A note on packing graphs without cycles of length up to five.

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Abstract

The following statement was conjectured by Faudree, Rousseau, Schelp and Schuster: if a graph G is a non-star graph without cycles of length $m \leq 4$ then G is a subgraph of its complement.
So far the best result concerning this conjecture is that every non-star graph G without cycles of length $m \leq 6$ is a subgraph of its complement. In this note we show that $m \leq 6$ can be replaced by $m \leq 5$.

1 Introduction

We deal with finite, simple graphs without loops and multiple edges. We use standard graph theory notation. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The order of G is denoted by $|G|$ and the size is denoted by $||G||$. We say that G is *packable in its complement* (G is packable, in short) if there is a permutation σ on $V(G)$ such that if xy is an edge in G then $\sigma(x)\sigma(y)$ is not an edge in G. Thus, G is packable if and only if G is a subgraph of its complement. In [2] the authors stated the following conjecture:

Conjecture 1 *Every non-star graph G without cycles of length $m \leq 4$ is packable.*

In [2] they proved that the above conjecture holds if $||G|| \leq \frac{6}{5}|G| - 2$. Woźniak proved that a graph G without cycles of length $m \leq 7$ is packable [6]. His result was improved by Brandt [1] who showed that a graph G without cycles of length $m \leq 6$ is packable. Another, relatively short proof of Brandt's result was given in [3]. In this note we prove the following statement.

Theorem 2 *If a graph G is a non-star graph without cycles of length $m \leq 5$ then G is packable.*

The basic ingredient for the proof of our theorem is the lemma presented below. This lemma is both a modification and an extension of Lemma 2 in [4].

Lemma 3 *Let G be a graph and $k \geq 1$, $l \geq 1$ be any positive integers. If there is a set $U = \{v_1, ..., v_{k+l}\} \subset V(G)$ of $k + l$ independent vertices of G such that*

1. *k vertices of U have degree at most l and l vertices of U have degree at most k;*
2. *vertices of U have mutually disjoint sets of neighbors, i.e. $N(v_i) \cap N(v_j) = \emptyset$ for $i \neq j$;*
3. *$G - U$ is packable*

then there exists a packing σ of G such that U is an invariant set of σ, i.e. $\sigma(U) = U$.

Proof. Let $G' := G - U$ and σ' be a packing of G'. Below we show that we can find an appropriate packing σ of G.
For any $v \in V(G')$ we define $\sigma(v) := \sigma'(v)$. Then let us consider a bipartite graph B with partition sets $X := \{v_1, ..., v_{k+l}\} \times \{0\}$ and $Y := \{v_1, ..., v_{k+l}\} \times \{1\}$. For $i, j \in \{1, ..., k + l\}$ the vertices

$(v_i, 0)$, $(v_j, 1)$ are joined by an edge in B if and only if $\sigma'(N(v_i)) \cap N(v_j) = \emptyset$. So, if $(v_i, 0)$, $(v_j, 1)$ are joined by an edge in B we can put $\sigma(v_i) = v_j$.
Without loss of generality we can assume that $k \leq l$. Note that if $\deg v_i \leq l$ in G then $\deg(v_i, 0) \geq k$ in B. Furthermore, if $\deg v_i \leq k$ in G then $\deg(v_i, 0) \geq l$ in B. Thus X contains k vertices of degree $\geq k$ and l vertices of degree $\geq l$. In the similar manner we can see that Y contains k vertices of degree $\geq l$ and l vertices of degree $\geq k$. In particular, every vertex in Y has degree $\geq k$. Let $S \subset X$. If $|S| \leq k$ then obviously $|N(S)| \geq |S|$. Suppose that $k < |S| \leq l$. Then there is at least one vertex of degree l in S thus $|N(S)| \geq l \geq |S|$. Finally, we show that if $|S| > l + 1$, then $N(S) = Y$. Indeed, otherwise let $(v_j, 1) \in Y$ be a vertex which has no neighbor in S. Thus $\deg(v_j, 1) \leq |X| - |S| \leq k + l - (l + 1) = k - 1$, a contradiction. Hence, for any $S \subset X$ we get $|S| \leq |N(S)|$. Therefore, by the famous Hall's theorem [5], there is a matching M in B. We define $\sigma(v_i) = v_j$ for $i, j \in \{1, ..., k + l\}$ such that $(v_i, 0)$, $(v_j, 1)$ are incident with the same edge in M. □

2 Proof of Theorem 2

Proof. Assume that G is a counterexample of Theorem 2 with minimal order. We choose an edge $xy \in E(G)$ with the maximal sum $\deg x + \deg y$ of degrees of its endvertices among all edges of G. Since G is not a star $\deg x \geq 2$ and $\deg y \geq 2$. Let U be the union of the sets of neighbors of x and y different from x, y. Define $k := \deg x - 1$, $l := \deg y - 1$. We may assume that $k \leq l$. Consider graph $G' := G - \{x, y\}$. Note that because of the choice of the edge xy, U contains k vertices of degree $\leq l$ and l vertices of degree $\leq k$ in G'. Moreover, since G has no cycles of length ≤ 5, the vertices of U are independent in G' and have mutually disjoint sets of neighbors in G'. By our assumption $G' - U$ is packable or it is a star.

Assume that $G' - U$ is packable. Thus, by Lemma 3, there is a packing σ' of G' such that $\sigma'(U) = U$. This packing can be easily modified in order to obtain a packing of G. Namely, note that there are vertices $v, w \in U$ where v is a neighbor of x and w is a neighbor of y such that $\sigma'(v)$ is a neighbor of x and $\sigma'(w)$ is a neighbor of y, or $\sigma'(v)$ is a neighbor of y and $\sigma'(w)$ is a neighbor of x. In the former case $(x\sigma'(v)y\sigma'(w))\sigma'$ is a packing of G and in the latter case $(x\sigma'(v))(y\sigma'(w))\sigma'$ is a packing of G. Thus we get a contradiction.

Assume now that $G' - U$ is a star (with at least one edge). Note that since G has no cycles of lengths up to five, every vertex from U has degree ≤ 2 in G. Moreover, G has a vertex which is at distance at least 3 from y. Let z denote a vertex which is not in U and is at distance 2 from x, or if such a vertex does not exist let z be any vertex which is at distance at least 3 from y. Furthermore, let W denote the set of neighbours of y. Consider a graph $G'' := G - \{y, z\}$. Thus W consists of l vertices of degree ≤ 1 in G'' and one vertex of degree $k \leq l$ in G''. Note that $G'' - W$ has an isolated vertex, namely a neighbour of x. Thus $G'' - W$ is not a star, hence it is packable. Moreover vertices from W are independent and have mutually disjoint sets of neighbours in G''. Thus by Lemma 3 there is a packing σ'' of G'' such that $\sigma''(W) = W$. Then $(yz)\sigma''$ is a packing of G. Therefore, we get a contradiction again, so the proof is finished. □

References

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