (again, the girth assumption implies that vertices from U_1 have disjoint neighborhoods). Finally, $\sigma = (x, a, y, z)\sigma_1'$ is a packing of G. In the case when G'' is a star let $z' \in V(G'')$ be a neighbor of z in G'' and let z'' denote a neighbor of z' in G'' (recall that $\deg z \geq 2$ and the star has at least 2 vertices). Note that because of the girth, z'' is not a neighbor of z in G. Furthermore, either z' or z'' is the center of the star G''. Hence, $\sigma'' = (z', z'', a')$ is a packing of $G'' + a'$. We extend σ'' to a packing of the whole G in the same way as previously. □

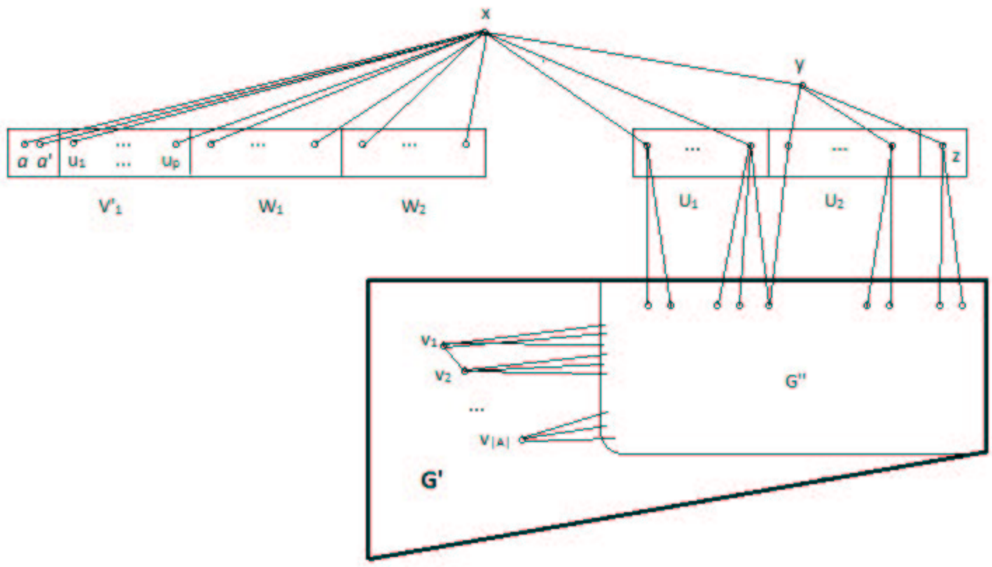
Lemma 17 *Let $k \geq 3$. Let G be a graph with minimum order n such that G is a non-star, non-packable graph with girth $g \geq 5$ and size $||G|| \leq \frac{2k-1}{k} n - \alpha_k(n)$. If two vertices of G of degree 1 have different neighbors then G has at most 20 vertices of degree 1.*

Proof. Let V_1 denote the set of all vertices of G with degree 1. Suppose for a contradiction, that $|N(V_1)| \geq 2$ and $|V_1| > 20$. By the same argument as in the proof of Lemma 13 we may assume that G contains a vertex x with $\deg x \geq 7$. Let $x_1, x_2 \in V_1$ and y_1, y_2, $y_1 \neq y_2$, be the neighbors of x_1 and x_2 respectively.

Note that y_1 and y_2 cover at most 7 edges. Indeed, otherwise $G' := G - \{x_1, x_2, y_1, y_2\}$ arises from G by deleting 4 vertices and at least 8 edges. Hence, $||G'|| \leq \frac{2k-1}{k}|G'| - \alpha_k(|G'|)$, by Proposition 10. Moreover, y_1 or y_2 has at least two neighbors in G'. Hence, by the girth assumption, G' is not a star. Thus, by the minimality assumption there is a packing σ' of G'. Then, $(x_1, y_1, x_2, y_2)\,\sigma'$ is a packing of G.

Therefore, $\deg y_1, \deg y_2 \leq 6$ and x is not a neighbor of any vertex from V_1. Moreover, $\deg y_1 + \deg y_2 \leq 8$ if $y_1 y_2$ is an edge of G, and $\deg y_1 + \deg y_2 \leq 7$ otherwise. In particular, y_2 has at most $7 - \deg y_1$ neighbors in V_1. Analogously, every vertex other than y_1 of G has at most $7 - \deg y_1$ neighbors in V_1. Let $V_1' \subset V_1 \setminus \{y_1\}$ be the set of all vertices of degree 1 which are at distance equal to 1 or 2 from y_1. Let $V_1'' = V_1 \setminus V_1'$. Thus, $|V_1'| \leq (\deg y_1 - 1)(7 - \deg y_1) + 1$. Hence, $|V_1''| \geq |V_1| - (\deg y_1 - 1)(7 - \deg y_1) - 1$. Since every vertex other than y_1 of G has at

Figure 1:



most $7 - \deg y_1$ neighbors in V_1, we have

$$|N(V_1'')| \geq \frac{|V_1| - (\deg y_1 - 1)(7 - \deg y_1) - 1}{7 - \deg y_1}.$$

Therefore, if $|V_1| \geq (\deg y_1 - 1)(7 - \deg y_1) + 1 + (\deg y_1 - 1)(7 - \deg y_1) + 1$ then $|N(V_1'')| \geq \deg y_1$, so we can find a set $W \subset V_1$ of $\deg y_1$ vertices of degree 1 which are independent, have different neighbors and are at distance at least 3 from y_1 (we include the case when $\deg y_1 = 1$). It is easy to check that the above statement is true if $|V_1| \geq 20$ since the largest number of vertices of degree 1 is needed when $\deg y_1 = 4$.

Consider now a graph $G'' := G - (W \cup \{x, x_1, y_1\})$. Note that in order to obtain G'' we remove from G, $\deg y_1 + 3$ vertices and at least $\deg y_1 + (\deg y_1 + \deg x - 1) \geq 2(\deg y_1 + 3)$ edges. Therefore, by Proposition 10, $||G''|| \leq \frac{2k-1}{k}|G''| + \alpha_k(|G''|)$. Hence, by the minimality assumption, there is a packing σ'' of G''. Furthermore, $(x, x_1)\sigma''$ is a packing of $G - (W \cup \{y_1\})$. Then, by Lemma 9, there is a packing of G, a contradiction. $\square$

3 Proof of Theorem 5

Proof. Fix k, $k \geq 3$. Assume that G is a counterexample to Theorem 5 with minimum order n. By Lemma 11, $n > 10k^2 \geq 90$. Moreover, by Lemma 13, G has no isolated vertices, and, by Lemmas 16 and 17, G has less than $4\sqrt{n} + 2$ vertices of degree 1. Let V_1 be the set of all vertices of degree 1 in G, so $|V_1| < 4\sqrt{n} + 2$.

Let S denote a most numerous set of independent vertices of degrees $2, ..., k$ which have mutually disjoint sets of neighbors. Note that $S \neq \emptyset$. Indeed, otherwise

$$\frac{4k-2}{k}n - 2\alpha_k(n) \geq 2||G|| = \sum_{v \in V(G)} \deg v > (4\sqrt{n} + 2) + (k+1)(n - 4\sqrt{n} - 2).$$

Hence,

$$\frac{k^2-4k+4}{k}(4\sqrt{n}+2)-4k(4k-5) > \frac{k^2-3k+2}{k}n,$$

which is not possible because $n > 4\sqrt{n}+2$ for $n > 90$, and $k^2-3k+2 > k^2-4k+4$ for $k \geq 3$.

By the girth assumption, $G-S$ is not a star. Moreover, by Proposition 10, $||G-S|| \leq \frac{2k-1}{k}|G-S| + \alpha_k(|G-S|)$. Thus, by the minimality assumption, $G-S$ is packable. Hence, by Lemma 9 (with $l=k$),

$$|S| < 2k. \tag{4}$$

Thus,

$$|N(S)| < 2k^2. \tag{5}$$

Let $V_j := \{v \in V(G) \setminus N(S) : \deg v = j\}$. By the definition of S, every vertex from $V_2 \cup ... \cup V_k$ has a neighbor in $N(S)$. Thus, $\sum_{v\in N(S)} \deg v \geq |V_2| + ... + |V_k|$. Therefore,

$$\begin{aligned}\frac{4k-2}{k}n - 2\alpha_k(n) \geq 2||G|| = \sum_{v\in V(G)} \deg v = \sum_{u\in N(S)} \deg u + \sum_{v\in V(G)\setminus N(S)} \deg v \\ \geq (|V_2| + ... + |V_k|) + |V_1| + 2|V_2| + ... + k|V_k| \\ + (k+1)\left(n - |V_1| - |V_2| - ... - |V_k| - |N(S)|\right),\end{aligned}$$

Thus, by (5),

$$(k-2)|V_2| + (k-3)|V_3| + ... + |V_{k-1}| > \frac{k^2-3k+2}{k}n - k|V_1| - 2k^2(k+1) + 2\alpha_k(n).$$

Clearly, $|N(N(S))| \geq |V_2| + |V_3| + ... + |V_{k-1}|$, hence

$$|N(N(S))| > \frac{k-1}{k}n - \frac{k}{k-2}|V_1| + \frac{2\alpha_k(n) - 2k^2(k+1)}{k-2}. \tag{6}$$

Thus, vertices from $N(S)$ cover at least $\frac{k-1}{k}n - \frac{k}{k-2}|V_1| + \frac{2\alpha_k(n)-2k^2(k+1)}{k-2}$ edges. Consider now the graph $G-N(S)$. Let T_1,...,T_p, with $|T_i| \geq |T_j|$ for $i < j$, denote connected components of $G-N(S)$ which are trees such that each vertex of T_i is incident with at most one vertex in $N(S)$. We call these components *minimal components* of $G-N(S)$. Let $R := G - N(S) - V(T_1) - ... - V(T_p)$. Let r denote the sum of the size of R and the number of all vertices in R which are joined (in G) with $N(S)$ by at least two edges. Since R does not contain minimal components, every component of R which is a tree contains a vertex joined with $N(S)$ by at least two edges. On the other hand, every component of R which is not a tree has at least as many edges as vertices. Hence, $r \geq |R|$. Moreover, r counts all edges in R and some edges between R and $N(S)$ which are not counted in inequality (6), because this inequality counts only the number of vertices in $N(N(S))$ and ignores the number of connections.

Note that there are exactly $n - |N(S)| - |R| - p$ edges in $\bigcup_{i=1}^{p} T_i$. Below we show that p is greater than $2|N(S)| - |R| + r + 1$. By the assumption and by inequality (6), the size of G satisfies:

$$\begin{aligned}\frac{2k-1}{k}n - \alpha_k(n) \geq ||G|| > \frac{k-1}{k}n - \frac{k}{k-2}|V_1| + \frac{2\alpha_k(n) - 2k^2(k+1)}{k-2} \\ + (n - |N(S)| - p - |R|) + r\end{aligned}$$

Thus, since $|N(S)| < 2k^2$,

$$\begin{aligned}p &> -\frac{k}{k-2}|V_1| + \frac{2\alpha_k(n) - 2k^2(k+1)}{k-2} - 2k^2 + 1 - |R| + r + \alpha_k(n) \\ &= -\frac{k}{k-2}|V_1| + \frac{k}{k-2}\alpha_k(n) - \frac{2k^2(2k-1)}{k-2} + 1 - |R| + r.\end{aligned}$$

Then the number p' of non-trivial (i.e. with at least one edge) minimal components satisfies

$$p' \geq p - |V_1| > -\frac{2k-2}{k-2}|V_1| + \frac{k}{k-2}\alpha_k(n) - \frac{2k^2(2k-1)}{k-2} + 1 - |R| + r.$$

Using the bound on $|V_1|$ we obtain

$$p' > 4k^2 + 1 - |R| + r > 2|N(S)| + 1 - |R| + r. \tag{7}$$

Since $r \geq |R|$, $T_1, ..., T_{2|N(S)|}$ are non-trivial minimal components of G. Let $G' := G[N(S) \cup V(T_1) \cup ... \cup V(T_{2|N(S)|})]$ and $G'' := G - G'$. Below we show that there exists a packing of G' such that the image of every vertex in $N(S)$ is not in $N(S)$. Let L be a set of maximum cardinality l of vertex-disjoint starry trees, such that each starry tree is formed of two of the trees T_i, $1 \leq i \leq 2|N(S)|$, and one vertex (the middle vertex) from $N(S)$. Let $H_1, ..., H_l$, $l \leq |N(S)|$, denote the starry trees. Suppose first that $l = |N(S)|$. Then we pack every starry tree in such a way that the middle vertex is the image of one of its neighbors in the same starry tree. Since $T_1, ..., T_{2|N(S)|}$ are non-trivial trees, every starry tree is not a star. Hence, the required packing exists by Lemma 12. Let σ_i be the required packing of H_i. We claim that the product $\sigma = \sigma_1 ... . \sigma_{|N(S)|}$ is a packing of G' as well. Since σ_i is a packing of H_i, only edges between different starry trees may spoil the packing of G'. Furthermore, every middle vertex is mapped on a non-middle vertex. Since there are no edges between T_i and T_j for $i \neq j$, the edges between middle vertices do not spoil the packing. It remains to check the edges of the form xy where x is the middle vertex of some starry tree and y is a non-middle vertex of another starry tree. However, since the middle vertex of each starry tree is the image of one of its neighbors in the same starry tree and this neighbor has no other neighbors outside its minimal component, these edges do not also spoil the packing. Suppose now, that $l < |N(S)|$. Again, we pack every starry tree in such a way that the middle vertex is the image of one of its neighbors. Moreover, since L is maximal, each remaining vertex of $N(S)$ has no neighbors in each of the remaining minimal components (otherwise, we would have an extra starry tree). Hence, by Theorem 8, each of the remaining vertices from $N(S)$ together with two non-trivial minimal components (not involved in any starry tree) can be packed without fixed points. We claim that the product of these packings is a proper packing of G'. Suppose for a contradiction that the image of an edge e in G' coincides with some other edge e' in G'. Using the previous argument, e' must join a vertex $z \in N(S)$ which is not in any starry tree from L with a non-middle vertex of some starry tree H. Moreover, e must join the middle vertex of H with some minimal component which is not in any starry tree from L. By replacing the middle vertices incident to e and e' we obtain more than l starry trees and we get a contradiction. Hence G' is packable.

Recall that $r \geq ||R||$. Furthermore, since $p \geq p'$, by (7) we have

$$\begin{aligned} ||G''|| = ||R \cup T_{2|N(S)|+1} \cup ... \cup T_p|| &= ||R|| + |T_{2|N(S)|+1}| + ... + |T_p| - (p - 2|N(S)|) \\ &< ||R|| + |T_{2|N(S)|+1}| + ... + |T_p| - (r - |R|) - 1 \\ &\leq |R| + |T_{2|N(S)|+1}| + ... + |T_p| - 1 \\ &= |R \cup T_{2|N(S)|+1} \cup ... \cup T_p| - 1 = |G''| - 1. \end{aligned}$$

Thus, by Theorem 1, G'' is packable.

Let σ', σ'' denote packings of G' and G'', respectively. Then $\sigma = \sigma'\sigma''$ is a packing of G. Suppose for a contradiction that the image of an edge xy in G coincides with some other edge $\sigma(x)\sigma(y)$ in G. Then $x, \sigma(x) \in V(G')$ and $y, \sigma(y) \in V(G'')$. By construction of G' and G'' we have that x and $\sigma(x)$ belong to $N(S)$. Then we get a contradiction, since the image of every vertex in $N(S)$ is not in $N(S)$. The packing σ contradicts the assumption that G was non-packable, so we deduce no counterexample to Theorem 5 exists. $\square$

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