On vertex stability with regard to complete bipartite subgraphs

Aneta Dudek and Andrzej Żak
Faculty of Applied Mathematics, AGH University of Science and Technology
al. Mickiewicza 30, 30-059 Kraków, Poland
e-mail: {dudekane,zakandrz}@agh.edu.pl

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Abstract

A graph $G$ is called $(H; k)$-vertex stable if $G$ contains a subgraph isomorphic to $H$ ever after removing any of its $k$ vertices. $Q(H; k)$ denotes the minimum size among the sizes of all $(H; k)$-vertex stable graphs. In this paper we complete the characterization of $(K_{m, n}; 1)$-vertex stable graphs with minimum size. Namely, we prove that for $m \geq 2$ and $n \geq m + 2$, $Q(K_{m, n}; 1) = mn + m + n$ and $K_{m, n} + K_1$ as well as $K_{m+1, n+1} - e$ are the only $(K_{m, n}; 1)$-vertex stable graphs with minimum size, confirming the conjecture of Dudek and Zwonek.

Key words: vertex stable, bipartite graph, minimal size
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1 Introduction

We deal with simple graphs without loops and multiple edges. We use the standard notation of graph theory, cf. [1]. The following notion was introduced in [2]. Let $H$ be any graph and $k$ a non-negative integer. A graph $G$ is called $(H; k)$-vertex stable if $G$ contains a subgraph isomorphic to $H$ ever after removing any of its $k$ vertices. Then $Q(H; k)$ denotes minimum size among the sizes of all $(H; k)$-vertex stable graphs. Note that if $H$ does not have isolated vertices then after adding to or removing from a $(H; k)$-vertex stable graph any number of isolated vertices we still have a $(H; k)$-vertex stable graph with the same size. Therefore, in the sequel we assume that no graph in question has isolated vertices.

There are two trivial examples of $(H; k)$-vertex stable graphs, namely $(k + 1)H$ (a disjoint union of $(k + 1)$ copies of $H$) and $H + K_k$ (a graph obtained from $H \cup K_k$ by joining all the vertices of $H$ to all the vertices of $K_k$). Therefore,

Proposition 1 $Q(H; k) \leq \min \left\{ (k + 1)|E(H)|, |E(H)| + k|V(H)| + \binom{k+1}{2} \right\}.$

On the other hand, the following is easily seen.

Proposition 2 Suppose that $H$ contains $k$ vertices which cover $q$ edges. Then $Q(H; k) \geq |E(H)| + q$.

Recall also the following

Proposition 3 ([2]) Let $\delta_H$ be a minimal degree of a graph $H$. Then in any $(H; k)$-vertex stable graph $G$ with minimum size, $\deg_G v \geq \delta_H$ for each vertex $v \in G$. 

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The exact values of \( Q(H;k) \) are known in the following cases: \( Q(C_i;k) = i(k+1), i = 3, 4, Q(K_4;k) = 5(k+1), Q(K_n;k) = i(\frac{n^2+k}{2}) \) for \( n \) large enough, and \( Q(K_{1,m};k) = m(k+1), Q(K_{n,n};1) = n^2+2n, Q(K_{n,n+1};1) = (n+1)^2, n \geq 2, \) see [2, 3]. In this paper we complete the characterization of \((K_{m,n};1)\) vertex stable graphs with minimum size. Namely, we prove the following theorem and hence confirm Conjecture 1 formulated in [3].

**Theorem 1** Let \( m, n \) be positive integers such that \( m \geq 2 \) and \( n \geq m+2 \). Then \( Q(K_{m,n};1) = mn + m + n \) and \( K_{m,n} \ast K_1 \) as well as \( K_{m+1,n+1} - c \), where \( c \in E(K_{m+1,n+1}) \), are the only \((K_{m,n};1)\)-vertex stable graphs with minimum size.

## 2 Proof of the main result

Proof of Theorem 1. Let \( m \geq 2 \) and \( n \geq m+2 \) be positive integers. Define \( G_1 := K_{m,n} \ast K_1 \) and \( G_2 := K_{m+1,n+1} - c \) where \( c \in E(K_{m+1,n+1}) \). Let \( G = (V,E) \) be a \((K_{m,n};1)\)-vertex stable graph with minimum size. Thus, by Proposition 1, \( |E(G)| \leq mn + m + n \). Clearly \( G \) contains a subgraph \( H \) isomorphic to \( K_{m,n} \). Let \( H = (X,Y;E_H) \) with vertex bipartition sets \( X, Y \) such that \( |X| = m \) and \( |Y| = n \). Let \( v \in X \). Since \( G \) is \((K_{m,n};1)\)-vertex stable, \( G - v \) contains a subgraph \( H' \) isomorphic to \( K_{m,n} \). Let \( H' = (X',Y';E_{H'}) \) with vertex bipartition sets \( X',Y' \) such that \( |X'| = m \) and \( |Y'| = n \). We denote \( x_1 = |X \cap X'|, x_2 = |X \cap Y'|, y_1 = |Y \cap X'|, y_2 = |Y \cap Y'| \). Hence \( x_1 + x_2 \leq m - 1, y_1 + y_2 \leq n, y_1 \leq m \). One can see that \( |E(G)| \geq 2mn - x_1y_2 - x_2y_1 \).

Consider the following linear programming problem with respect to \( y_1 \) and \( y_2 \):

\[
\begin{align*}
y_1 & \leq m \\
y_1 + y_2 & \leq n \\
y_1 & \geq 0 \\
y_2 & \geq 0 \\
c & = x_1y_2 + x_2y_1 \rightarrow \max
\end{align*}
\]

where \( x_1 \) and \( x_2 \) are parameters such that \( x_1, x_2 \geq 0, x_1 + x_2 \leq m - 1 \).

![Fig. 1. Geometrical interpretation of the linear programming problem](image-url)
The proof falls into two cases.

**Case 1 :** \( x_1 < x_2 \)

In this case \( y_1 = m, y_2 = n - m \), \( c = x_2m + x_1(n - m) \) is the unique optimal solution of the above linear programming problem. This can be easily checked using a geometrical interpretation of the linear programming problem, see Fig. 1. Thus \( |E(G)| \geq 2mn - x_2m - x_1(n - m) \) and the inequality is strict if \( y_1 \neq m \) or \( y_2 \neq n - m \). We assume that \( x_1 + x_2 = m - 1 \) because otherwise the size of \( G \) may only increase. Then

\[
|E(G)| \geq 2mn - m^2 + m + x_1(2m - n) := f(x_1).
\]

**Subcase 1a :** \( n > 2m \)

Then \( f(x_1) \) is decreasing. Furthermore, \( x_1 < \frac{m-1}{2} \) since \( x_1 < x_2 \). Thus

\[
|E(G)| > f \left( \frac{m - 1}{2} \right) = \frac{3}{2}mn + \frac{1}{2}n \geq mn + m + n.
\]

Thus \( |E(G)| > mn + m + n \), a contradiction.

**Subcase 1b :** \( n < 2m \)

Then \( f(x_1) \) is increasing. Thus

\[
E(G) \geq f(0) = 2mn - m^2 + m \geq mn + m + n
\]

with equality if and only if \( m = 2 \) and \( n = 4 \), which is not possible in this subcase.

**Subcase 1c :** \( n = 2m \)

In this case

\[
E(G) \geq mn + m + n
\]

with equality if and only if \( m = 2 \), \( n = 4 \), \( y_1 = y_2 = 2 \). Recall that \( x_1 < x_2 \) whence \( x_1 = 0 \) and \( x_2 = 1 \). Let \( u \in Y' \setminus (X \cup Y) \). Thus \( |E(G)| \geq 12 + \deg u \). Hence \( \deg u = 2 \) and \( |V(G)| = 7 \) because otherwise \( |E(G)| > mn + m + n \). However, then \( G \) is not \( (K_{2,4};1) \)-stable. Indeed let \( w \) be a neighbor of \( u \). Then \( G - w \) does not contain any subgraph isomorphic to \( K_{2,4} \) since \( G - w \) has 6 vertices and one of them has degree 1. Therefore Case 1 is not possible.

**Case 2 :** \( x_1 \geq x_2 \)

In this case \( c = x_1n \) is the optimal solution of the above linear problem, see Fig. 1. Therefore, \( |E(G)| \geq 2mn - x_1n \). If \( x_1 \leq m - 2 \) then \( |E(G)| \geq 2mn - (m - 2)n = mn + 2n > mn + m + n \). Hence we may assume that \( x_1 = m - 1 \) and \( x_2 = 0 \). Thus there is only one vertex, say \( u \), such that \( u \in X' \setminus X \).

**Subcase 2a :** \( y_2 = n \)

Thus, \( u \) have \( n \) neighbors in \( Y \). Note that \( |V(G)| \leq m + n + 2 \). Indeed, otherwise by Proposition 3, \( |E(G)| \geq mn + n + 2m - 1 > mn + m + n \). Consider now a graph \( G'' := G - w \) where \( w \in Y \). Clearly \( G - w \) contains a subgraph \( H'' \) isomorphic to \( K_{m,n} \). Let \( H'' = (X'', Y'', E_{H''}) \) with vertex bipartition sets \( X'', Y'' \) such that \( |X''| = m \) and \( |Y''| = n \). Let \( x_1' = |X \cap X''|, x_2' = |X \cap Y''|, y_1' = |Y \cap X''|, y_2' = |Y \cap Y''| \).

Suppose first that \( |V(G)| = m + n + 2 \) and \( u, u_1 \in V(G) \setminus (X \cup Y) \). Since \( |E(G)| \leq mn + m + n \), \( \deg u_1 = m \) and \( \deg u = n + 1 \). In particular, \( u_1 \notin X'' \) and \( u \) has no neighbor in \( X \). Furthermore, \( |E(G)| \geq mn + n + m + x'_1, x'_2 + y'_1, y'_2 \). Thus, \( x'_1 = 0 \) or \( x'_2 = 0 \), and \( y'_1 = 0 \) or \( y'_2 = 0 \). We distinguish two possibilities.
1. \( x'_1 = 0 \). Then \( y'_1 \neq 0 \). Indeed, otherwise \( X'' = \{u, u_1\} \), a contradiction with previous observation that \( u_1 \notin X'' \). Hence, \( y'_2 = 0 \). Thus, \( x'_2 = m \) and \( u, u_1 \in Y'' \) (so \( n = m+2 \)). Consequently, \( y'_2 = m \). However, then \( G \) is not \((K_{m,m+2};1)\)-stable. Indeed, let \( w_1 \) be a neighbor of \( u_1 \), \( w_1 \in X'' \subset Y \). Then \( G - w_1 \) consists of a subgraph isomorphic to \( K_{m+1,m+1} \) plus one vertex (namely \( u_1 \)) and \( m-1 \) edges incident to it. Therefore, \( G - w_1 \) does not contain any subgraph isomorphic to \( K_{m,m+2} \).

2. \( x'_1 \neq 0 \). Then \( x'_2 = 0 \) and \( u \notin Y'' \). Consequently, \( u_1 \in Y'' \) and \( y'_2 \neq 0 \). Hence \( y'_1 = 0 \). It is easy to see now that \( G \cong G_2 \).

Assume now that \( |V(G)| = m + n + 1 \). Hence \( x'_1 + x'_2 = m \) and \( y'_1 + y'_2 = n - 1 \). We have the next two possibilities.

3. \( x'_1 + y'_1 = m \). Then \( |E(G)| \geq mn + x'_1 x'_2 + y'_1 y'_2 + \deg u \geq mn + x'_1 x'_2 + y'_1 y'_2 + n + x'_1 \).

Hence
\[
|E(G)| \geq mn + (m - x'_1)(n - 1 - m + 2x'_1) + n + x'_1 =: f_1(x'_1), \quad 0 \leq x'_1 \leq m.
\]

It is not difficult to see that \( f_1(x'_1) \) obtains the smallest value for \( x'_1 = 0 \) or \( x'_1 = m \) only. Thus, \( |E(G)| \geq \min\{f_1(0), f_1(m)\} \). Note that \( f_1(0) = 2mn + n - m - m^2 \geq mn + m + n \) with equality if and only if \( n = m + 2 \). However, then there is a vertex \( y \in Y'' \) such that \( G - y \cong K_{m+1,m+1} \) so \( G - y \) does not contain any subgraph isomorphic to \( K_{m,m+2} \).

Furthermore, \( f(m) \geq mn + n + m \). Thus, \( |E(G)| \geq mn + m + n \) with equality if and only if \( x_1 = m \). Then \( G \cong G_1 \).

4. \( x'_2 + y'_2 = n \). Then \( |E(G)| \geq mn + x'_1 x'_2 + y'_1 y'_2 + n + x'_2 \). Hence,
\[
|E(G)| \geq mn + (m - x'_2)x'_2 + (x'_2 - 1)(n - x'_2) + n + x'_2 =: f_2(x'_2), \quad 1 \leq x'_2 \leq m.
\]

One can see that \( f_2(x'_2) \) obtains the smallest value for \( x'_2 = 1 \) or \( x'_2 = m \) only. Thus,
\[
|E(G)| \geq \min\{f_2(1), f_2(m)\}.
\]

Note that \( f_2(1) = mn + n + m \). Then \( G \cong G_1 \). On the other hand, \( f_2(m) = 2mn + 2m - m^2 > mn + m + n \).

Subcase 2b : \( y_2 < n \)

Thus, there is a vertex \( z \in Y' \) such that \( z \in V(G) \setminus (X \cup Y) \). This clearly forces \( m-1 \) neighbors of \( z \) in \( X \setminus \{v\} \). Consider now a graph \( G - v_1, v \neq v_1 \in X \). We repeat all preceding arguments to the graph \( G - v_1 \). Consequently, \( G \cong G_i, i = 1, 2, \) or there is a vertex \( z_1 \in V(G) \setminus (X \cup Y) \) which has \( m-1 \) neighbors in \( X \setminus \{v_1\} \). If \( z = z_1 \) then \( z \) has \( m \) neighbors in \( X \) and \( G \cong G_1 \) if \( u \in Y \) or \( G \cong G_2 \) otherwise. If \( z \neq z_1 \) then either \( \deg z + \deg z_1 \geq 2m + 1 \) if both vertices \( z \) and \( z_1 \) are involved in a \( K_m \) contained in \( G - v \) or \( G - v_1 \), or \( \deg u \geq n + 1 \) otherwise. Thus, \( |E(G)| \geq mn + 2m - 1 + n > mn + m + n \).

3 Concluding remarks

In [2, 3] it is proved that \( Q(K_{1,n}; k) = (k + 1)n \). However, for \( n \geq 3 \) the extremal graphs are not characterized.

Proposition 4 Let \( G \) be a \((K_{1,n}; k)\)-vertex stable graph with minimum size, \( n \geq 3 \). Then \( G = (k + 1)K_{1,n} \).

Proof. The proof is by induction on \( k \). The thesis is obvious for \( k = 0 \). Assume that \( k > 0 \). Let \( G \) be a \((K_{1,n}; k)\)-vertex stable graph with minimum size. Hence, \( |E(G)| = (k + 1)n \). Note that each \((K_{1,n}; k)\)-vertex stable graph contains \( k + 1 \) vertices of degree at least \( n \). Let \( v \in V(G) \) with \( \deg v \geq n \). Thus, \( G - v \) is \((K_{1,n}; k-1)\)-vertex stable graph with \( |E(G - v)| \leq kn \). Hence, \( |E(G - v)| = kn \) and \( \deg v = n \). By the induction hypothesis \( G - v = kK_{1,n} \). Note that \( v \) is not a neighbor of any vertex of degree \( n \). Suppose on the contrary, that \( uv \in E(G) \) and \( \deg u = n \). Then
$G - u$ contains only $k - 1$ vertices of degree greater than or equal to $n$ whence is not $(K_1, n; k - 1)$-vertex stable, a contradiction. Thus, $G$ contains $k + 1$ independent vertices of degree exactly $n$. We will show that these vertices have pairwise disjoint sets of neighbors. Indeed, otherwise let $x$ be a common neighbor of two vertices of degree $n$. Thus, again, $G - x$ has only $k - 1$ vertices of degree greater than or equal to $n$, a contradiction.

In the following table we present the complete characterization of $(K_m, n; 1)$-vertex stable graphs with minimum size.

<table>
<thead>
<tr>
<th>$m; n$</th>
<th>$Q(K_m, n; 1)$</th>
<th>All $(K_m, n; 1)$-vertex stable graphs with minimum size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 1, n = 1$</td>
<td>2</td>
<td>$2K_{1,1}$</td>
</tr>
<tr>
<td>$m = 1, n = 2$</td>
<td>4</td>
<td>$K_{2,2}, 2K_{1,2}$</td>
</tr>
<tr>
<td>$m = 1, n \geq 3$</td>
<td>$2n$</td>
<td>$2K_{1,n}$</td>
</tr>
<tr>
<td>$m = 2, n = 2$</td>
<td>8</td>
<td>$K_{2,2} * K_{1,2}, K_{3,3} - e, 2K_{2,2}$</td>
</tr>
<tr>
<td>$m \geq 2, n = m + 1$</td>
<td>$(m + 1)^2$</td>
<td>$K_{m+1,m+1}$</td>
</tr>
<tr>
<td>$m \geq 3, n = m$</td>
<td>$m^2 + 2m$</td>
<td>$K_{m,m} * K_{1,m+1,m+1} - e$</td>
</tr>
<tr>
<td>$m \geq 2, n \geq m + 2$</td>
<td>$mn + m + n$</td>
<td>$K_{m,n} * K_{1,m+1,n+1} - e$</td>
</tr>
</tbody>
</table>

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References

