On packable digraphs

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Abstract

One of the classical results in packing theory states that every graph of order n and size less than or equal to $n-2$ is packable in its complement. Moreover, the bound is sharp because the star is not packable. A similar problem arises for digraphs, namely, to find the maximal number $f_D(n)$ such that every digraph of order n and size less than or equal to $f_D(n)$ is packable. So far it is known that $\frac{7}{4}n - 81 \leq f_D(n) \leq 2n-3$ where the upper bound is sharp. In this paper we prove that $f_D(n) = 2n - o(n)$.

1 Introduction

We deal with finite, directed graphs without loops or multiple arcs. We use standard graph theory notation. Let D be a digraph with a vertex set $V(D)$ and an arc set $A(D)$. For any vertex x in $V(D)$ let us denote by $d^+(v)$ the outer degree of x. By $d^-(x)$ we denote the inner degree of x. The degree of a vertex x, denoted by $d(x)$, is defined by $d(x) = d^+(x) + d^-(x)$. By $N(x)$ we denote the set of vertices incident with x in D. If xy and yx belong to $A(D)$, then we say that x and y are joined by a *symmetric arc*. For two permutations σ', σ'' a permutation $\sigma = \sigma'\sigma''$ is the product of σ' and σ''.

Let G be a graph (a digraph) with a vertex set $V(G)$. The order of G is denoted by $|G|$ and the size is denoted by $||G||$. We say that G is *packable in its complement* (G is packable, in short) if there is a permutation σ on $V(G)$ such that if xy is an an edge (an arc) of G, then $\sigma(x)\sigma(y)$ is not an edge (an arc) in G. A graph (a digraph) is *self-complementary* if it is isomorphic to its complement. Obviously, every self-complementary graph is packable.

The problem of finding the maximum number $f_G(n)$ such that every graph G of order $|G| = n$ and size $||G|| \leq f_G(n)$ is packable was independently solved in [2, 3, 6].

Theorem 1 *Let G be a graph of order n such that $||G|| \leq n-2$. Then G is packable.*

The example of the star shows that Theorem 1 cannot be improved by raising the size of G. However it can be improved in other ways. The following theorem was proved in [7]:

Theorem 2 *Let G be a graph of order n such that $||G|| \leq n-2$. Then G is packable without fixed points, i.e. $\sigma(x) \neq x$ for every $x \in V(G)$.*

A similar problem arises for digraphs with a corresponding function $f_D(n)$. If a digraph D has only symmetric arcs, then by Theorem 1, D is packable if $||D|| \leq 2n-4$. On the other hand, the example of a complete bipartite digraph D' with $V(D') = \{v_1, ..., v_n\}$ and $A(D') = \{(v_1, v_j), (v_j, v_1); j = 2, ..., n\}$ shows that $f_D(n) \leq 2n-3$. This leads to the following conjecture which in a stronger form was formulated in [1].

Conjecture 3 *Let D be a digraph of order n and size $||D|| \leq 2n-3$. Then D is packable.*

The first result related to Conjecture 3 was that every digraph of order $n \geq 3$ and size at most n is contained in a self-complementary digraph of order n, see [1]. Hence every such digraph is packable. Clearly, a digraph of order 2 and size equal to 0 or 1 is packable, hence we obtain:

Remark 4 *Every digraph D with $||D|| < |D|$ is packable.*

The bound on the size was improved in [8].

Theorem 5 *Let D be a digraph of order n such that $||D|| \leq \frac{3}{2}(n-2)$. Then D is contained in a self-complementary digraph of order n.*

So far the best known result concerning Conjecture 3 was presented in [4].

Theorem 6 *Let D be a digraph of order n and size $||D|| \leq \frac{7}{4}n - 81$. Then D is packable without fixed points, i.e. $\sigma(x) \neq x$ for every $x \in V(D)$.*

Therefore $\frac{7}{4}n - 81 \leq f_D(n) \leq 2n-3$. We shall prove that $f_D(n) = 2n - o(n)$. Namely we shall prove the following:

Theorem 7 *Let D be any digraph of order n and size $||D|| \leq 2n - 10n^{2/3} - 7$. Then D is packable.*

Note that our new lower bound is better than the previous one for $n \geq 63107$.

Finally, we recall another classical result in packing theory [6] for it will be used in the proof of Theorem 7.

Theorem 8 *Let G_1 and G_2 be graphs of order n such that $2\Delta(G_1)\Delta(G_2) < n$. Then the complete graph K_n contains edge disjoint copies of G_1 and G_2.*

Sketch of the proof of Theorem 7. The proof is by induction on n. Let D be a digraph of order n and size at most $2n - 10n^{2/3} - 7$. If $2n - 10n^{2/3} - 7 \leq \frac{7}{4}n - 81$ then D is packable by Theorem 5. The above inequality holds for $22 \leq n \leq 63107$ (but $2n - 10n^{2/3} - 7 < 0$ for $n < 22$). From now on we assume that D is a digraph of order $n \geq 63107$ and size $||D|| = \lfloor 2n - 10n^{2/3} - 7 \rfloor$, and we assume the theorem holds for every digraph with order less than n. Moreover, by Theorem 8 we may assume that $\Delta(D) \geq 177$.

The paper is organized as follows. In the next section we prove some preliminary lemmas. They will be needed in the main part of the proof of Theorem 7 presented in the third section.

2 Lemmas

We start with a lemma which is a slight modification of Lemma 2 in [4]. In the new form it can be applied more widely.

Lemma 9 *Let G be a digraph and $k \geq 1$ be any positive integer. If there is a set $U = \{v_1, ..., v_{2k}\} \subset V(G)$ of $2k$ independent vertices of G such that*

1. *vertices of U have degrees at most k;*
2. *vertices of U have mutually disjoint sets of neighbors, i.e. $N(v_i) \cap N(v_j) = \emptyset$ for $i \neq j$;*
3. *there is a packing σ' of $G - U$,*

then there exists a packing σ of G.

For completness we give the proof of the above lemma although it is analogous to the proof of Lemma 2 in [4].
Proof. Let $G' := G - U$ and let σ' be a packing of G'. Below we show that we can find a packing σ of G.
For any $v \in V(G')$ let us define $\sigma(v) := \sigma'(v)$. Then let us consider a bipartite graph H with partite sets $A = U \times \{0\}$ and $B = U \times \{1\}$. For $i, j \in \{1, ..., 2k\}$ the vertices $(v_i, 0)$, $(v_j, 1)$ are joined by an edge in H if and only if $\sigma'(N(v_i)) \cap N(v_j) = \emptyset$. So, if $(v_i, 0)$, $(v_j, 1)$ are joined by an edge in H we can put $\sigma(v_i) = v_j$. Because $d(v_i) \leq k$ for $i \in \{1, ..., 2k\}$, $d((v_i, 0)) \geq 2k - k = k$ and $d((v_i, 1)) \geq 2k - k = k$. Let $S \subset A$. If $|S| \leq k$ then obviously $|N(S)| \geq |S|$. Notice that if $|S| > k$, then $N(S) = B$. Indeed, otherwise let $(v_j, 1) \in B$ be a vertex which has no neighbour in S. Thus $d((v_j, 1)) \leq |A| - |S| \leq 2k - (k + 1) = k - 1$, a contradiction. Hence, in any case $|S| \leq |N(S)|$. Thus, by Hall's theorem there is a matching M in H. Therefore we can define $\sigma(v_i) = v_j$ for $i, j \in \{1, ..., 2k\}$ such that $(v_i, 0)$, $(v_j, 1)$ are incident with the same edge in M. □

Let T_1, T_2 be vertex-disjoint digraphs such that they do not contain any symmetric arc and their underlying graphs are trees (we include isolated vertices as trivial trees). Let x be a vertex belonging neither to the vertex set of T_1 nor T_2 and let B be any non-empty set of nonsymmetric arcs such that if an arc uv belongs to B then $u = x$ or $v = x$. A digraph $H = (V, A)$ we call *a starry tree* if $V = V(T_1) \cup V(T_2) \cup \{x\}$ and $A = A(T_1) \cup A(T_2) \cup B$. We call the vertex x *a middle vertex* of H. Note that a starry tree need not be connected.

Lemma 10 *Let H be a starry tree. Then there is a packing of H such that the middle vertex of H is the image of its neighbor.*

Proof. The proof is by induction on $|T_1| + |T_2|$. If $|T_1| + |T_2| = 2$, then the existence of an adequate packing is obvious.
Assume that $|T_1| + |T_2| \geq 3$. Without loss of generality we may assume that $|T_1| \geq 2$. Let l be a leaf in T_1 and let l' be the neighbor of l other than x. We distinguish two cases:

Case 1. The middle vertex x is not joined with l.

Case 2. The middle vertex x is joined with l; we assume that there is an arc xl in H since the case with lx in H is analogous.

In Case 1, by the induction hypothesis, there exists a packing σ' of $H' = H - \{l\}$ such that x is the image of its neighbor. Then $\sigma = (l, l')\sigma'$ is an adequate packing of H if l' is a fixed point of σ' or otherwise, $\sigma = (l)\sigma'$ is an adequate packing of H.

Consider Case 2. By Remark 4 there exist packings σ_1 and σ_2 of T_1 and T_2, respectively. Note that every subdigraph of T_i is also packable. For every vertex $u \in V(T_i)$ let σ_i^u denote a packing of $T_i - \{u\}$. Let $H_1 = H[V(T_1) \cup \{x\}]$ and $H_2 = H[V(T_2) \cup \{x\}]$ be two induced subdigraphs of H.

There are two possibilities:

- There is a vertex $y \in T_2$, which is not joined with x or yx is an arc in T_2.
- For every vertex $y \in T_2$ there is an arc xy in T_2.

In the first situation let σ_1' be a packing of $T_1 - \{l, l'\}$. Then $(x, l)(l', y)\sigma_1'\sigma_2^y$ is an adequate packing of H. In the second situation, no matter how the arcs in T_2 are oriented, there is a sink s in H_2. Moreover, x is a source in H_2. Then $\sigma = (x, s)\sigma_2^s\sigma_1$ is an adequate packing of H. □

Lemma 11 *If there is an isolated vertex in D, then D is packable.*

Proof. Let y be a vertex such that $d(y) = 0$. Recall that there exists a vertex $u \in V(D)$ with $d(u) \geq 177$. Note, that $D' = D - \{u, y\}$ has small enough size (we delete 2 vertices and at least 177 arcs from D). Then, by the induction hypothesis there is a packing σ' of a graph $D' := D - \{u, y\}$. Then $(u, y)\sigma'$ is a packing of D. □

Lemma 12 *If D contains at least eight vertices with degree 1, then D is packable.*

Proof. Let $W = \{v_1, ..., v_8\} \subset V(D)$ be a set of vertices such that $d(v_1) = ... = d(v_8) = 1$. Let y_i be the only neighbor of v_i. Let u be a vertex such that $d(u) \geq 177$. We distinguish two cases:

Case 1. There is $i \in \{1, ..., 8\}$ such that $y_i \in W$. Then by the induction hypothesis, $D' = D - \{v_i, y_i, u\}$ is packable. Let σ' be a packing of D'. Then $\sigma = (v_i, y_i, u)\sigma'$ is a packing of D.

Case 2. W is a set of independent vertices. If there is $i \in \{1, ..., 8\}$ such that $d(y_i) \geq 4$, then by the induction hypothesis there is a packing σ_1 of $D_1 = D - \{v_i, y_i\}$ and $(v_i, y_i)\sigma_1$ is a packing of D.

Assume that $d(y_i) < 4$ for each $i = 1, ..., 8$. In particular, u is not a neighbor of any v_i. Suppose next that a vertex y is a common neighbor of two vertices $v, v' \in W$. Then there exists a packing σ_2 of $D_2 = D - \{v, v', y, u\}$, whence $(v, u)(v', y)\sigma_2$ is a packing of D.

Consequently, we assume that $y_i \neq y_j$ if $i \neq j$. Let $w \in W$ and let z be the neighbor of w. Recall that $d(z) \leq 3$. Therefore, there are at least 5 vertices $v^1, ..., v^5 \in W$ such that $N(v^i) \cap N(z) = \emptyset$. Consider now digraphs $D_3 = D - \{v^1, ..., v^5, w, z, u\}$ and $D_4 = D - \{v^1, ..., v^5, z\}$. Clearly, D_3 satisfies the induction assumption, hence there exists a packing σ_3 of D_3. Then $\sigma_4 = (u,w)\sigma_3$ is a packing of D_4. Vertices $v^1, ..., v^5, z$ are independent vertices which have degrees less than or equal to 3 and with mutually disjoint sets of neighbours. Thus, by Lemma 9, D is packable. □

3 Proof of Theorem 7

Proof. Let $k = \lfloor n^{1/3} \rfloor$. We assume that $d(v) \geq 2$ for each vertex v in D except at most seven of degree one because otherwise D is packable by Lemma 12.

Let us consider a set K of vertices in D with degrees greater than or equal to 2 and less than or equal to k. We choose the set S with maximum cardinality among all sets of independent vertices in K which have disjoint sets of neighbors. By Lemma 9, $|S| \leq 2k$. Hence $|N(S)| \leq 2k^2 \leq 2n^{2/3}$. Let $V_j := \{v \in V(D) \setminus N(S) : d(v) = j\}$. Clearly, every vertex from $V_2 \cup ... \cup V_k$ has a neighbor in $N(S)$. Furthermore, the number m of vertices of degree greater than k does not exceed $2n^{2/3}$. Indeed

$$4n - 20n^{2/3} - 14 \geq 2||D|| \geq 7 + 2(n - 7 - m) + m\left(n^{1/3} - 1\right),$$

hence

$$m \leq \frac{2n - 20n^{2/3} - 7}{n^{1/3} - 3} < 2n^{2/3}.$$

Therefore

$$|N(N(S))| \geq |V_2 \cup ... \cup V_k| \geq n - 7 - m - |N(S)| > n - 7 - 4n^{2/3}. \quad (1)$$

Thus, vertices from $N(S)$ cover at least $n - 7 - 4n^{2/3}$ arcs.

Let $\mathcal{C}$ be the set of components in $D - N(S)$. Let $\mathcal{T} = \{T \in \mathcal{C} : (xy \in A(T) \Rightarrow yx \notin A(T)) \wedge (||T|| = |T| - 1) \wedge (\forall x \in V(T) : |N(x) \cap N(S)| \leq 1)\}$. So, every component T in $\mathcal{T}$ does not contain symmetric arcs and the underlying graph of T is a tree (in particular, we consider an isolated vertex as a trivial tree). Furthermore, each vertex of T is incident with at most one arc joining it and a vertex in $N(S)$. We call components in $\mathcal{T}$ *minimal components* of $D - N(S)$. Let $R := \mathcal{C} - \mathcal{T}$. Let r denote the sum of the number of arcs in R and the number of components in R such that they do not contain symmetric arcs and their underlying graphs are trees. Then $r \geq |R|$. Moreover, r counts all arcs in R and some arcs between R and $N(S)$ which are not counted in inequality (1). Indeed, each of the considered $r - ||R||$ components of R contains a vertex which is joined with $N(S)$ by at least two arcs (and only one of them is counted in (1)) because components of R are not minimal.

Let p denote the cardinality of $\mathcal{T}$. Note that there are at least $n - |N(S)| - |R| - p$ arcs in $\mathcal{T}$. Below we show that p is at least $2|N(S)| - |R| + r$. By the assumption and by inequality (1), the size of D satisfies:

$$\begin{aligned} 2n - 10n^{2/3} - 7 \geq ||D|| &\geq n - 7 - 4n^{2/3} + (n - |N(S)| - p - |R|) + r \\ &> 2n - 6n^{2/3} - 7 - p - |R| + r. \end{aligned}$$

Thus

$$p > 4n^{2/3} - |R| + r \geq 2|N(S)| - |R| + r. \tag{2}$$

In the next part of the proof we part a digraph D into two vertex-disjoint subdigraphs D' and D'' this way that D' and D'' are packable. Moreover, we can extend these packings into a packing of D.

Let us denote p minimal components in $\mathcal{T}$ by $T_1, ..., T_p$. Let $D' = D[V(T_1) \cup ... \cup V(T_{2|N(S)|}) \cup N(S)]$. Below we show that there exists a packing of D' such that the image of every vertex in $N(S)$ is not in $N(S)$. If in D' there are $|N(S)|$ vertex-disjoint starry trees $H_1, ..., H_{2|N(S)|}$ with middle vertices in $N(S)$, then we pack every starry tree as in Lemma 10. Let σ_i denote a packing of H_i. We claim that $\sigma = \sigma_1 ... \sigma_{2|N(S)|}$ form a packing of D' as well. Note that only arcs between different starry trees (but not between their middle vertices) may spoil the packing of D'. These arcs are of the form xy where x is the middle vertex of some starry tree and y is a non-middle vertex of another starry tree. However, since the middle vertex of each starry tree is the image of its neighbor in the same starry tree and this neighbor has no other neighbors outside its minimal component, this is indeed a packing of D'. Otherwise, suppose that $l < |N(S)|$ is the largest number of vertex-disjoint starry trees in D' with middle vertices in $N(S)$ and let L with cardinality l denote some set of such starry trees. This time we pack starry trees from L as in Lemma 10. By Theorem 2, each of the remaining vertices from $N(S)$ together with two minimal components can be packed without fixed points. We claim that the product of these packings is a proper packing of D'. Suppose for the contrary that the image of an arc a in D' coincides with some other arc a' in D'. Hence a' must join a vertex $z \in N(S)$ which is not in any starry tree from L with a non-middle vertex of some starry tree H. Moreover, a must join the middle vertex of H with some minimal component which is not in any starry tree from L. Thus D' contains more than l starry trees and we get a contradiction. Hence D' is packable.

Below we show that $D'' = D - D'$ is also packable. Note that $D'' = R \cup T_{2|N(S)|+1} \cup ... \cup T_p$. Then, by inequality (2):

$$\begin{aligned} ||D''|| = ||R \cup T_{2|N(S)|+1} \cup ... \cup T_p|| &= ||R|| + |T_{2|N(S)|+1}| + ... + |T_p| - (p - 2|N(S)|) \\ &< ||R|| + |T_{2|N(S)|+1}| + ... + |T_p| - (r - |R|) \\ &\leq |R| + |T_{2|N(S)|+1}| + ... + |T_p| \\ &= |R \cup T_{2|N(S)|+1} \cup ... \cup T_p| = |D''|. \end{aligned}$$

Thus, by Remark 4, D'' is packable.

Let σ', σ'' denote packings of D' and D'', respectively. Then $\sigma = \sigma'\sigma''$ is a packing of D. Suppose for the contrary that the image of an arc xy in D coincides with some other arc $\sigma(x)\sigma(y)$ in D. Then $x, \sigma(x) \in V(D')$ and $y, \sigma(y) \in V(D'')$. By construction of D' and D'' it implies that x and $\sigma(x)$ belong to $N(S)$. Then we get a contradiction, since the image of every vertex in $N(S)$ is not in $N(S)$. $\square$

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