

Hamilton saturated hypergraphs of essentially minimum size

Andrzej Ruciński*

Department of Discrete Mathematics
Adam Mickiewicz University
Poznań, Poland

rucinski@amu.edu.pl

Andrzej Żak †

Faculty of Applied Mathematics
AGH University of Science and Technology
Kraków, Poland

zakandr@agh.edu.pl

Submitted: June 28, 2012; Accepted: ?, 2013; Published: XX

Mathematics Subject Classifications: 05C65

Abstract

For $1 \leq \ell < k$, an ℓ -overlapping cycle is a k -uniform hypergraph in which, for some cyclic vertex ordering, every edge consists of k consecutive vertices and every two consecutive edges share exactly ℓ vertices. A k -uniform hypergraph H is ℓ -Hamiltonian saturated, $1 \leq \ell \leq k - 1$, if H does not contain an ℓ -overlapping Hamiltonian cycle $C_n^{(k)}(\ell)$ but every hypergraph obtained from H by adding one more edge does contain $C_n^{(k)}(\ell)$. Let $sat(n, k, \ell)$ be the smallest number of edges in an ℓ -Hamiltonian saturated k -uniform hypergraph on n vertices. Clark and Entinger proved in 1983 that $sat(n, 2, 1) = \lceil \frac{3n}{2} \rceil$. In this talk we prove that $sat(n, k, \ell) = \Theta(n^\ell)$ for $\ell = 1$ as well as for all $k \geq 5$ and $\ell \geq 0.8k$.

1 Introduction

The notion of a hypergraph cycle can be ambiguous. In this paper we are *not* concerned with the Berge cycles as defined by Berge in [1] (see also [12]). Instead, given integers $1 \leq \ell < k$, we define an ℓ -overlapping cycle as a k -uniform hypergraph in which, for some cyclic ordering of its vertices, every edge consists of k consecutive vertices, and every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly ℓ vertices. The notion of an ℓ -overlapping path is defined similarly. Note that the number of edges of an ℓ -overlapping cycle with s vertices is $s/(k - \ell)$ (and thus, s is divisible by $k - \ell$). The two extreme cases of $\ell = 1$ and $\ell = k - 1$ are referred

*Research supported by the Polish NSC grant N201 604940 and the NSF grant DMS-1102086

†Research partially supported by the Polish Ministry of Science and Higher Education.

to as, respectively, *loose* and *tight* cycles (paths). We denote an ℓ -overlapping cycle on s vertices by $C_s^{(k)}(\ell)$.

An ℓ -overlapping Hamiltonian cycle in a n -vertex k -graph H is any subhypergraph of H isomorphic to $C_n^{(k)}(\ell)$. If H contains an ℓ -overlapping Hamiltonian cycle then H itself is called ℓ -Hamiltonian. A tight Hamiltonian cycle was introduced in the seminal paper by Katona and Kierstad [16] under the name of a *Hamiltonian chain*. Since the appearance of [16], ℓ -Hamiltonian cycles have been studied intensively in the context of Dirac-type properties (for a survey see [17]), Ramsey properties (e.g., in [13, 14]), random hypergraphs ([10, 6, 7]). However, the saturation problem for Hamiltonian cycles in hypergraphs is mentioned only in a survey paper by Katona [15].

Given a k -uniform hypergraph H (or, shortly, a k -graph) and a k -element set $e \in H^c$, where H^c is the complement of H , we denote by $H + e$ the hypergraph obtained from H by adding e to its edge set. A k -graph H is ℓ -Hamiltonian saturated, $1 \leq \ell \leq k - 1$, if H is not ℓ -Hamiltonian but for every $e \in H^c$ the k -graph $H + e$ is such. The largest number of edges in an ℓ -Hamiltonian saturated k -graph on n vertices, that is, the Turán number for the cycle $C_n^{(k)}(\ell)$, denoted by $ex(n, C_n^{(k)}(\ell))$, has been determined recently in [11]. It turned out that

$$ex(n, C_n^{(k)}(\ell)) = \binom{n-1}{k} + ex(n-1, P),$$

where $P = P(k, \ell)$ is the $(k-1)$ -uniform, $(\ell-1)$ -overlapping path with $\lfloor \frac{k}{k-\ell} \rfloor$ edges. In particular, for graphs ($k=2$) the largest size of a Hamiltonian saturated graph is $\binom{n-1}{2} + 1$. This value is realized by a unique graph consisting of a clique on $n-1$ vertices and a pedant vertex. Note that this is the only Hamiltonian saturated graph with minimum degree 1.

In this paper we are interested in the other extreme. For n divisible by $k-\ell$, let $sat(n, k, \ell)$ be the *smallest* number of edges in an ℓ -Hamiltonian saturated k -graph on n vertices. In the case of graphs, Clark and Entringer proved in 1983 that $sat(n, 2, 1) = \lceil \frac{3n}{2} \rceil$ for n large enough.

For k -graphs with $k \geq 3$ it seems to be quite hard to obtain such precise results. Therefore, the emphasis is put on the order of magnitude of $sat(n, k, \ell)$. It was observed in [15] that $sat(n, k, k-1) = \Omega(n^{k-1})$. After some preliminary results in [8, 9], the second author showed recently that for $k \geq 2$, $sat(n, k, k-1) = \Theta(n^{k-1})$, see [18]. Here we extend that result to ℓ -overlapping Hamiltonian cycles for several other values of ℓ . Our main result is the following.

Theorem 1.1. *For all $k \geq 3$ and $\ell = 1$, as well as for all $\frac{4}{5}k \leq \ell \leq k-1$*

$$sat(n, k, \ell) = \Theta(n^\ell).$$

We conjecture that Theorem 1.1 holds for all k and $1 \leq \ell \leq k-1$.

2 Preliminaries

The next two sections contain proofs of the upper bound in Theorem 1.1. Here we give a simple proof of the lower bound.

Proposition 2.1. *For all $k \geq 2$ and $1 \leq \ell \leq k - 1$*

$$\text{sat}(n, k, \ell) = \Omega(n^\ell).$$

Proof. If H is an ℓ -saturated k -graph with n vertices and m edges then for every nonedge $e \in H^c$ there is an edge $f \in H$ such that $|e \cap f| = \ell$ (in fact, there are two such edges f , since e has to close an ℓ -overlapping cycle). But for every $f \in H$, the number of k -element subsets e which satisfy $|e \cap f| = \ell$ is exactly

$$\binom{k}{\ell} \binom{n-k}{k-\ell}.$$

Thus, every $f \in H$ can intersect this way at most $\binom{k}{\ell} \binom{n-k}{k-\ell}$ nonedges e . Hence,

$$\left(\binom{n}{k} - m \right) \leq m \times \binom{k}{\ell} \binom{n-k}{k-\ell}$$

which implies that $m = \Omega(n^\ell)$. □

In the rest of the paper we assume that G is a graph on the vertex set $\{1, \dots, n\}$. Let $c(G)$ denote the number of components of G . Given a subset $T \subseteq V(G)$, let $G[T]$ be the subgraph of G induced by T .

Fact 2.2. *Let k, ℓ , and Δ be constants. If $\Delta(G) \leq \Delta$ then the number of k -element subsets $T \subseteq V(G)$ with $c(G[T]) \leq \ell$ is $O(n^\ell)$.*

Proof. The number of k -element subsets $T \subseteq V(G)$ with $c(G[T]) \leq \ell$ is at most

$$(n(k-1)!\Delta^{k-1})^\ell = O(n^\ell). \quad \square$$

Given a graph G and an integer sequence $\mathbf{a} = (a_1, \dots, a_n)$, the \mathbf{a} -blow-up of G is the k -graph H with

$$V(H) = \bigcup_{i=1}^n U_i, \quad |U_i| = a_i,$$

$$H = \bigcup_{ij \in G} K^{(k)}(U_i \cup U_j)$$

where $K^{(k)}(U)$ is the complete k -graph on U and the sets U_i are pairwise disjoint. If $a_i = a$ for all $i = 1, \dots, n$, then we simply write a -blow-up instead of \mathbf{a} -blow-up. For a subset $S \subset V(H)$, let

$$\text{tr}(S) = \{i \in V(G) : U_i \cap S \neq \emptyset\}.$$

Furthermore, set

$$c(S) = c(G[tr(S)]).$$

The following result is an immediate corollary of Fact 2.2.

Corollary 2.3. *Let a_1, \dots, a_n, k, ℓ , and Δ be constants. If $\Delta(G) \leq \Delta$ and H is an \mathbf{a} -blow-up of G then the number of k -element subsets $S \subseteq V(H)$ with $c(S) \leq \ell$ is $O(n^\ell)$. \square*

2.1 Hamiltonian cycle saturated graphs

The proofs of the upper bounds in Theorem 1.1 are constructive. The starting points of our constructions are sparse Hamiltonian saturated *graphs*, also known as maximally non-Hamiltonian graphs. Probably the best known Hamiltonian saturated graphs of minimum size are Isaac's snarks J_k which are 3-regular, connected, bridgeless graphs with chromatic index four, and the number of vertices $n = 4k$. In a series of papers Clark, Crane, Entringer and Shapiro [3, 4, 5] constructed Hamiltonian saturated graphs (by a modification of Isaac's snarks) with minimum possible size for all sufficiently large n .

Theorem 2.4 ([5]). *For all even $n \geq 36$ as well as all odd $n \geq 53$ there exists a Hamiltonian saturated graph of order n and size $\lceil 3n/2 \rceil$.*

In order to obtain the right order of magnitude for $sat(n, k, \ell)$ for all values of ℓ considered in the paper, we will need Hamiltonian saturated graphs with bounded maximum degree. (Due to the asymptotic nature of our result, the numerical value of the bound does not matter to us.) By analyzing the construction in [5] one can see that the Hamiltonian saturated graphs obtained there do have bounded maximum degree. An alternative way, which we prefer, is by combining Theorem 2.4 with the following result of Bondy.

Theorem 2.5 ([2]). *Let G be a Hamiltonian saturated graph with $n \geq 7$ vertices. If for some $0 \leq m \leq n$ the graph G has m vertices of degree 2, then $|E(G)| \geq (3n + m)/2$.*

Corollary 2.6. *For all $n \geq 52$ there exists a Hamiltonian saturated graph G of order n with $\Delta(G) \leq 5$.*

Proof. By Theorem 2.4 for all $n \geq 52$ there exists a Hamiltonian saturated graph G with n vertices and at most $(3n + 1)/2$ edges. Clearly, $\delta(G) \geq 2$. Hence, by Theorem 2.5, there is at most one vertex of degree 2 in G , and, consequently, no vertex of degree greater than 5. \square

3 The loose case : $\ell = 1$

In this Section we prove Theorem 1.1 for $\ell = 1$. We begin with a simple lemma.

Lemma 3.1. *If a graph G is not Hamiltonian then the $(k - 1)$ -blow-up H of G is not 1-Hamiltonian.*

Proof. Suppose that H contains a 1-Hamiltonian cycle $C_H = \{e_1, \dots, e_n\}$. Define $f : C_H \rightarrow G$ by $f(e_s) = \{i, j\} \in G$, where $e_s \in H[U_i \cup U_j]$. Since $|H[U_i \cup U_j] \cap C_H| \leq 1$ for all $1 \leq i < j \leq n$, the mapping f is one-to-one. Furthermore, $C_G = \{f(e_s), s = 1, \dots, n\}$ is a connected, spanning subgraph of G . Moreover, $\delta(C_G) \geq 2$. Indeed, fix $i \in \{1, \dots, n\}$, recall that $|U_i| = k - 1$, and observe that every subset of $k - 1$ vertices of C_H intersects at least two edges of C_H . Thus, C_G is a Hamiltonian cycle in G , a contradiction. \square

In view of Proposition 2.1, in order to prove Theorem 1.1 for $\ell = 1$ it suffices to construct for every sufficiently large N divisible by $k - 1$, a 1-Hamiltonian saturated k -graph H with N vertices and $O(N)$ edges. Let H_1 be a $(k - 1)$ -blow-up of a Hamiltonian saturated graph G with n vertices, $n = \frac{N}{k-1}$, and $\Delta(G) = O(1)$. (By Corollary 2.6 G exists.) By Lemma 3.1, H_1 is not 1-Hamiltonian and $|V(H)| = N$. Set $V = V(H_1)$ and let

$$H_2 = \left\{ e \in \binom{V}{k} : tr(e) \text{ is a clique in } G \right\}.$$

Since for every $e \in H_1$ the set $tr(e)$ spans an edge of G , we have $H_1 \subseteq H_2$. Finally, let H be a maximal k -graph on the vertex set V such that $H_1 \subseteq H \subseteq H_2$ and H is not 1-Hamiltonian. By Corollary 2.3, $|H| \leq |H_2| = O(N)$. The following lemma completes the proof of Theorem 1.1 in the case $\ell = 1$.

Lemma 3.2. *For every $e \in H^c$, $H + e$ is 1-Hamiltonian.*

Proof. By the maximality of H we may restrict our attention to only those e for which $tr(e)$ is not a clique. Fix one pair $\{i, j\} \notin G$ such that $tr(e) \supset \{i, j\}$. Without loss of generality (w.l.o.g.) we may assume that $i = 1$ and $j = 2$. Since G is Hamiltonian saturated, $G + \{1, 2\}$ has a Hamiltonian cycle containing the edge $\{1, 2\}$. Let C_G be a Hamiltonian cycle in $G + \{1, 2\}$ corresponding, w.l.o.g., to a cyclic ordering $(1, \dots, n)$. Set $r_s = |e \cap U_s|$, $s = 1, 2, \dots, n$. We build a 1-Hamiltonian cycle $C_H = \{e_1, \dots, e_n\}$ in H by ‘tracing’ C_G . In doing so, we make sure that the last vertex of each edge e_i belongs to the set U_{i+1} and that $U_{i+1} \subseteq e_1 \cup e_i \cup e_{i+1}$.

Formally, we construct C_H as follows. (See Fig. 1 for an illustration.)

- Let $e_1 = e$ and choose $v_1 \in e_1 \cap U_1$ and $v_2 \in e_1 \cap U_2$.
- Further, let $e_2 \in H[U_2 \cup U_3]$ with $e_1 \cap e_2 = \{v_2\}$ and $|e_2 \cap U_2| = k - r_2$. Note that

$$|e_2 \cap U_3| = k - |e_2 \cap U_2| = r_2 \leq k - 1 - r_3 = |U_3 \setminus e_1|$$

and $U_2 \subset e_1 \cup e_2$. Choose $v_3 \in e_2 \cap U_3$.

- Subsequently, for $3 \leq t \leq n - 1$, let $e_t \in H[U_t \cup U_{t+1}]$ with $e_{t-1} \cap e_t = \{v_t\}$, $e_t \cap e_1 = \emptyset$, and $|e_t \cap U_t| = k - \sum_{s=2}^t r_s$. Note that

$$|e_t \cap U_{t+1}| = \sum_{s=2}^t r_s \leq k - 1 - r_{t+1} = |U_{t+1} \setminus e_1|,$$

because $\sum_{s=1}^n r_s = k$ and $r_1 \geq 1$. Moreover, $U_t \subset e_1 \cup e_t \cup e_{t+1}$. Set $v_{t+1} \in e_t \cap U_{t+1}$.

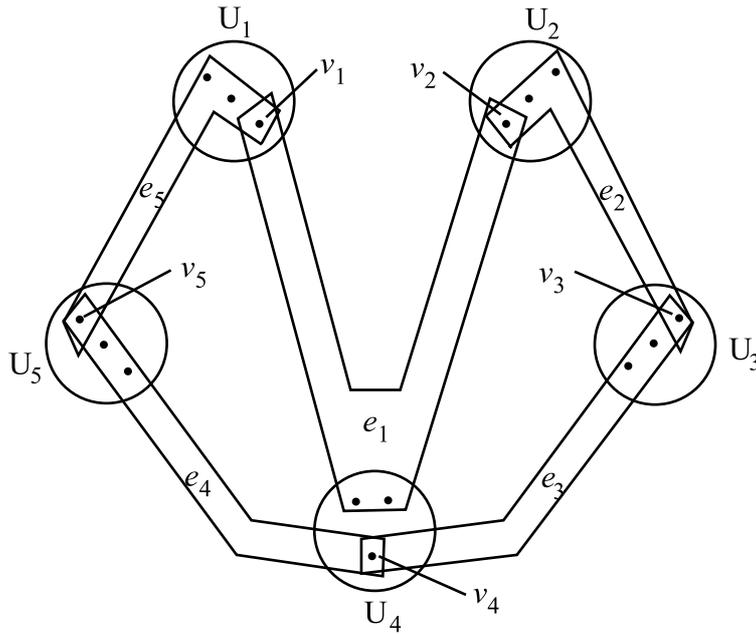


Figure 1: An illustration to the proof of Lemma 3.2: $n = 5$, $k = 4$

- Finally, let $e_n \in H[U_n \cup U_1]$ with $e_{n-1} \cap e_n = \{v_n\}$, $|e_n \cap U_n| = k - \sum_{s=2}^n r_s$, and $e_n \setminus U_n = (U_1 \setminus e_1) \cup \{v_1\}$. Note that

$$|e_n \cap U_1| = \sum_{s=2}^n r_s = k - r_1 = 1 + |U_1 \setminus e_1|$$

and $U_1 \subset e_n \cup e_1$.

Thus, indeed, $C_H = \{e_1, \dots, e_n\}$ is a 1-Hamiltonian cycle in H . □

4 The case $\ell \geq 4k/5$

In this Section we prove our main result, that is, Theorem 1.1 for $\ell \geq \frac{4}{5}k$. Let $\mathbf{a} = (a_1, \dots, a_n)$, where

$$2k - \ell + 1 \leq a_i \leq 4\ell - 2k + 1, \quad i = 1, \dots, n. \quad (1)$$

Note that under our assumption on ℓ we do have $2k - \ell + 1 \leq 4\ell - 2k + 1$, and that for all $1 \leq \ell \leq k - 1$,

$$a_i \leq 2\ell - 1. \quad (2)$$

Let G be an n -vertex Hamiltonian saturated graph with n sufficiently large and $\Delta(G) = O(1)$, guaranteed by Corollary 2.6, and let H_1 be the \mathbf{a} -blow-up k -graph of G with

$$V = V(H_1) = \bigcup_{i=1}^n U_i, \quad \text{where } |U_i| = a_i, \quad i = 1, \dots, n.$$

Observe that for each $e \in H_1$, the set $tr(e)$ is either a vertex or an edge of G and thus $c(e) = 1$. Given a set $S \subseteq V$, let

$$\min(S) = \min\{i : i \in tr(S)\} = \min\{i : U_i \cap S \neq \emptyset\}.$$

Further, let

$$H_2 = \left\{ e \in \binom{V}{k} : |e \cap U_{\min(e)}| \geq k - l + 1, c(e) \geq k - l + 1 \text{ and } |e \cap U_{\min(e)}| + c(e) \geq l + 2 \right\}.$$

Since for every $e \in H_2$ we have $c(e) \geq k - l + 1 \geq 2$, the k -graphs H_1 and H_2 are edge-disjoint.

Lemma 4.1. $H_1 \cup H_2$ is not ℓ -Hamiltonian.

Proof. Suppose that $H_1 \cup H_2$ contains an ℓ -Hamiltonian cycle C_H .

Case 1. Assume first that $C_H \subseteq H_1$ and define a subgraph C_G of G as the set of all 2-element traces $tr(e)$ of the edges e of C_H . Formally,

$$C_G = \{tr(e) : e \in C_H \text{ and } |tr(e)| = 2\}.$$

We are going to arrive at a contradiction by showing that C_G is a Hamiltonian cycle in G . Since, clearly, C_G is a connected, spanning subgraph of G , it is enough to prove that C_G is 2-regular.

Let us fix $i \in \{1, \dots, n\}$. As C_H has to enter and leave the set U_i at some point, there exist an edge $e \in C_H$ and an index $j \neq i$ such that $tr(e) = \{i, j\}$. Let P be a longest ℓ -overlapping path in C_H (a segment of C_H) containing e and with $\bigcup_{f \in P} tr(f) = \{i, j\}$. Further, let e', e'' be the two edges of C_H which intersect $V(P)$ each in ℓ vertices and set $A' = e' \cap V(P)$ and $A'' = e'' \cap V(P)$ (see Fig. 2). Since on the one hand $tr(A') \subseteq \{i, j\}$ while, on the other hand, $A' \subset e'$ and $|tr(e') \cap \{i, j\}| = 1$ (and the same is true for A'') we have $|tr(A')| = |tr(A'')| = 1$. However, $tr(A') \neq tr(A'')$. Indeed, if, say, $A' \cup A'' \subseteq U_i$ then, by (2), we would have $A' \cap A'' \neq \emptyset$ and consequently

$$e \subseteq V(P) \subseteq A' \cup A'' \subseteq U_i,$$

a contradiction with the choice of e .

In conclusion, if for some $e \in C_H$ we have $tr(e) = \{i, j\}$ then there is a set $A \subset U_i$ with $|A| = \ell$ which on the cycle C_H is connected to e by an ℓ -overlapping path consisting of vertices from $U_i \cup U_j$ only. Moreover, the edge, say e' , extending A along C_H in the opposite direction (away from e) satisfies $tr(e') = \{j', i\}$, where $j' \neq i, j$, and so $N_{C_G}(i) \supseteq \{j, j'\}$. To show that C_G is indeed 2-regular, suppose to the contrary that there exist edges $e_1, e_2, e_3 \in C_H$ with $tr(e_s) = \{i, j_s\}$, $s = 1, 2, 3$, where j_1, j_2, j_3 are mutually distinct and different from i . Let A_s , $s = 1, 2, 3$, be the sets described above (with respect to e_s). Since $|A_s| = \ell$, again by (2), the sets A_1, A_2, A_3 intersect pairwise. Assume w.l.o.g that A_1 is located (along C_H) between e_1 and e_2 . Then A_3 cannot intersect A_1 , a contradiction. (See Fig. 3 for an illustration of this proof.)

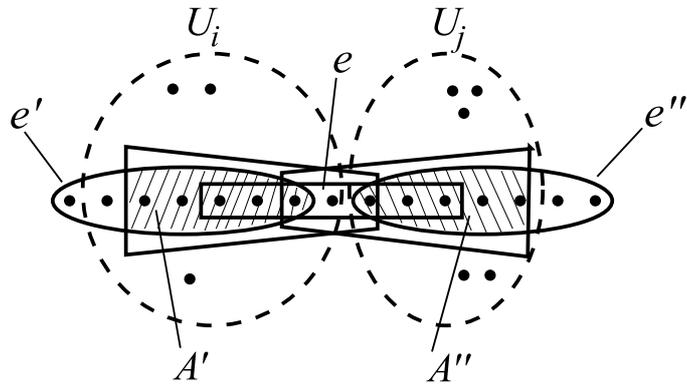


Figure 2: An illustration to the proof of Lemma 4.1: $k = 7$, $\ell = 5$, $|U_i| = |U_j| = 10$, the path P consists of 3 “quadrangular” edges.

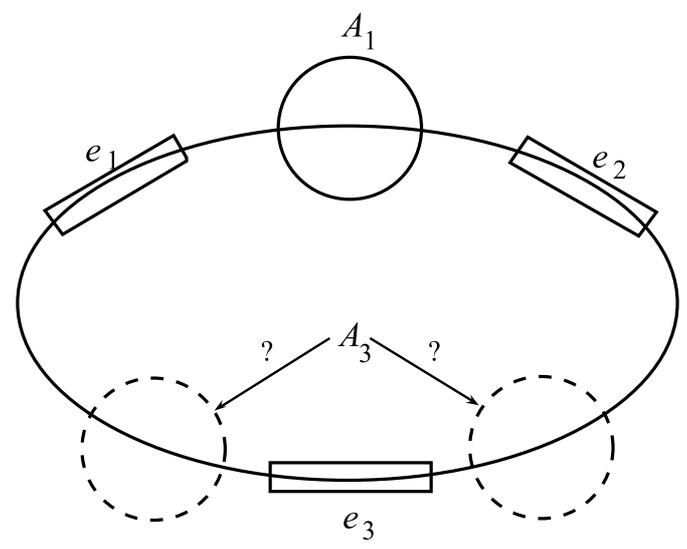


Figure 3: An illustration to the proof of Lemma 4.1

Case 2. Assume that C_H contains a segment of more than k^2 consecutive edges from H_2 (that is, an ℓ -overlapping path in $C_H \cap H_2$). Let e_1, \dots, e_{s-1} , $s > k^2 + 1$, be such a segment. Recall that $|e \cap U_{\min(e)}| \geq k - \ell + 1$ for every $e \in H_2$, while $|e_t \cap e_{t+1}| = \ell$ for all $t = 1, \dots, s - 2$. These two facts imply that $\min(e_t) = \min(e_{t+1})$ for $t = 1, \dots, s - 2$, and so $|e_t \cap U_i| \geq k - \ell + 1$ for some $i \in [1, n]$ and all $t = 1, \dots, s - 1$. On the other hand, observe that a vertex can belong to at most k edges of C_H . Hence,

$$|U_i| \geq |(e_1 \cup \dots \cup e_{s-1}) \cap U_i| \geq \frac{1}{k}(s-1)(k-\ell+1) \geq \frac{2}{k}(s-1) > 2k > |U_i|,$$

a contradiction.

Case 3. Assume that $H_2 \cap C_H \neq \emptyset$ but the longest segment in C_H of consecutive edges from H_2 has length at most k^2 . Let e_1, \dots, e_{s-1} , $2 \leq s \leq k^2 + 1$, be such a segment. Then $e_m \in H_1$ and $e_s \in H_1$. As in Case 2, $|e_t \cap U_i| \geq k - \ell + 1$ for some $i \in [1, n]$ and all $t = 1, \dots, s - 1$. Consequently, $e_m \cap U_i \neq \emptyset$ as well as $e_s \cap U_i \neq \emptyset$. By the definition of H_1 , each of $tr(e_m)$ and $tr(e_s)$ is either the singleton $\{i\}$ or an edge of G containing vertex i and thus, $c(e_m) = c(e_s) = 1$. In view of this and the inequality $c(e_1) \geq k - \ell + 1$ we have $(e_1 \setminus e_m) \cap U_i = \emptyset$. Analogously, $(e_{s-1} \setminus e_s) \cap U_i = \emptyset$. Moreover, in fact $c(e_1) = c(e_{s-1}) = k - \ell + 1$. Therefore, by the third criterion in the definition of H_2 ,

$$|U_i \cap (e_1 \cap e_m)| = |U_i \cap e_1| \geq \ell + 2 - (k - \ell + 1) = 2\ell - k + 1$$

and, similarly,

$$|U_i \cap (e_{s-1} \cap e_s)| \geq 2\ell - k + 1.$$

Observe that for large n , $e_m \cap e_s = \emptyset$. Indeed, if $e_m \cap e_s \neq \emptyset$, then, necessarily $e_1 \subseteq e_s \cup e_m$, and consequently, $c(e_1) = 1$ – a contradiction with the definition of H_2 . Hence, we have

$$|U_i| \geq |U_i \cap (e_1 \cap e_m)| + |U_i \cap (e_{s-1} \cap e_s)| \geq 4\ell - 2k + 2,$$

a contradiction with (1). □

Let

$$H_3 = \left\{ e \in \binom{V}{k} : c(e) \leq \ell \right\}.$$

Recall that for all $e \in H_1$ we have $c(e) = 1$, while for all $e \in H_2$ we have $c(e) \leq \ell$. Thus, $H_1 \cup H_2 \subseteq H_3$. Finally, let H be a maximal k -graph on the vertex set V such that $H_1 \cup H_2 \subseteq H \subseteq H_3$ and H is not ℓ -Hamiltonian. By Corollary 2.3,

$$|H| \leq |H_3| = O(N^\ell). \tag{3}$$

We will next show that H is ℓ -Hamiltonian saturated.

Lemma 4.2. *For every $e \in H^c$, $H + e$ is ℓ -Hamiltonian.*

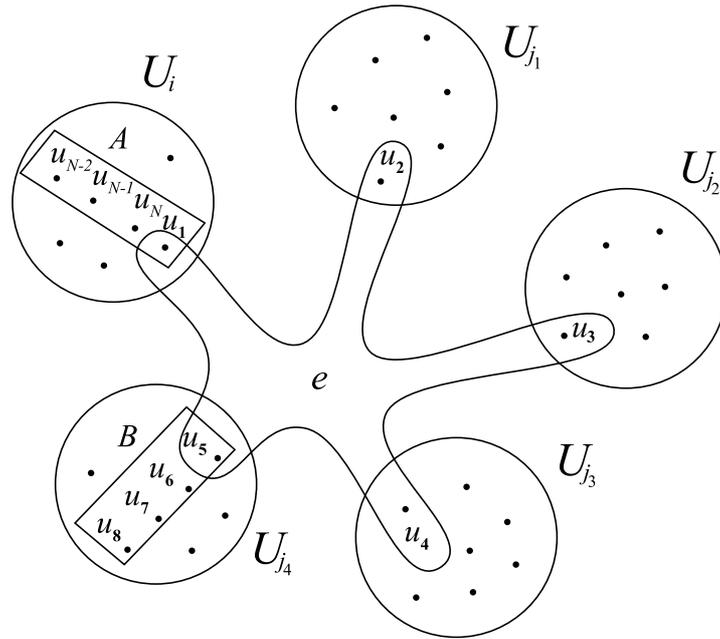


Figure 4: The construction of P in the tight case: $k = 5$, all $|U_j| = 7$.

Proof. By the definition of H the thesis holds for each e with $c(e) \leq \ell$. Hence, we may assume that $c(e) \geq \ell + 1$. We will build an ℓ -overlapping Hamiltonian cycle

$$C_H = (e_1, \dots, e_m) = (u_1, \dots, u_N), \quad m = \frac{N}{k - \ell},$$

in $H + e$ using the Hamiltonian saturation of G . As the general proof is a bit complicated, we will first assume that $\ell = k - 1$, in which case the construction can be simplified. This way, avoiding tedious details, we will be able to exhibit the main ideas quite clearly.

The tight case : $\ell = k - 1$. We have $k + 2 \leq a_j = |U_j| \leq 2k - 3$ for all $j = 1, \dots, n$. Since $c(e) \geq k$, the set $tr(e)$ is, in fact, an independent k -element set in G . Let

$$tr(e) = \{i < j_{k-1} < \dots < j_1\}$$

and $e = (u_1, \dots, u_k)$, where $u_1 \in U_i$ and $u_{1+t} \in U_{j_t}$ for $t = 1, \dots, k - 1$. We construct first a tight path $P \subseteq H_2 + e$ extending e in both directions, so that the two ends A and B of P are $(k - 1)$ -tuples contained in, respectively, U_i and $U_{j_{k-1}}$. To do so, let u_{k+t} , $t = 1, \dots, k - 2$, be any vertices of $U_{j_{k-1}}$ different from u_k , whereas u_{N-t} , $t = 0, 1, \dots, k - 3$, be any vertices of U_i different from u_1 . Then

$$P = (u_{N-k+3}, \dots, u_N, u_1, \dots, u_{2k-2}).$$

(See Fig. 4 for an illustration of this construction.)

To see that P is a tight path in $H + e$ with ends $A = (u_{N-k+3}, \dots, u_N, u_1)$ and $B = (u_k, \dots, u_{2k-2})$, note that for each $q = 2, \dots, k-1$ the edge $e_q = (u_q, \dots, u_{q+k-1})$ satisfies: $\min(e_q) = j_{k-1}$, $|e_q \cap U_{j_{k-1}}| = q$, $c(e_q) = k-q+1$, and thus $e_q \in H_2$. Similarly, for $q = 0, \dots, k-3$, the edges $e_{m-q} = (u_{N-q}, \dots, u_N, u_1, \dots, u_{k-q-1})$, for which $\min(e_{m-q}) = i$, also belong to H_2 .

Recall that $ij_{k-1} \notin G$ and thus, by the Hamiltonian saturation of G there is a Hamiltonian path Q from i to j_{k-1} in G . We connect the ends of P , that is, the sets A and B , by a tight path P' in $H_1 \subseteq H$, tracing the path Q in G in such a way that every time Q visits a vertex v of G we add to P' all vertices of $U'_v = U_v \setminus V(P)$. Since, $|U'_v| \geq |U_v| - 1 \geq k-1$ (with some margin), we can always do so by using only the edges of H_1 .

General case. For $\ell \leq k-2$ the situation becomes more complicated and the above simple construction of the ℓ -overlapping path P fails. For instance, if u_{k-1} and u_k are in the same component of $G[tr(e)]$ and $k-2$ is divisible by $k-\ell$, then $c(e_{(k-2)/(k-\ell)+1}) = 1$, and so $e_{(k-2)/(k-\ell)+1} \notin H_2$. Nevertheless we manage to follow the same idea by slightly modifying the above construction.

Recall that $c(e) \geq \ell + 1$. Let $j_1 > j_2 > \dots > j_\ell > i = \min(e)$ be some $\ell + 1$ elements of $tr(e)$, belonging to different components of $G[tr(e)]$ and including $i = \min(e)$. Further, let $e_1 = e = (u_1, \dots, u_k)$, where $u_1 \in U_i$ and $u_{k-\ell+t} \in U_{j_t}$, $t = 1, \dots, \ell$, while $u_2, \dots, u_{k-\ell}$ remain unspecified.

Our plan is, again, first to construct a path $P \subseteq H_2 + e$ extending e in both directions (Part 1), and then to complete C_H by connecting the ends of P by a path $P' \subseteq H_1$ (Part 2). The path P' will follow a Hamiltonian path Q in G which together with the pair $\{i, j_\ell\}$ forms a Hamiltonian cycle in $G + \{i, j_\ell\}$.

Part 1. Let integers q and r be defined by

$$(q+1)(k-\ell) + r = k, \quad 1 \leq r \leq k-\ell. \quad (4)$$

The ℓ -path P will consist of $3q+5$ edges, e_{m-2q}, \dots, e_{q+4} , and thus, of $k + (3q+4)(k-\ell)$ vertices, $u_{N-k+2r-\ell+1}, \dots, u_N, u_1, \dots, u_{4k-2\ell-r}$. The edges are determined by the vertices as they begin at every $(k-\ell)$ th vertex. (Note that $k+\ell-2r = (2q+1)(k-\ell)$, and thus e_1 , the $(2q+2)$ nd edge of P does coincide with e .)

We now list all the vertices of P , that is, for each index x we specify the set U_j from which we (arbitrarily) select a vertex u_x .

1. For $N-k+2r-\ell+1 \leq x \leq N-k+2r$ we select $u_x \in U_i$; thus, P begins with ℓ vertices of U_i ; we denote their set by I_1 .
2. For $N-k+2r+1 \leq x \leq N-k+\ell$ we select $u_x \in U_{j_t}$, where $t = x - (N-k) - 1$; this segment of P has exactly one vertex from each set $U_{j_{2r+t}}$, $t = 0, \dots, \ell-2r-1$; we denote this set by M_1 ("M" like in mixed).
3. For $N-k+\ell+1 \leq x \leq N$ we select $u_x \in U_i$; thus, P returns to U_i for $k-\ell$ steps; we denote this set *enlarged* by u_1 , the first vertex of e_1 , by I_2 .

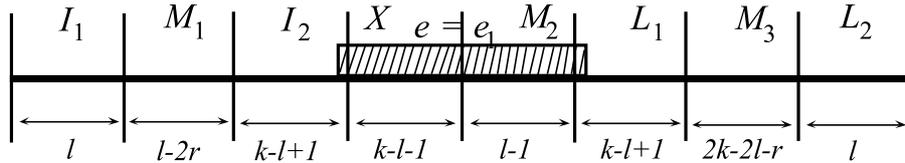


Figure 5: The construction of the ℓ -path P

4. The next k vertices of P are the vertices of $e = e_1$, namely u_1, \dots, u_k ; we set $X = \{u_2, \dots, u_{k-\ell}\}$ and $M_2 = \{u_{k-\ell+1}, \dots, u_{k-1}\}$ (we know nothing about the elements of X).
5. For $k+1 \leq x \leq 2k-\ell$ we select $u_x \in U_{j_\ell}$; thus, P traverses through some $k-\ell$ vertices of U_{j_ℓ} ; we denote this set *enlarged* by u_k , the last vertex of e_1 , by L_1 .
6. For $2k-\ell+1 \leq x \leq 4k-3\ell-r$ we select $u_x \in U_{j_t}$, where $t = x - 2k + \ell$; this segment of P has exactly one vertex from each set U_{j_t} , $t = 1, \dots, 2k-2\ell-r$; we denote this set by M_3 .
7. For $4k-3\ell-r+1 \leq x \leq 4k-2\ell-r$ we select $u_x \in U_{j_\ell}$; thus, P ends with ℓ vertices from U_{j_ℓ} ; we denote their set by L_2 .

The construction of the path P is illustrated in Fig. 5.

Let us now estimate how many vertices of each set U_j are used by the above constructed path P . Set $r_j = |V(P) \cap U_j|$, $j = 1, \dots, n$.

Fact 4.3. $r_i \leq 2k - \ell$, $r_{j_\ell} \leq 2k - \ell$ and $r_j \leq k - \ell + 2$ for all $j \notin \{i, j_\ell\}$.

Proof. Note that since $c(e_1) \geq \ell + 1$, $|e_1 \cap U_j| \leq k - \ell$ for each $j \in [1, n]$. In addition, P uses $\ell + (k - \ell) = k$ vertices of both, U_i and U_{j_ℓ} , and at most two vertices of each set U_{j_t} , $t = 1, \dots, \ell - 1$. \square

Fact 4.4. Every edge of P belongs to $H_2 + e$.

Proof. Let us split the edges of P into those appearing “before e ” (b.e.) and “after e ” (a.e.) Formally, set

$$P = B_e \cup \{e\} \cup A_e,$$

where $B_e = \{e_{m-2q}, \dots, e_m\}$ and $A_e = \{e_2, \dots, e_{q+4}\}$ (recall that $e_1 = e$). We will give the proof first for the a.e. edges and then for the b.e. edges. Set

$$I = I_1 \cup I_2 \quad M = M_1 \cup M_2 \cup M_3 \quad L = L_1 \cup L_2.$$

Let $f \in P - e$.

Case $f \in A_e$: In this case, $f \cap X = \emptyset$ and $\min(f) = j_\ell$. Consequently, $f \cap U_{\min(f)} = f \cap L$ and our first goal (c.f. the definition of H_2) is to show that

$$|f \cap L| \geq k - \ell + 1. \quad (5)$$

Observe that either $f \supset L_1$, in which case (5) is true, or $f \subset L \cup M_3$ and so,

$$|f \cap L| = k - |M_3| = k - (2k - 2\ell - r) = 2\ell - k + r \geq k - \ell + 1,$$

because $r \geq 1$ and $\ell \geq \frac{2}{3}k$. Thus (5) holds again.

As a next step we will show that

$$c(f) \geq k - \ell + 1. \quad (6)$$

Note that

$$|f \cap M| = |f \cap (M_2 \cup M_3)| \leq k - |L_1| = k - (k - \ell + 1) = \ell - 1.$$

Hence, the elements of $f \cap M_2$ and $f \cap M_3$ come from different sets U_j , $j \in \{1, \dots, \ell - 1\}$ and, consequently, $c(f) = |f \cap M| + 1$, where we add 1 because of U_{j_ℓ} (recall that the set $\{j_1, \dots, j_\ell\}$ is independent in G). If $f \supset M_3$ then

$$c(f) \geq |M_3| + 1 = (2k - 2\ell - r) + 1 \geq k - \ell + 1,$$

since $r \leq k - \ell$. Otherwise, $f \cap L_2 = \emptyset$ and

$$c(f) = k - |L_1| + 1 = \ell \geq k - \ell + 1$$

and (6) holds again. Since, clearly,

$$|f \cap U_{\min(f)}| + c(f) = |f| + 1 = k + 1 \geq \ell + 2,$$

$f \in H_2$, and so $A_e \subseteq H_2$.

Case $f \in B_e$: In this case, $\min(f) = i$. Consequently, $f \cap U_{\min(f)} = f \cap I$ and our first goal is to show that

$$|f \cap I| \geq k - \ell + 1. \quad (7)$$

Observe that the first (that is, with the smallest index) vertex of f coincides with, or is to the left of the first vertex of I_2 . Thus, $f \supset I_2$ or

$$|f \cap I| \geq k - |M_1| = k - \ell + 2r \geq k - \ell + 2,$$

and in either case (7) holds.

As a next step we will prove that

$$c(f) \geq k - \ell + 1. \quad (8)$$

Note that

$$|f \cap M| = |f \cap (M_1 \cup M_2)| \leq k - |I_2| - |X| = k - 2(k - \ell) \leq \ell - 1.$$

Hence, the elements of $f \cap M_1$ and $f \cap M_2$ come from different sets U_j , $j \in \{1, \dots, \ell - 1\}$ and, again, $c(f) = |f \cap M| + 1$. If $f \cap I_1 = \emptyset$ then

$$|f \cap M| \geq k - |I_2| - |X| = k - 2(k - \ell) \geq k - \ell,$$

because $\ell \geq \frac{2}{3}k$. If $f \supset M_1$ then

$$|f \cap M| \geq |M_1| = \ell - 2r \geq k - \ell,$$

because $r \leq k - \ell$ and $\ell \geq \frac{3}{4}k$. Otherwise, that is, when $f \cap I_1 \neq \emptyset$ but $f \not\supset M_1$,

$$|f \cap M| = k - |f \cap I_1| \geq k - \ell.$$

Thus, (8) holds in all cases.

It remains to prove that

$$|f \cap U_{\min(f)}| + c(f) \geq \ell + 2. \tag{9}$$

Recall that $\min(f) = i$ and $|f \cap U_i| = |f \cap I|$, while $c(f) = |f \cap M| + 1$. Since, clearly,

$$|f \cap I| + |f \cap M| + |f \cap X| = |f| = k,$$

we have

$$|f \cap U_{\min(f)}| + c(f) \geq k + 1 - |f \cap X| \geq k + 1 - (k - \ell - 1) = \ell + 2.$$

Hence, $f \in H_2$ and, consequently, $B_e \subseteq H_2$. This completes the proof of Fact 4.4. \square

Part 2. Recall that $\{i, j_\ell\}$ is not an edge of G . Hence, by the Hamiltonian saturation property of G , there is a Hamiltonian path Q from j_ℓ to i in G . As in the loose ($\ell = 1$) and tight ($\ell = k - 1$) cases treated earlier, we build the rest of C_H by ‘tracing’ Q . Each time we visit a vertex $x \in V(Q)$ we consecutively include to C_H all vertices from $U_x \setminus V(P)$ (in any order). This way we create an ℓ -path P' consisting of k -tuples $e_{q+5}, \dots, e_{m-2q-1}$.

Note that by Fact 4.3 and the lower bound in (1), we have

$$|U_x \setminus V(P)| = |U_x| - r_x \geq (2k - \ell + 1) - (k - \ell + 2) = k - 1 \tag{10}$$

for each $x \in V(Q) \setminus \{i, j_\ell\}$. Hence, $|tr(e_j)| \leq 2$, for all $j = q + 5, \dots, m - 2q + 1$. Moreover, for each such j with $|tr(e_j)| = 2$ the pair $tr(e_j)$ is an edge of G . Therefore, $e_j \in H_1$, for $j = q + 5, \dots, m - 2q + 1$. In conclusion, $C_H = P \cup P'$ is an ℓ -Hamiltonian path in $H_1 \cup H_2 + e \subseteq H + e$, which completes the proof of Lemma 4.2. \square

The conclusion of the proof of Theorem 1.1. In order to prove Theorem 1.1 for $\ell \geq \frac{4}{5}k$ we need to construct, for every sufficiently large N divisible by $k - \ell$, an ℓ -Hamiltonian saturated k -graph H with N vertices and $O(N^\ell)$ edges. Assume first that

$\ell > \frac{4}{5}k$. As then $2k - \ell + 2 \leq 4\ell - 2k + 1$ we may use as the sizes $a_i = |U_i|$ both numbers, $2k - \ell + 1$ and $2k - \ell + 2$. It is well known that every number $N \geq N_0 = N_0(k, \ell)$ (the Frobenius number) can be expressed as a sum of these two numbers. For an N divisible by $k - \ell$, let us fix one such partition

$$N = a_1 + \cdots + a_n, \quad 2k - \ell + 1 \leq a_i \leq 2k - \ell + 2,$$

and let H be as in Lemma 4.2. Then, by (3), H indeed is an ℓ -Hamiltonian saturated k -graph with N vertices and $O(N^\ell)$ edges.

In the critical case $\ell = \frac{4}{5}k$, we need to refine our previous estimates a bit. Assume that for some integer $p \geq 1$, we have $k = 5p$ and $\ell = 4p$. Then, by (4), $r = p$, and so, $2r = 2p > 2k - 2\ell - r = p$. Thus, every index $j \in \{j_1, \dots, j_{\ell-1}\}$ appears at most once in the set $M_1 \cup M_3$, and consequently, we can improve the bound on r_j from Fact 4.3 down to $k - \ell + 1$. This implies, in turn, that the crucial estimate $|U_x| - r_x \geq k - 1$ from Part 2 of the construction of the cycle C_H in the proof of Lemma 4.2 (see (10)) remains valid even for sets U_x with $|U_x| = 2k - \ell$. Note that the lower bound in (1) was not used in any other part of the proof. We may thus complete the proof as before, expressing N this time as

$$N = a_1 + \cdots + a_n, \quad 2k - \ell \leq a_i \leq 2k - \ell + 1.$$

□

5 Remarks and open problems

Note that in the case $\ell = k - 1$ our Theorem 1.1, as stated, covers only $k \geq 5$. However, in the proof of Lemma 4.2 we could have $k \leq a_j = |U_j| = k + 1$. Indeed, then we still have $|U'_j| \geq |U_j| - 1 \geq k - 1$, while the punch-line inequality in the proof of Lemma 4.1, that is, $|U_i| \leq k + 1 \leq 4\ell - 2k + 1 = 2k - 3$ holds already for $k \geq 4$. So, in fact, our proof of Theorem 1.1 works also in the case $k = 4, \ell = 3$. Moreover, for $k = 3$, by fixing $|U_j| = 3$ for all j , the proofs of both lemmas, Lemma 4.1 and Lemma 4.2, go through and yield that $\text{sat}(3n, 3, 2) = \Theta(n^2)$. As we mentioned in the Introduction, it has been proved in [18], via a different construction, that $\text{sat}(n, k, k - 1) = \Theta(n^{k-1})$ for all $k \geq 3$.

A big open problem is to extend our result to all $1 \leq \ell \leq k - 1$, that is, to prove the following conjecture.

Conjecture 5.1. *For all $1 \leq \ell \leq k - 1$, $k \geq 2$, $\text{sat}(n, k, \ell) = \Theta(n^\ell)$.*

The smallest open case is $k = 4, \ell = 2$.

6 Acknowledgements

We thank the reviewer for carefully reading our manuscript and for giving detailed comments and suggestions that have been helpful to improve the manuscript.

References

- [1] C. Berge. Nombres de coloration de l'hypergraphe h -parti complet. *Springer Lecture Notes in Math.*, 411:13–20, 1975.
- [2] J. A. Bondy. Variations on the hamiltonian theme. *Canad. Math. Bull.*, 15:57–62, 1972.
- [3] L. Clark, R. Crane, R. Entringer and H. Shapiro. On smallest maximally non-Hamiltonian graphs. In *Proceedings of the seven-teenth Southeastern international conference on combinatorics, graph theory, and computing (Boca Raton, Fla., 1986)*, 53:215–220, 1986.
- [4] L. Clark and R. Entringer. Smallest maximally non-Hamiltonian graphs. *Period. Math. Hungar.*, 14(1):57–68, 1983.
- [5] L. H. Clark, R. C. Entringer and H. D. Shapiro. Smallest maximally nonhamiltonian graphs II. *Graphs Combin.*, 8:225–231, 1992.
- [6] Andrzej Dudek and A. Frieze. Loose Hamilton Cycles in Random Uniform Hypergraphs. *Electron. J. Combin.*, 18:P48, 2011.
- [7] Andrzej Dudek and A. Frieze. Tight Hamilton cycles in random uniform hypergraphs. *Random Structures Algorithms*, 42(3):374–385, 2013.
- [8] Aneta Dudek, G. Y. Katona and A. Żak. Hamilton-chain saturated hypergraphs. *Discrete Math.*, 310:1172–1176, 2010.
- [9] Aneta Dudek and A. Żak. On Hamilton-chain saturated uniform hypergraphs. *Discrete Math. Theor. Comput. Sci.*, 14:21–28, 2012.
- [10] A. Frieze. Loose Hamilton Cycles in Random 3-Uniform Hypergraphs. *Electron. J. Combin.*, 17:N28, 2010.
- [11] R. Glebov, Y. Person and W. Weps. On extremal hypergraphs for Hamiltonian cycles. *European J. Combin.*, 33:544–555, 2012.
- [12] A. Gyárfás, G. Sárközy and E. Szemerédi. Monochromatic Hamiltonian 3-tight Berge cycles in 2-colored 4-uniform hypergraphs. *J. Graph Theory*, 63:288–299, 2010.
- [13] P. Haxell, T. Łuczak, Y. Peng, V. Rödl, A. Ruciński, M. Simonovits and J. Skokan. The Ramsey number for hypergraph cycles I. *J. Combin. Theory Ser. A*, 113:67–83, 2006.
- [14] P. Haxell, T. Łuczak, Y. Peng, V. Rödl, A. Ruciński and J. Skokan. The Ramsey number for 3-uniform tight hypergraph cycles. *Combin. Probab. Comput.*, 18:165–204, 2009.
- [15] G. Y. Katona. Hamiltonian chains in hypergraphs, A survey. *Graphs, Combinatorics, Algorithms and its Applications*, (ed. S. Arumugam, B. D. Acharya, S. B. Rao), Narosa Publishing House 2004.
- [16] G. Y. Katona and H. Kierstead. Hamiltonian chains in hypergraphs. *J. Graph Theory*, 30:205–212, 1999.

- [17] V. Rödl and A. Ruciński. Dirac-type questions for hypergraphs – a survey (or more problems for Endre to solve). *Bolyai Soc. Math. Studies 21 (2010), An Irregular Mind (Szemerédi is 70)*, Bolyai Soc. Math. Studies 21:561–590, 2010.
- [18] A. Żak. Growth order for the size of smallest hamiltonian chain saturated uniform hypergraphs. *European J. Combin.*, 34:724–735, 2013.