Lower bound on the size of $(H;1)$-vertex stable graphs

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Abstract

A graph G is called $(H;k)$-*vertex stable* if G contains a subgraph isomorphic to H even after removing any k of its vertices. By $\mathrm{stab}(H;k)$ we denote the minimum size among the sizes of all $(H;k)$-vertex stable graphs. In this paper we present a general result concerning $(H;1)$-vertex stable graphs. Namely, for an arbitrary graph H we give a lower bound for $\mathrm{stab}(H;1)$ in terms of the order, connectivity and minimum degree of H. The bound is nearly sharp.

1 Introduction

By a word graph we mean a simple graph without loops and multiple edges. A multigraph is a graph in which multiple edges (but not loops) are allowed. Given a graph G, $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. Furthermore, $|V(G)|$ is the order of G and $|E(G)|$ is the size of G. Let H be any graph and k a non-negative integer. A graph G is called $(H;k)$-*vertex stable* if G contains a subgraph isomorphic to H ever after removing any k of its vertices. Then $\mathrm{stab}(H;k)$ denotes the minimum size among the sizes of all $(H;k)$-vertex stable graphs. Note that if H does not have isolated vertices then after adding to or removing from a $(H;k)$-vertex stable graph any number of isolated vertices we still have a $(H;k)$-vertex stable graph with the same size. Therefore, in the sequel we assume that no graph in question has isolated vertices.

The notion of $(H;k)$-vertex stable graphs was introduced in [2]. So far the exact value of $\mathrm{stab}(H;k)$ is known in the following cases: $\mathrm{stab}(K_{1,m};k) = m(k+1)$, $\mathrm{stab}(C_i;k) = i(k+1)$, $i = 3,4$, $\mathrm{stab}(K_4;k) = 5(k+1)$, see [2], and $\mathrm{stab}(K_5;k) = 7(k+1)$ for $k \geq 5$ [5], $\mathrm{stab}(K_n;k) = \binom{n+k}{2}$ for $n \geq 2k-2$ [6]. Furthermore, $\mathrm{stab}(K_{m,n};1) = mn+m+n$ if $n \geq m+2, m \geq 2$, see [4], and $\mathrm{stab}(K_{n,n+1};1) = (n+1)^2$ for $n \geq 2$, $\mathrm{stab}(K_{n,n};1) = n^2+2n$ for $n \geq 2$, see [3]. Moreover, in all the above examples vertex stable graphs with minimum size are characterized. On the other hand,

$$n + \lceil 2\sqrt{n-1} \rceil \leq \mathrm{stab}(C_n;1) \leq n + \lceil 2\sqrt{n-1} \rceil + 1, \tag{1}$$

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the lower bound being attained for infinitely many n's, see [1]. An upper and a lower bound on $\text{stab}(C_n; k)$ for sufficiently large n is also presented therein.

So far, the above problem has been considered only for restricted families of graphs. In this paper we present a more general result. It is easy to see that $\text{stab}(H; 1) \geq m + \delta \geq \delta(\frac{n}{2} + 1)$ for a graph H of order n with m edges and minimum degree δ.

Theorem 1 *If H is a graph of order $n \geq 6$, minimal degree $\delta \geq 1$ and connectivity $\kappa \geq 0$, then*

$$\text{stab}(H; 1) \geq \frac{\delta}{2}(n - \kappa + 1) + \sqrt{\delta\kappa(n - \kappa + 1)} + \frac{\kappa}{2}$$

Note, that since the lower bond (1) is attained for infinitely many n's, our new bound is sharp in these cases. In Section 3, we present more infinite families of graphs for which our new bound is (almost) attained.

2 Proof of Theorem 1

By $N_G(x)$ we denote the set of vertices adjacent with x in G. For a vertex set X, the set $N_G(X)$ denotes the external neighborhood of X in G, i.e.

$$N_G(X) = \{y \in V(G) \setminus X : y \text{ is adjacent with some } x \in X\}.$$

Recall the following observation.

Proposition 2 ([2]) *Let G be $(H; k)$-vertex stable graph with minimum size. Then each vertex as well as each edge of G is contained in some copy of H. In particular, for each vertex $v \in G$ there is $\deg_G v \geq \delta_H$, where δ_H denotes the minimum degree of H.*

Proof of Theorem 1. Let G be a $(H; 1)$ stable graph with minimum size and let $|V(G)| = v$. Let $x_1, ..., x_m \in V(G)$ be vertices of degree greater than or equal to $\delta + 1$ in G. By Proposition 2 all other vertices of G have degree δ. Let $C_1, ..., C_q$ be the components of $G - \{x_1, ..., x_m\}$.

Suppose first, that there exist a component $C = C_j$ for some $j \in \{1, .., q\}$ such that $|N_G(V(C))| \leq \kappa - 1$. By Proposition 2, C intersects with some copy of H. Note, that this copy of H may contain only vertices from $C \cup N_G(V(C))$. Indeed, otherwise H contains a cutting set of cardinality less than $|N_G(V(C))| \leq \kappa - 1$, a contradiction. Thus, C contains at least $n + 1 - \kappa$ vertices,

$$|C| \geq n + 1 - \kappa. \tag{2}$$

Note that after removing from G any vertex $x_i \in N_G(V(C))$ each vertex of C is not any longer a vertex of H. Indeed, after removing x_i its neighbors in G have degree less than δ. Thus, they cannot be in H. Hence, their neighbors would have degrees less than δ in H. Thus, the latter vertices cannot be in H neither, and so on. Therefore, since G is $(H; 1)$-stable, $G - C$ contains a copy of H. Thus, $|E(G - C)| \geq |E(H)| \geq \frac{n\delta}{2}$. Hence, by (2),

$$|E(G)| \geq \frac{n\delta}{2} + \frac{|C|\delta}{2} \geq \frac{\delta}{2}(n + 1 - \kappa) + \frac{n\delta}{2} \geq \frac{\delta}{2}(n - \kappa + 1) + \sqrt{\delta\kappa(n - \kappa + 1)} + \frac{\kappa}{2},$$

because $\delta \geq \kappa$ and $n \geq 6$.

Hence we may assume that $|N_G(V(C_j))| \geq \kappa$ for all $j = 1, ..., q$. Thus, if $m \leq \kappa$ then, in G, every x_i, $i = 1, ..., m$, is connected with a vertex of each component C_j, $j = 1, ..., q$. Therefore, for $u \in \{x_1, ..., x_m\}$ a copy of H contained in $G - u$ may contain only vertices from $\{x_1, ..., x_m\} \setminus \{u\}$. Since $m \leq \kappa$, $\delta \leq \kappa - 2$, a contradiction.

So we may assume that $m > \kappa$. Let $A(x_i) \subset V(G)$ denote the set of all vertices which are in those components C_j, $j = 1, ..., q$, that satisfy $x_i \in N_G(V(C_j))$. Note, that

$$\sum_{i=1}^{m} |A(x_i)| \geq \kappa(v - m), \tag{3}$$

because $|N_G(V(C_j))| \geq \kappa$ for every j (so every vertex from $V(G) \setminus \{x_1, ..., x_m\}$ belongs to at least κ sets $A(x_i)$). Let $M = \max_i |A(x_i)| = |A(x_t)|$ for some $t \in \{1, ..., m\}$. Thus,

$$m \cdot M \geq \sum_{i=1}^{m} |A(x_i)| \text{ , whence, by(3),}$$
$$M \geq \kappa \frac{v - m}{m}. \tag{4}$$

Note that since G is $(H; k)$ stable, $G - x_t$ contains a copy of H. By the same reason as previously, this copy cannot contain any vertex from $A(x_t)$. Thus, $v - |A(x_t)| - 1 = v - M - 1 \geq n$. Hence,

$$v \geq (n + 1 - \kappa)\frac{m}{m - \kappa}. \tag{5}$$

Furthermore,

$$2|E(G)| \geq (\delta + 1)m + \delta(v - m) \text{ hence}$$
$$|E(G)| \geq \frac{m}{2} + \frac{v\delta}{2} \geq \frac{m}{2} + (n + 1 - \kappa)\frac{m\delta}{2(m - \kappa)}. \tag{6}$$

Let $f(x) := x/2 + (n + 1 - \kappa)\frac{\delta x}{2(x-\kappa)}$, $x > \kappa$. By simple computations, one can see that f has minimum in $x_0 = \sqrt{\delta\kappa(n + 1 - \kappa)} + \kappa$. Hence, $|E(G)| \geq f(x_0) = \frac{\delta}{2}(n - \kappa + 1) + \sqrt{\delta\kappa(n - \kappa + 1)} + \frac{\kappa}{2}$. □

3 Tightness

Example. Let δ be an even positive integer. Let $t = p^2\delta$ for some integer $p \geq 2$. We will construct a graph $H(t)$, such that $\text{stab}(H(t); 1)$ is near the lower bound from Theorem 1.

Then

$$V(H(t)) := V_0 \cup V_0' \cup V_1 \cup V_1' \cup \ldots \cup V_{t-1} \cup V_{t-1}'$$

with $|V_i| = |V_i'| = \delta/2$ for all $i = 0, \ldots, t - 1$. Note that $|V(H(t))| = n = t\delta$. The set of edges is defined in following way. For all $i = 0, \ldots, t - 1$:

1. there is a clique K_δ built on $V_i \cup V_i'$,
2. there is a perfect matching between V_i' and V_{i+1} ($i + 1$ taken modulo t), see Fig. 1.

Claim 1 $H(t)$ is δ-regular.

Claim 2 The vertex-connectivity of $H(t)$ is equal to δ.

Proof. Take any $x, y \in V(H(t))$. We show that there are δ vertex-independent paths between x and y. Let $x \in V_i \cup V_i'$ and $y \in V_j \cup V_j', i \leq j$. Since $V_i \cup V_i'$ makes up a δ-clique, every vertex of this set can be used in a different path. A half of these paths go through $V_{i+1}, V_{i+1}', V_{i+2}, V_{i+2}' \ldots, V_{j-1}'$ and the rest go through $V_{i-1}', V_{i-1}, V_{i-2}', V_{i-2} \ldots, V_{j+1}$ (all indices taken modulo t). In the end all paths reach the clique built on $V_j \cup V_j'$ and, finally the vertex y. □

Claim 3 Let $n = |H(t)|$. Then

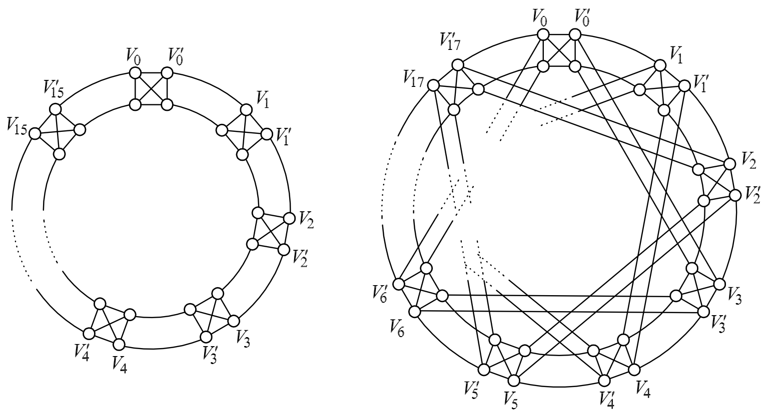
$$\frac{(n-\delta+2)\delta}{2} + \delta\sqrt{n-\delta+1} \leq \text{stab}(H(t); 1) \leq \frac{(n+1)\delta}{2} + \delta\sqrt{n}$$

Proof. The lower bound follows from Theorem 1. Consider the following graph $G(t+p)$.

$$V(G) = V_0 \cup V_0' \cup V_1 \cup V_1' \cup \ldots \cup V_{t+p-1} \cup V_{t+p-1}'$$

with $|V_i| = |V_i'| = \frac{\delta}{2}$ for all $i = 0, \ldots, t+p-1$. The set of edges is defined in following way. For all $i = 0, \ldots, t+p-1$:

1. There is a clique K_δ built on $V_i \cup V_i'$.
2. There is a perfect matching between V_i' and V_{i+1} ($i+1$ taken modulo $t+p$).
3. There is a perfect matching between V_{ip}' and V_{ip+p+1} for all $i = 0, \ldots, \frac{t}{p}$ (indices taken modulo $t+p$), see Fig. 1.



$H(16)$ $G_1(18)$

Figure 1:

Note that $|G(t+p)| = (t+p)\delta$. Moreover, without edges from item 3, the graph is δ-regular. Hence,

$$||G(t+p)|| = \frac{(t+p)\delta^2}{2} + \frac{\delta(t+p)}{2p}.$$

Since $p = \sqrt{\frac{t}{\delta}}$ we obtain

$$||G(t+p)|| = \frac{t\delta^2}{2} + \frac{\delta\sqrt{t\delta}}{2} + \frac{\delta\sqrt{t\delta}}{2} + \frac{\delta}{2} = \frac{n\delta}{2} + \delta\sqrt{n} + \frac{\delta}{2}.$$

Now we have to prove that $G(t+p)$ is in fact $(H(t), 1)$-stable. Without loss of generality we can consider a graph $G(t+p) - x$ where $x \in V_i \cup V_i'$, $i \in \{1, \ldots, p\}$. Then it can be seen that

$$H(t) \subset G(t+p)[V_0 \cup V_0' \cup V_{p+1} \cup V_{p+1}' \cup \ldots \cup V_{t+p-1} \cup V_{t+p-1}'] \subset G(t+p) - x.$$

□

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