Near packings of graphs

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Abstract

A *packing* of a graph G is a set $\{G_1, G_2\}$ such that $G_1 \cong G$, $G_2 \cong G$, and G_1 and G_2 are edge disjoint subgraphs of K_n. Let $\mathcal{F}$ be a family of graphs. A *near packing admitting* $\mathcal{F}$ of a graph G is a generalization of a packing. In a near packing admitting $\mathcal{F}$, the two copies of G may overlap so the subgraph defined by the edges common to both copies is a member of $\mathcal{F}$. In the paper we study three families of graphs (1) $\mathcal{E}_k$ – the family of all graphs with at most k edges, (2) $\mathcal{D}_k$ – the family of all graphs with maximum degree at most k, and (3) $\mathcal{C}_k$ – the family of all graphs that do not contain a subgraph of connectivity greater than or equal to $k+1$. By $m(n, \mathcal{F})$ we denote the maximum number m such that each graph of order n and size less than or equal to m has a near-packing admitting $\mathcal{F}$. It is well known that $m(n, \mathcal{C}_0) = m(n, \mathcal{D}_0) = m(n, \mathcal{E}_0) = n-2$ because a near packing admitting $\mathcal{C}_0$, $\mathcal{D}_0$ or $\mathcal{E}_0$ is just a packing. We prove some generalization of this result, namely we prove that $m(n, \mathcal{C}_k) \approx (k+1)n$, $m(n, \mathcal{D}_1) \approx \frac{3}{2}n$, $m(n, \mathcal{D}_2) \approx 2n$. We also present bounds on $m(n, \mathcal{E}_k)$. Finally, we prove that each graph of girth at least five has a near packing admitting $\mathcal{C}_1$ (i.e. a near packing admitting the family of the acyclic graphs).

1 Introduction

In this paper we use the term *graph* to refer to simple graphs without loops or multiple edges. The vertex and edge set of a graph G is denoted by $V(G)$ and $E(G)$, respectively. The maximum degree of G is denoted by $\Delta(G)$. A graph is called k-connected if any two of its vertices can be joined by k internally vertex disjoint paths. A complete graph K_1 is

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0-connected. By $N_G(x)$ we denote the set of vertices adjacent with x in G. For a vertex set X, the set $N_G(X)$ denotes the external neighbourhood of X in G, i.e.

$$N_G(X) = \{y \in V(G) \setminus X : y \text{ is adjacent with some } x \in X\}.$$

The degree of a vertex x is the number of vertices adjacent to x and is denoted by $d_G(x)$.

Definition 1. Let G_1 and G_2 be two graphs such that $|V(G_1)| = |V(G_2)| = n$. A *packing* of G_1 and G_2 is a pair of edge-disjoint subgraphs $\{H_1, H_2\}$ of K_n such that $H_1 \cong G_1$ and $H_2 \cong G_2$.

Definition 2. Let $\mathcal{F}$ be any family of graphs and let G_1, G_2 be two graphs such that $|V(G_1)| = |V(G_2)| = n$. A *near packing admitting* $\mathcal{F}$ of G_1 and G_2 is a pair of subgraphs $\{H_1, H_2\}$ of K_n such that $H_1 \cong G_1$ and $H_2 \cong G_2$, and the subgraph having edges $E(H_1) \cap E(H_2)$ is a member of $\mathcal{F}$.

Given a graph G and a permutation σ of $V(G)$, by $\sigma(G)$ we denote the graph with $V(\sigma(G)) = V(G)$ and such that $\sigma(u)\sigma(v) \in E(\sigma(G))$ if and only if $uv \in E(G)$ for any $u, v \in V(G)$. The spanning subgraph of G having edges $E(G) \cap E(\sigma(G))$ is denoted by G^*_σ (abbreviated to G^* if no confusion arises). Thus, in case when $G_1 \cong G_2 \cong G$ the problem of finding a near packing admitting $\mathcal{F}$ of G_1 and G_2 is equivalent to the problem of finding a permutation σ of $V(G)$ such that $G^*_\sigma \in \mathcal{F}$. Such a permutation σ of $V(G)$ is called *a near packing of* G *admitting* $\mathcal{F}$.

We consider three families of graphs : (1) $\mathcal{E}_k$ being the family of all graphs with with at most k edges, (2) $\mathcal{D}_k$ being the family of all graphs with maximum degree at most k, and (3) $\mathcal{C}_k$ being the family of all graphs that do not contain a subgraph of connectivity greater than or equal to $k+1$. Notice that $\mathcal{D}_0 = \mathcal{C}_0 = \mathcal{E}_0$ is a family of edgeless graphs. Furthermore $\mathcal{C}_1$ is a family of acyclic graphs and $\mathcal{C}_1 \cap \mathcal{D}_2$ is a family of linear forests (i.e. disjoint unions of paths).

Let $\mathcal{F}$ be any family of graphs. By $m(n, \mathcal{F})$ we denote the maximum number m such that each graph of order n and size less than or equal to m has a near-packing admitting $\mathcal{F}$. A classic result in this area, obtained independently in [1, 2, 7], states that

Theorem 3 ([1, 2, 7])**.** *$m(n, \mathcal{C}_0) = m(n, \mathcal{D}_0) = m(n, \mathcal{E}_0) = n - 2$,*

because a near packing admitting $\mathcal{C}_0$, $\mathcal{D}_0$ or $\mathcal{E}_0$ is just a packing. Our aim is to prove some generalizations of Theorem 3. For every $k \geqslant 1$, we determine $m(n, \mathcal{C}_k)$ up to a constant depending only on k. We find the problem concerning near packings admitting $\mathcal{D}_k$ considerably harder. We determine only $m(n, \mathcal{D}_1)$ up to a constant, while $m(n, \mathcal{D}_2)$ is determined assymptotically. We also give bounds on $m(n, \mathcal{E}_k)$.

The notion of a near packing was introduced by Eaton [3] in order to obtain some investigations concerning the following conjecture of Bollobás and Eldridge:

Conjecture 4 ([1])**.** *If $|V(G_1)| = |V(G_2)| = n$ and $(\Delta(G_1) + 1) \cdot (\Delta(G_2) + 1) \leqslant n + 1$, then there is a packing of G_1 and G_2.*

The following theorem is a special case of a more general result proved by Eaton.

Theorem 5 ([3]). *If $|V(G_1)| = |V(G_2)| = n$ and $(\Delta(G_1)+1)\cdot(\Delta(G_2)+1) \leqslant n+1$, then there is a near packing admitting $\mathcal{D}_1$ of G_1 and G_2.*

We also investigate another conjecture of graph packing by Faudree, Rousseau, Schelp and Schuster [4]:

Conjecture 6. *For every non-star graph G of girth at least 5, there is a packing of two copies of G.*

In particular, Conjecture 6 is true for sufficiently large planar graphs [6]. On the other hand, the statement from the above conjecture is true if G is a non-star graph of girth at least six [5]. In this paper we prove that the statement is true if the term 'packing' is replaced by the term 'near packing admitting $\mathcal{C}_1$'. This result is in some sense best possible, since for every permutation σ of $V(K_{n,n})$ with $n \geqslant 3$, $K^*_{n,n}$ contains a cycle C_4.

2 Preliminaries

Lemma 7. *Let G be a graph and $k, l, q \geqslant 0$ integers. Suppose that G contains an independent set $U \subset V(G)$ that satisfies the following conditions:*

1. *$d_G(u) \leqslant k$ for each $u \in U$,*
2. *$|N_G(u) \cap N_G(v)| \leqslant q$ for every $u, v \in U$.*

If $|U| \geqslant \frac{2(k-q)}{l-q+1}$, then for every permutation σ' of $V(G) \setminus U$ there exists a permutation σ of $V(G)$ such that $\sigma|_{G-U} = \sigma'$ and $d_{G^*_\sigma}(u) \leqslant l$ for each $u \in U$.

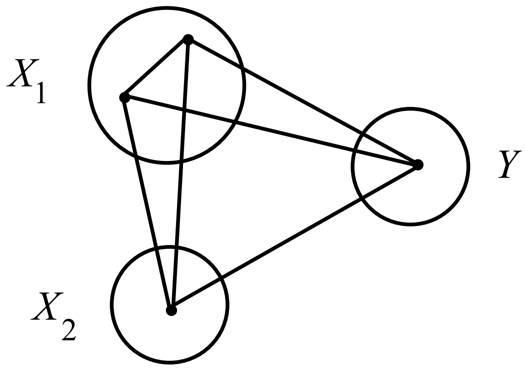
Proof. Let $G' := G - U$ and σ' be any permutation of $V(G')$. Below we show that we can extend σ' to a permutation σ as required of G.
For any $v \in V(G')$ let us define $\sigma(v) := \sigma'(v)$. Then let us consider a bipartite graph B with partition sets $X := U \times \{0\}$ and $Y := U \times \{1\}$. For $u, v \in U$ the vertices $(u, 0)$, $(v, 1)$ are joined by an edge in B if and only if $|\sigma'(N_G(u)) \cap N_G(v)| \leqslant l$. So, if $(u, 0)$, $(v, 1)$ are joined by an edge in B we can put $\sigma(u) = v$. In other words, if $(u, 0)$, $(v, 1)$ are not neighbors in B, then $|\sigma'(N(u)) \cap N(v)| \geqslant l + 1$. Therefore, since $|N_G(u) \cap N_G(v)| \leqslant q$ and $d_G(u) \leqslant k$ for $u \in U$, we have $d_B((u, 0)) \geqslant |U| - \frac{k-q}{l-q+1} \geqslant \frac{k-q}{l-q+1}$, by the assumption on $|U|$. Similarly, $d_B((v, 1)) \geqslant \frac{k-q}{l-q+1}$.

Let $S \subset X$. If $|S| \leqslant |U| - \frac{k-q}{l-q+1}$ then obviously $|N_B(S)| \geqslant |S|$. Notice that if $|S| > |U| - \frac{k-q}{l-q+1}$ then $N_B(S) = Y$. Indeed, othervise let $(v, 1) \in Y$ be a vertex which has no neighbour in S. Thus,

$$d_B((v, 1)) \leqslant |A| - |S| = |U| - |S| < |U| - (|U| - \frac{k-q}{l-q+1}) = \frac{k-q}{l-q+1},$$

a contradiction. Hence, in any case $|S| \leqslant |N(S)|$. Thus, by the Hall's theorem there is a matching M in G. Therefore we can define $\sigma(u) = v$ for $u, v \in U$ such that $(u, 0)$, $(v, 1)$ are incident with the same edge in M. □

Figure 1: $K^+_{2,1,1}$



Proposition 8. *Let G be a graph of order n and size m with $m \leqslant an - f(n)$, where a is a real number and $f(n)$ is a non-decreasing function. If $U \subset V(G)$ and vertices from U cover at least $a|U|$ edges, then*

$$m' \leqslant an' - f(n'),$$

where n' and m' are respectively the order and the size of $G - U$.

Proof.

$$\begin{aligned} m' &\leqslant an - f(n) - a|U| = a(n - |U|) - f(n) \\ &\leqslant a(n - |U|) - f(n - |U|) = an' - f(n'), \end{aligned}$$

because $f(n) \geqslant f(n - |U|)$. □

3 Near packings admitting $\mathcal{C}_k$

Recall that $m(n, \mathcal{C}_0) = n - 2$. We start with the following construction. Let $K^+_{s,k-s,k-s}$ denote a graph with vertex set $V(K^+_{s,k-s,k-s}) = X_1 \cup X_2 \cup Y$ such that X_1, X_2, Y are pairwise disjoint and $|X_1| = s$, $|X_2| = |Y| = k - s$. Furthermore, $E(K^+_{s,k-s,k-s}) = E_1 \cup E_2$, where $E_1 = \{xy : x \in X_1 \cup X_2, y \in Y\}$ and $E_2 = \{xz : x \in X_1, z \in X_1 \cup X_2\}$. In other words, $K^+_{s,k-s,k-s}$ arises from a tripartite graph (with partition sets X_1, X_2 and Y) by adding all possible edges having two endpoints in X_1, see Figure 1. It is easily seen that any two vertices of $K^+_{s,k-s,k-s}$ are joined by at least k internally vertex disjoint paths, so $K^+_{s,k-s,k-s}$ is k connected. In what follows $\bar{G}$ denotes the complement of a graph G, i.e. a graph on the same vertex set as G and with the property that $e \in E(\bar{G})$ if and only if $e \notin E(G)$.

Lemma 9. $m(n, \mathcal{C}_k) \leqslant (k+1)n - (k+1)\frac{k+2}{2} - 1$

Proof. Let $G = \overline{K_{k+1}} + K_{n-k-1}$. Clearly, $|E(G)| = (k+1)n - (k+1)\frac{k+2}{2}$. We will show that G does not have a near packing admitting $\mathcal{C}_k$. Consider an arbitrary permutation σ of $V(G)$. Let $S \subset V(K_{k+1})$ be a maximal set of vertices with the property that $\sigma(S) \subset V(K_{k+1})$. Let $|S| = s$. Then, G^*_σ contains a $K^+_{s,k+1-s,k+1-s}$ with $X_1 = S$, $Y = V(K_{k+1}) \setminus S$ and $X_2 \subset V(K_{n-k-1})$. □

Theorem 10. $m(n, \mathcal{C}_k) \geqslant (k+1)n - 4k(k+1)^2 - 2$

Proof. For $k = 0$ the result follows from Theorem 3. Fix $k \geqslant 1$ and let $c_k = 4k(k+1)^2 + 2$. We will prove that each graph of order n and size at most $(k+1)n - c_k$ has a near packing admitting $\mathcal{C}_k$.

Suppose that G is a counterexample with minimum order n. Let m denote the size of G, so $m \leqslant (k+1)n - c_k$. Note that if $n \leqslant 4(k+1)^2$, then

$$\begin{aligned} m &\leqslant (k+1)n - c_k = kn - c_k + n \\ &\leqslant k(4(k+1)^2) - (4k(k+1)^2 + 2) + n = n - 2. \end{aligned}$$

Hence G has a near packing admitting $\mathcal{C}_k$, by Theorem 3, which contradicts our assumption on G. Thus, we may assume that $n \geqslant 4(k+1)^2 + 1$. Furthermore, if $\Delta(G) \leqslant 2(k+1) - 1$ then $(\Delta(G) + 1)^2 \leqslant 4(k+1)^2 < n + 1$. Hence, G has a near packing admitting $\mathcal{C}_k$ by Theorem 5 (because $\mathcal{D}_1 \subset \mathcal{C}_k$), a contradiction again. Therefore, we may assume that $\Delta(G) \geqslant 2(k+1)$. Let $w \in V(G)$ with $d_G(w) \geqslant 2(k+1)$.

Suppose first that G contains a vertex u with $d_G(u) \leqslant k$. By Proposition 8 and by the minimality assumption, $G' := G - \{u, w\}$ has a near packing σ' admitting $\mathcal{C}_k$. We claim that $\sigma := (u, w)\sigma'$ is a near packing of G admitting $\mathcal{C}_k$. Indeed, since $d_G(u) \leqslant k$ then $d_{G^*}(u) \leqslant k$ as well as $d_{G^*}(w) \leqslant k$. Hence, neither u nor w can be in a subgraph of G^* of connectivity $k+1$ or more. Moreover, since $\sigma|_{G'}$ is a near packing of G' admitting $\mathcal{C}_k$, then $G^* - \{u, w\}$ does not contain a subgraph of connectivity $k+1$ or more, neither. Therefore, σ is a near packing of G admitting $\mathcal{C}_k$.

Thus, we may assume that $d_G(u) \geqslant k+1$ for every $u \in V(G)$. Let S be a maximum set of vertices of G such that S is independent, $k+1 \leqslant d_G(u) \leqslant 2k+1$ for each $u \in S$, and $|N_G(u) \cap N_G(w)| \leqslant k$ for every $u, w \in S$. Since S is independent, by Proposition 8 and by the minimality assumption, $G - S$ has a near packing σ'' admitting $\mathcal{C}_k$. By Lemma 7 (with k, l, q replaced by $2k+1, k, k$, respectively), if $|S| \geqslant 2k+2$ then there is a permutation σ of G, such that $\sigma|_{G-S} = \sigma''$ and $d_{G^*}(u) \leqslant k$ for every $u \in S$. Simirarly as before, we can see that σ is a near packing of G admitting $\mathcal{C}_k$, a contradiction.

Therefore $|S| \leqslant 2k+1$ and so $|N_G(S)| \leqslant (2k+1)^2$. Let $V_j = \{v \in V(G) \setminus N_G(S) : d_G(v) = j\}$. Note that by the definition of S, we have $|N_G(S) \cap N_G(u)| \geqslant k+1$ for every $u \in V_{k+1} \cup \cdots \cup V_{2k+1}$. Hence, vertices from $N_G(S)$ are incident (in common) to at least

$(k+1)(|V_{k+1}|+\cdots+|V_{2k+1}|)$ edges. Thus,

$$\begin{aligned}
&(2k+2)n-8k(k+1)^2-4 \geqslant 2m\\
&=\sum_{u\in N_G(S)} d_G(u)+\sum_{v\in V(G)\setminus N_G(S)} d_G(v)\\
&\geqslant (k+1)(|V_{k+1}|+\dots|V_{2k+1}|)+(k+1)|V_{k+1}|+\cdots+(2k+1)|V_{2k+1}|\\
&+(2k+2)(n-|V_{k+1}|+\dots|V_{2k+1}|-|N_G(S)|)\\
&\geqslant (2k+2)n-(2k+2)(2k+1)^2,
\end{aligned}$$

a contradiction. Hence, we deduce no counterexample to Theorem 10 exists. □

Theorem 11. *Every graph of girth at least 5 has a near packing admitting $\mathcal{C}_1$.*

Proof. Let G be a minimum counterexample to Theorem 11. Let $u\in V(G)$ with $d_G(u)=\Delta(G)$. Let $G'=G-u$ and $U=N_G(u)$. By the girth assumption, U is an independent set in G' (as well as in G), and $N_{G'}(x)\cap N_{G'}(y)=\emptyset$ for every $x,y\in U$. By the minimality assumption $G'':=G'-U$ has a near packing σ'' admitting $\mathcal{C}_1$. Moreover, $|U|=\Delta(G)$ and $d_{G'}(u)\leqslant\Delta(G)-1$. Hence, by Lemma 7 (with $k=\Delta(G)-1, l=1, q=0$), G' has a near packing σ' such that $\sigma'|_{G''}=\sigma''$ and $d_{G'^*}(u)\leqslant 1$ for each $u\in U$. Thus, since G''^* is acyclic, G'^* is also acyclic. Let u be any vertex from U. It is easy to see that the permutation σ such that $\sigma(u)=x$, $\sigma(x)=u$ and $\sigma(y)=\sigma'(y)$ for every $y\in V(G)\setminus\{u,x\}$ is a near packing of G admitting $\mathcal{C}_1$, a contradiction. □

4 Near packings admitting $\mathcal{D}_k$

Recall that $m(n,\mathcal{D}_0)=n-2$.

Lemma 12. $m(n,\mathcal{D}_k)\leqslant\left\lceil\frac{(k+2)(n-1)}{2}\right\rceil-1$.

Proof. Let H be a k-regular graph of order $n-1$ provided that k is even or $n-1$ is even. Otherwise, let H be a graph with all but one vertices having degree k and one vertex having degree $k+1$. Let $G=K_1+H$ and $V(K_1)=\{u\}$. It is easily seen that for any permutation σ of $V(G)$, the vertex u (as well as its image) has degree at least $k+1$ in G^*_σ. Thus, G does not have a near packing admitting $\mathcal{D}_k$. Furthermore, $E(G)=\frac{(k+1)(n-1)+n-1}{2}=\frac{(k+2)(n-1)}{2}$ if k is even or $n-1$ is even, or $E(G)=\frac{(k+1)(n-2)+(k+2)+(n-1)}{2}=\frac{(k+2)(n-1)+1}{2}$ otherwise. □

We are tempted to propose the following conjecture

Conjecture 13.
$$\frac{k+2}{2}n-c_1(k)\leqslant m(n,\mathcal{D}_k)\leqslant\frac{k+2}{2}n-c_2(k),$$
where $c_i(k)$ are constants depending only on k.

The next theorem confirms Conjecture 13 for $k=1$.

Theorem 14. $m(n, \mathcal{D}_1) \geqslant \frac{3}{2}n - 10$

Proof. Let G be a counterexample of minimum order n. Without loss of generality we assume that $m := |E(G)| = \frac{3}{2}n - 10$. Note that if $n \leqslant 16$ then $\frac{3}{2}n - 10 \leqslant n - 2$. Thus, by Theorem 3, G has a packing which contradicts our assumption on G. Hence, we may assume that $n \geqslant 17$. Furthermore, if $\Delta(G) \leqslant 3$, then $(\Delta(G)+1)^2 \leqslant 16 < n+1$, so G has a near packing admitting $\mathcal{D}_1$ by Theorem 5. Thus, we may assume that $\Delta(G) \geqslant 4$. Let $w \in V(G)$ with $d_G(w) \geqslant 4$.

Suppose first that G has a vertex u with $d_G(u) = 0$. Then, by Proposition 8 and by the minimality assumption, $G_1 := G - \{u, w\}$ has a a near packing σ_1 admitting $\mathcal{D}_1$. Clearly, $(u, w)\sigma_1$ is a near packing of G admitting $\mathcal{D}_1$.

So we may assume that G has no isolated vertex. Suppose now that G has a vertex u with $d_G(u) = 1$ and let v be the neighbor of u. If $d_G(v) \geqslant 3$ then, by Proposition 8 and by the minimality assumption, $G_2 := G - \{u, v\}$ has a near packing σ_2 admitting $\mathcal{D}_1$. Clearly, $(u, v)\sigma_2$ is a near packing admitting $\mathcal{D}_1$ of G. Similarly, if $d_G(v) = 1$ then $(u)(w, v)\sigma_3$ is a near packing admitting $\mathcal{D}_1$ of G where σ_3 is a near packing admitting $\mathcal{D}_1$ of $G - \{u, v, w\}$ (σ_3 exists by the minimality assumption). Thus we may assume that $d_G(v) = 2$. Let x be the neighbor of v different from u. If $x \neq w$ then $(u)(v, w, x)\sigma_4$ is a near packing admitting $\mathcal{D}_1$ of G where σ_4 is a near packing admitting $\mathcal{D}_1$ of $G - \{u, v, w, x\}$ (σ_4 exists by the minimality assumption). Finally, if $x = w$ then $(u)(v, w)\sigma_5$ is a near packing admitting $\mathcal{D}_1$ of G where σ_5 is a near packing admitting $\mathcal{D}_1$ of $G - \{u, v, w\}$ (σ_5 exists by the minimality assumption).

Therefore, we may assume that $d_G(u) \geqslant 2$ for each $u \in V(G)$. Let $S \subset V(G)$ be a maximal set such that S is independent in G, $d_G(v) = 2$ for every $v \in S$, and $N_G(u) \cap N_G(v) = \emptyset$ for every $u, v \in S$. Note that $S \neq \emptyset$. By Proposition 8 and by the minimality assumption, $G - S$ has a near packing σ' admitting $\mathcal{D}_1$. Note that if $|S| \geqslant 4$, then by Lemma 7 (with $k = 2$, $q = 0$ and $l = 0$), there exists a near packing of G admitting $\mathcal{D}_1$, a contradiction with the assumption on G. Thus, $|S| \leqslant 3$ and so $|N_G(S)| \leqslant 6$. Let $V_j = \{v \in V(G) \setminus N_G(S) : d_G(v) = j\}$. Note that by the definition of S, we have $|N_G(S) \cap N_G(u)| \geqslant 1$ for every $u \in V_2$. Therefore,

$$\begin{aligned} 3n - 20 = 2m &= \sum_{u \in N_G(S)} d_G(u) + \sum_{v \in V(G) \setminus N_G(S)} d_G(v) \\ &\geqslant |V_2| + 2|V_2| + 3(n - |V_2| - |N_G(S)|) \geqslant 3n - 18, \end{aligned}$$

a contradiction. Hence, we deduce no counterexample to Theorem 14 exists. □

The following result provides some evidence for Conjecture 13 in case when $k = 2$.

Theorem 15 ([8])**.** *$m(n, \mathcal{D}_2) \geqslant 2n - 10n^{2/3} - 7$.*

5 Near packings admitting $\mathcal{E}_k$

The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ together with all the edges joining V_1 and V_2.

Lemma 16. *If* $n \geqslant 2k+2$ *then* $m(n, \mathcal{E}_{2k}) \leqslant \left\lceil \frac{(k+2)(n-1)}{2} \right\rceil - 1$.

Proof. Let H be a k-regular graph of order $n-1$ provided that k is even or $n-1$ is even. Otherwise, let H be a graph with all but one vertices having degree k and one vertex having degree $k+1$. Let $G = K_1 + H$ and $V(K_1) = \{u\}$. It is easily seen that for any permutation σ of $V(G)$, the vertex u as well as $\sigma(u)$ has degree at least $k+1$ in G^*_σ. Thus, if $u \neq \sigma(u)$ then G^*_σ has at least $2k+1$ edges. If $u = \sigma(u)$ then u has degree $n-1$ in G^*_σ. Since $n \geqslant 2k+2$, G^*_σ has at least $2k+1$ edges. Therefore, G does not have a near packing admitting $\mathcal{E}_{2k}$. Furthermore, $E(G) = \frac{(k+1)(n-1)+n-1}{2} = \frac{(k+2)(n-1)}{2}$ if k is even or $n-1$ is even, or $E(G) = \frac{(k+1)(n-2)+(k+2)+(n-1)}{2} = \frac{(k+2)(n-1)+1}{2}$ otherwise. □

Theorem 17. $m(n, \mathcal{E}_k) \geqslant \sqrt{\frac{k}{2} n(n-1)}$.

Proof. Let G be a graph of order n and size m. We will prove that if $m \leqslant \sqrt{\frac{k}{2} n(n-1)}$ then there is a near-packing of G admitting $\mathcal{E}_k$. Consider the probability space whose $n!$ points are the permutations of $V(G)$. For any two edges $e, f \in E(G)$ let X_{ef} denote the indicator random variable with value 1 if f is an image of e. Then

$$E(X_{ef}) = Prob(X_{ef} = 1) = \frac{2(n-2)!}{n!} = \binom{n}{2}^{-1}.$$

Let $X = \sum_{e,f \in E(G)} X_{ef}$. Thus, by the linearity of expectation, we have

$$E(X) = \sum_{e,f \in E(G)} E(X_{ef}) \leqslant m^2 \binom{n}{2}^{-1} \leqslant k.$$

This implies that there exists a permutation σ of $V(G)$ such that G^*_σ has at most k edges. Thus, σ is a near packing of G admitting $\mathcal{E}_k$. □

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