A note on $k$-placeable graphs

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Abstract

Let $G$ be a graph of order $n$. We prove that if the size of $G$ is less than or equal to $n - 2(k - 1)^3$ then the complete graph $K_n$ contains $k$ edge-disjoint copies of $G$. The case when $k = 2$ is the well known theorem of Sauer and Spencer 1978.

1 Introduction

We deal with finite, simple graphs without loops and multiple edges. We use standard graph theory notation. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The order of $G$ is denoted by $|G|$ and the size is denoted by $||G||$. By $N_G(x)$ we denote the set of vertices incident with $x$ in $G$.

Suppose $G_1, ..., G_k$ are graphs of order $n$. We say that there is a $k$-packing of graphs $G_1, ..., G_k$ (into $K_n$) if there are injections $\sigma_i : V(G_i) \to V(K_n)$ such that $\sigma_i^*[E(G_i)] \cap \sigma_j^*[E(G_j)] = \emptyset$ for $i \neq j$, where the map $\sigma_i^* : E(G_i) \to E(K_n)$ is induced by $\sigma_i$. If all $G_i$’s are isomorphic to a graph $G$ then a $k$-packing of $G_i$’s is called a $k$-placement of $G$. When there is a $k$-placement of $G$ we say that $G$ is $k$-placeable. The following theorem was independently obtained in [1, 2, 5].

Theorem 1 ([1, 2, 5]) Let $G$ be a graph of order $n$ such that $||G|| \leq n - 2$. Then $G$ is 2-placeable.

In this paper we prove

Theorem 2 Let $k$ be a positive integer and $G$ be a graph of order $n \geq 2(k - 1)^3$. If $||G|| \leq n - 2(k - 1)^3$ then $G$ is $k$-placeable.

Our result is related to the following conjecture of Bollobas and Eldridge [1].

Conjecture 3 ([1]) Let $G_1, ..., G_k$ be graphs of order $n$ such that $||G_i|| \leq n - k$, $i = 1, ..., k$. Then there is a $k$-packing of $G_1, ..., G_k$.

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The case \( k = 2 \) and \( k = 3 \) of this conjecture was proved in [5] and [4], respectively. Though our constant differs significantly from the one from Conjecture 3, as far as we know, for \( k \geq 4 \) our bound on the size of a graph is best known to guarantee its \( k \)-placeability.

# 2 Lemmas

Recall another classic result from graph packing theory.

**Theorem 4** ([6]) Let \( G_1 \) and \( G_2 \) be graphs of order \( n \) such that \( 2\Delta(G_1)\Delta(G_2) < n \). Then there is a 2-packing of \( G_1 \) and \( G_2 \).

**Corollary 5** Let \( k \) be a positive integer and \( G \) be a graph of order \( n \) such that \( 2(k-1)\Delta^2(G) < n \). Then \( G \) is \( k \)-placeable.

Proof. We proceed by induction on \( k \). The case when \( k = 1 \) is obvious. Suppose that \( k > 1 \) and there is a \((k-1)\)-placement \( \sigma_1, \ldots, \sigma_{k-1} \) of \( G \). Let \( G_{k-1} := (V(G), \sigma_1^*[E(G)] \cup \ldots \cup \sigma_{k-1}^*[E(G)]) \). Note that \( \Delta(G_{k-1}) \leq (k-1)\Delta(G) \). Thus, by Theorem 4, it follows that there is a 2-packing of \( G \) and \( G_{k-1} \). Therefore, \( G \) is \( k \)-placeable.

**Lemma 6** Let \( G \) be a graph and \( x \in V(G) \). If \( G \) has a set \( S \) of \( k-1 \) isolated vertices and \( G - (S \cup \{x\}) \) is \( k \)-placeable then \( G \) is \( k \)-placeable.

Proof. Let \( \sigma_1', \ldots, \sigma_k' \) be a \( k \)-placement of \( G' := G - (S \cup \{x\}) \). Let \( S = \{y_1, \ldots, y_{k-1}\} \). Then \( \sigma_1 = \sigma_1', \sigma_i = (y_{i-1}, x)\sigma_i', i = 2, \ldots, k \), is a \( k \)-placement of \( G \).

**Lemma 7** Let \( G \) be a graph and \( d \geq 1 \) be any positive integer. If there is a set \( U = \{v_1, \ldots, v_{(2k-2)d}\} \subset V(G) \) of \((2k-2)d\) independent vertices of \( G \) such that

1. vertices of \( U \) have degrees at most \( d \) each;
2. vertices of \( U \) have mutually disjoint sets of neighbors, i.e. \( N(v_i) \cap N(v_j) = \emptyset \) for \( i \neq j \);
3. there is a \( k \)-placement of \( G - U \),

then there exists a \( k \)-placement of \( G \).

Proof. Let \( G' := G - U \) and let \( \sigma_1', \sigma_2', \ldots, \sigma_k' \) be a \( k \)-placement of \( G' \). We will prove that for each \( 1 \leq m \leq k \) there is an \( m \)-placement \( \sigma_1, \ldots, \sigma_m \) of \( G \) in which \( U \) is an invariant set, i.e. \( \sigma_i^*[U] = U \) for each \( i = 1, \ldots, m \). The proof is by induction on \( m \). The case \( m = 1 \) is obvious with \( \sigma_1 = \sigma_1' \) for example.

Suppose that \( m > 1 \). Let \( G_{m-1} := (V(G), \sigma_1^*[E(G)] \cup \ldots \cup \sigma_{m-1}^*[E(G)]) \). Below we show that we can find a \( 2 \)-packing \( \sigma_m \) of \( G_{m-1} \) and \( G \).

Note that every vertex \( x \in U \) has degree at most \((m-1)d\) in \( G_{m-1} \). Moreover, since vertices from \( U \) have disjoint sets of neighbors in \( G \), every vertex of \( G_{m-1} \) can have at most \( m-1 \) neighbours in \( U \). For any \( v \in V(G') \) let us define \( \sigma_m(v) := \sigma_m'(v) \). Then let us consider a bipartite graph \( H \) with partite sets
A = U \times \{0\} and B = U \times \{1\}. For i, j \in \{1, ..., (2k - 2)d\} the vertices (v_i, 0), (v_j, 1) are joined by an edge in H if and only if \sigma_m'(N_G(v_i)) \cap N_{G^{m-1}}(v_j) = \emptyset.

So, if (v_i, 0), (v_j, 1) are joined by an edge in H we can put \sigma_m(v_i) = v_j.

Because deg_{G_i} v_i \leq d for i \in \{1, ..., (2k - 2)d\} and each vertex x \in G^{m-1} has at most m - 1 neighbors in U, deg_{H^G}(v_i, 0) \geq (2k - 2)d - (m - 1)d = (2k - m - 1)d

(every neighbor of v_i \in V(G) can block at most m - 1 possible places for v_j).

Moreover since deg_{G^{m-1}} v_j \leq (m - 1)d for j \in \{1, ..., (2k - 2)d\}, deg_{H^G}(v_j, 1) \geq (2k - 2)d - (m - 1)d = (2k - m - 1)d

(every vertex v_j \in V(G^{m-1}) can block places for at most (m - 1)d vertices of the m-th copy of G).

Let S \subset A. If |S| \leq (2k - m - 1)d then obviously |N(S)| \geq |S|. Notice that if |S| > (2k - m - 1)d, then N(S) = B. Indeed, otherwise let (v_j, 1) \in B be a vertex which has no neighbour in S. Thus (2k - m - 1)d \leq deg_{H^G}(v_j, 1) \leq |A| - |S| < (2k - 2)d - (2k - m - 1)d = (m - 1)d. Hence, 2k - m - 1 < m - 1 so m > k, a contradiction. Therefore, in any case |S| \leq |N(S)|. Thus, by Hall’s theorem there is a matching M in H. Therefore we can define \sigma_m(v_i) = v_j for i, j \in \{1, ..., (2k - 2)d\} such that (v_i, 0), (v_j, 1) are incident with the same edge in M.

\[ \text{3 Proof of Theorem 2} \]

Proof. The case when \(k = 1\) is obvious and the case when \(k = 2\) is Theorem 1. Assume that \(k \geq 3\). The proof is by induction on \(n\). The theorem is obvious for \(n = 2(k - 1)^3\). Assume that \(n > 2(k - 1)^3\). If \(2(k - 1)\Delta^2(G) < n\) then \(G\) is \(k\)-placeable by Corollary 5.

Suppose that \(2(k - 1)\Delta^2(G) \geq n\). Thus \(\Delta(G) > k - 1\) because \(n > 2(k - 1)^3\).

Let \(V_0\) and \(V_1\) denote sets of vertices of \(G\) which have degree equal to 0 and 1 respectively. The proof falls into two cases.

Case 1. Vertices of degree 1 have at least \(4k - 4\) neighbors, i.e. \(|N(V_1)| \geq 4k - 4\). Then we can find \(2k - 2\) independent vertices \(x_1, ..., x_{2k-2}\) of degree 1 that have pairwise different neighbors. Let \(G' := G - \{x_1, ..., x_{2k-2}\}\). Then \(|G'| = n - (2k - 2)\) and \(||G'\| \leq n - 2(k - 1)^3 - (2k - 2) = |G'|-2(k-1)^3\).

Hence, \(G'\) is \(k\)-placeable by the induction hypothesis. Thus, \(G\) is \(k\)-placeable by Lemma 7.

Case 2. Vertices of degree 1 have at most \(4k - 5\) neighbors, i.e. \(|N(V_1)| \leq 4k - 5\). Then

\[
2n - 4(k - 1)^3 = 2||G|| = \sum_{v \in V(G)} \deg v =
\]

\[
= \sum_{v \in V_0} 0 + \sum_{v \in V_1} 1 + \sum_{v \in N(V_1)} \deg v + \sum_{v \in V(G) \setminus (V_0 \cup V_1 \cup N(V_1))} \deg v \geq
\]

\[
\geq 0 + |V_1| + |V_1| + 2(n - |V_0| - |V_1| - (4k - 5)) =
\]

\[
= 2n - 2|V_0| - 8k + 10
\]

Therefore

\[
|V_0| \geq 2(k - 1)^3 - 4k + 5 \geq k - 1,
\]

3
if $k \geq 3$. Let $y_1, \ldots, y_{k-1}$ be isolated vertices of $G$ and let $x \in V(G)$ with $\deg x \geq k$. Let $G' := G - \{y_1, \ldots, y_{k-1}, x\}$. Note that $||G'|| \leq n - 2(k-1)^3 - k = |G'| - 2(k-1)^3$. Thus $G''$ is $k$-placeable by the induction hypothesis. Hence, by Lemma 6, $G$ is $k$-placeable, too. \hfill \Box

References


