On embedding graphs with bounded sum of size and maximum degree

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Abstract

We say that a graph is *embeddable* if it is a subgraph of its complement. One of the classic results on graphs embedding says that each graph on n vertices with at most $n-2$ edges is embeddable. The bound on the number of edges cannot be increased because, for example, the star on n vertices is not embeddable. The reason of this fact is the existence of a vertex with very high degree. In this paper we prove that by forbidding such vertices, one can significantly increase the bound on the number of edges. Namely, we prove that if $\Delta(G) + |E(G)| \leq 2n - f(n)$, where $f(n) = o(n)$, then G is embeddable. Our result is asymptotically best possible, since for the star S_n (which is not embeddable) we have $\Delta(S_n) + |E(S_n)| = 2n - 2$. As a corollary, we obtain that a digraph embedding conjecture by Benhocine and Wojda 1985 is true for digraphs with sufficiently many symmetric arcs.

1 Introduction

We deal with finite, simple graphs without loops or multiple edges. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$. The *order* of G is the number of vertices of G and is denoted by $|G|$. The *size* of G is the number of edges of G and is denoted by $||G||$. By $N_G(x)$ we denote the set of vertices adjacent to x in G. The *degree* (in G) of a vertex x is denoted by $d_G(x)$ and is equal to $|N_G(x)|$. The *maximum degree* of G is denoted by $\Delta(G)$ and is equal

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to the maximum among degrees of all vertices of G. For a vertex set X, the set $N_G(X)$ denotes the *external neighbourhood* of X in G, i.e.

$$N_G(X) = \{y \in V(G) \setminus X : y \text{ is adjacent to some } x \in X\}.$$

We say that G is *embeddable in its complement* (G is embeddable, in short) if there is a permutation σ on $V(G)$ such that if xy is an edge in G, then $\sigma(x)\sigma(y)$ is not an edge in G. Thus, G is embeddable if and only if G is a subgraph of its complement. If there exists a map σ with $\sigma(x) \neq x$ for every vertex $x \in V(G)$, then we say that G is *fixed-point-free* embeddable. In the sequel we use the permutation cycle notation.

One of the classical results in the theory of graph embedding is the following theorem, proved independently in [2, 3, 9].

Theorem 1 ([2, 3, 9]) *Every n-vertex graph having at most $n-2$ edges is embeddable.*

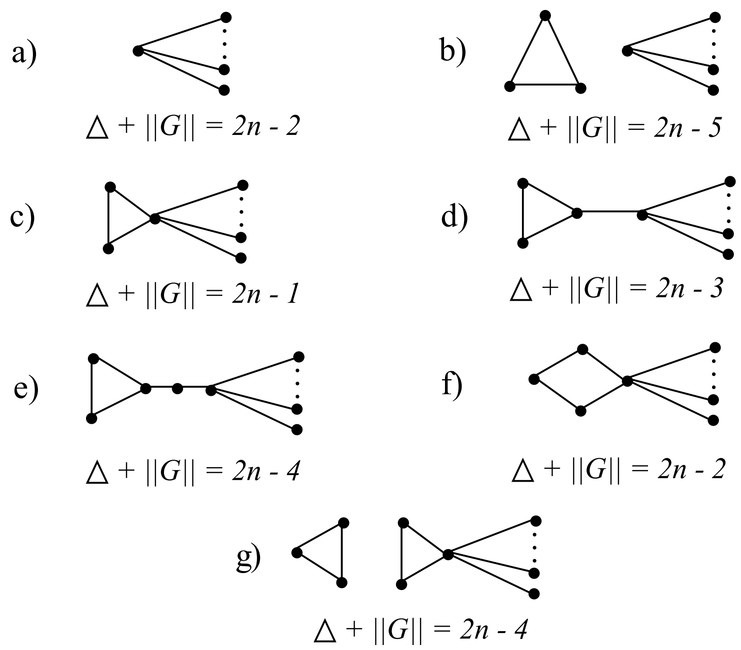
This theorem cannot be improved by raising the size of G since, for example, a star on n vertices is not embeddable. In [4] and [5], all non-embeddable graphs with order n and size $n-1$ or n are presented, see also [11]. Among them there are 7 infinite families, see Figure 1. It is clear from the examples that the strong restriction on the number of edges in Theorem 1 is a result of the existence of a vertex with very high degree. It seems to be very likely that by forbidding such vertices, one can significantly improve the bound on the size of a graph in the statement of Theorem 1. We confirm this intuition by proving the following theorem.

Theorem 2 *Let G be an n-vertex graph. If $||G|| + \Delta(G) \leq 2n - 14n^{2/3} - 20$ then G is embeddable.*

Note that the bound in Theorem 2 is asymptotically best possible. Indeed, it cannot be larger than $2n-6$ which follows from Figure 1 b). Furthermore, it cannot be essentially improved even with a stronger bound on the maximum degree. Indeed, consider the following example.

Example Let $V_1, ..., V_{t-1}$ be pairwise disjoint subsets with $|V_i| = t$ for $i = 1, ..., t-2$ and $|V_{t-1}| = n - t(t-2)$. Furthermore, let $x \in V_{t-1}$. Let G be a graph with $V(G) = V_1 \cup \cdots \cup V_{t-1}$ such that each V_i, $i = 1, ..., t-2$, induces a clique, V_{t-1} induce a star with center x and there are no other edges in G. Observe that G is not embeddable if n is sufficiently large. Indeed, suppose that σ is an embedding of G. If $\sigma(x) \in V_i$ for some $i \in \{1, ..., t-2\}$, then the remaining vertices of V_i must be images of vertices from different sets V_j, $j \neq t-1$. However, there are not enough sets V_j. Suppose that $\sigma(x) \in V_{t-1}$. If $\sigma(x) \neq x$, then x must be an image of a vertex from some set V_i with $i \in \{1, ..., t-2\}$. Thus, the remaining vertices of V_i have to be mapped on vertices from different sets V_j with $j \neq t-1$. However, there are not enough such sets V_j. Finally, if $\sigma(x) = x$, then the neighbors of x have to be mapped on the vertices from $V_1 \cup \cdots \cup V_{t-2}$, which is impossible if n is sufficiently large.

Figure 1: Infinite families of non-embeddable graphs of order n and size $n-1$ or n



Furthermore, $\Delta(G) = n - t(t-2) - 1$, $||G|| = \frac{t(t-1)}{2}(t-2) + n - t(t-2) - 1$. Hence, $\Delta(G) + ||G|| = 2n - 2 + t(t-2)\frac{t-5}{2}$.

Therefore, the coefficient 2 in Theorem 2 cannot be increased. Note that for $t = 4$, the bound on $\Delta(G) + ||G||$ is strongest. Let, $g(n)$ denote the maximum number such that every graph G of order n satisfying $\Delta(G) + ||G|| \leq g(n)$ is embeddable. Thus, we have that $g(n) = 2n - o(n)$. We conjecture that $g(n) = 2n - 7$, i.e.

Conjecture 3 *Let G be an n-vertex graph. If $||G|| + \Delta(G) \leq 2n - 7$ then G is embeddable.*

2 The idea of the proof of Theorem 2

We apply the idea that was used in [7] and [8]. We suppose that G is a counterexample of minimal order n. In Section 3 we prove some basic properties of G. The main proof has 5 parts.

Part a). Using the basic properties of G and Lemma 4, we first prove that G has a small set $U \subset V(G)$ that covers almost n edges.

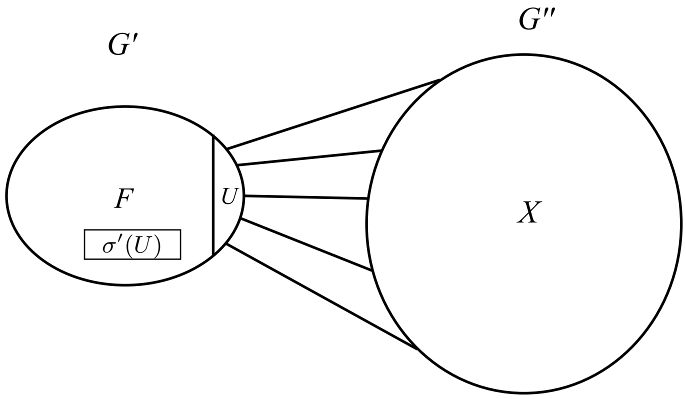
Part b). As a consequence, we obtain that $G - U$ has fewer edges than vertices. Hence, $G - U$ has sufficiently many components that are trees. This allows us to partition $V(G)$ into three sets U, F and X such that $G[F]$ (the subgraph of G induced by F) is a forest with $3|U|$ components. The key property is that there are no edges between F and X.

Part c). Let $G' = G[U \cup F]$ and $G'' = G[X]$, see Figure 2. Since, G' has a very specific structure, it is possible to prove that it has an embedding σ' such that $\sigma'(U) \subset F$.

Part d) Here we prove that G'' is a sparse graph, so it has an embedding σ'' by Theorem 1.

Part e) Since $\sigma'(U) \subset F$ and there are no edges between F and X, the product $\sigma'\sigma''$ is an embedding of G, which is a contradiction with the assumption on G.

Figure 2: A partition of $V(G)$



3 Lemmas

We use the following result from [6].

Lemma 4 ([6]) *Let G be a graph and k, l non-negative integers. If G has an independent set U of cardinality $k + l$ such that*

1. *U has k vertices with degree at most l, and its other l vertices have degree at most k,*
2. *the neighborhoods of the vertices of U are pairwise disjoint,*
3. *there is an embedding σ' of $G - U$,*

then there exists an embedding σ of G.

We will need also the following two known results, the former one is a special case of a more general theorem from [9].

Theorem 5 ([9]) *Every n-vertex graph G satisfying $2(\Delta(G))^2 < n$ is embeddable.*

Theorem 6 ([10]) *Every n-vertex graph having at most $n - 2$ edges is fixed-point-free embeddable.*

For convenience, let $\alpha(n) = 14n^{2/3} + 20$. In many places in the proofs we will use the following observation.

Proposition 7 *Let G be a graph of order n such that $||G|| + \Delta(G) \leq 2n - \alpha(n)$. If G' is a graph that arises from G by deleting m vertices and at least $2m$ edges, then $||G'|| + \Delta(G') \leq 2n' - \alpha(n')$, where n' is the order of G'.*

Proof. . Note that $\alpha(n)$ is increasing with respect to n. Thus,

$$||G'|| + \Delta(G') \leq 2n - \alpha(n) - 2m = 2(n-m) - \alpha(n) \leq 2n' - \alpha(n').$$

□

Lemma 8 *Let G be a graph of order n such that $||G|| + \Delta(G) \leq 2n - \alpha(n)$. If $n \leq 2744$, then G is embeddable.*

Proof. Note that if $n \leq 2744$ then $2n - 14n^{2/3} - 20 \leq n - 2$. Hence G is embeddable by Theorem 1. □

Lemma 9 *Let G be a graph of order n such that $||G|| + \Delta(G) \leq 2n - \alpha(n)$. If $\Delta(G) \leq 37$, then G is embeddable.*

Proof. If $n \leq 2744$, then G is embeddable by Lemma 8. So we may assume that $n \geq 2745$. Note that if $\Delta(G) \leq 37$, then $2(\Delta(G))^2 < 2745 \leq n$. Hence G is embeddable by Theorem 5. □

A *starry tree* is a graph H such that (1) $V(H)$ can be partitioned into four nonempty sets W_1, W_2, W_3 and $\{x\}$ that each induce a tree, (2) there is at least one edge incident to x, (3) all edges not belonging to the trees induced by W_1, W_2 and W_3 are incident to x and (4) there are no edges between x and $W_2 \cup W_3$. A vertex x we call *a middle vertex* of H. Note that a starry tree is always disconnected.

Lemma 10 *Every starry tree admits an embedding such that its middle vertex is the image of one of its neighbors.*

Proof. Let H be a starry tree and let T_1, T_2 and T_3 be trees induced by W_1, W_2 and W_3 respectively. The proof is by induction on $|T_1| + |T_2| + |T_3|$. If $|T_1| + |T_2| + |T_3| = 3$, then the existence of an embedding as required is obvious. Assume that $|T_1| + |T_2| + |T_3| \geq 4$. We distinguish two cases:
Case 1. There exists a leaf l in T_1 such that the middle vertex x is adjacent to l.
Case 2. No leaf of T_1 is adjacent to x.

Consider Case 1. Let $u \in W_2$, $v \in V(T_3)$ be vertices such that $T_2 - u$ or $T_3 - v$ either is disconnected or has at most one vertex. Thus, by Theorem 1 (or trivially in the latter situation), there is an embedding σ_2 of $T_2 - \{u\}$ and there is an embedding σ_3 of $T_3 - \{v\}$.

Suppose first that $|T_1| = 1$ with $W_1 = \{l\}$. Then, the product $(l, x, v, u)\sigma_2\sigma_3$ is an embedding as required of H. Suppose next that $|T_1| = 2$ with $W_1 = \{l, l'\}$. Then, $(l, x, u)(l', v)\sigma_2\sigma_3$ is an embedding as required of H.

So we may assume that $|T_1| \geq 3$. Let l' be the neighbor of l in T_1. Let $y \in W_1$ such that $T_1 - \{l, l', y\}$ either is disconnected or has at most one vertex. Thus, by Theorem 1 (or trivially in the latter situation), there is an embedding σ_1 of $T_1 - \{l, l', y\}$. Then the product $(l, x, u, y)(l', v)\sigma_1\sigma_2\sigma_3$ is an embedding as required of H.

Consider Case 2. Let L be the set of the leaves of T_1, $L = \{l_1, ..., l_s\}$. Note, that $|T_1| \geq 3$. Suppose first that all the leaves of T_1 have a common neighbor y. Since H is a starry tree (so there is at least one edge incident to x) and the leaves of T_1 are not joined with x, xy is an edge of H. Let $H' = H - L$. Clearly, H' is a starry tree. Thus, by the induction hypothesis there is an embedding as required σ' of H'. Furthermore, since y is the only neighbor of x in H', we have $\sigma'(y) = x$. In particular y is not a fixed point of σ'. Thus the product $(l_1) \cdots (l_s)\sigma'$ (i.e. $l_1, ..., l_s$ are fixed points) is an embedding as required of H.

So we may assume that there are $l_1, l_2 \in L$ with neighbors (in T_1) y_1, y_2, respectively, such that $y_1 \neq y_2$, $y_1 \neq l_2$ and $y_2 \neq l_1$ (recall that $|T_1| \geq 3$ in this case). Let $H'' = H - \{l_1, l_2\}$. Clearly, H'' is a starry tree. Hence, by the induction hypothesis, there is an embedding as required σ'' of H''. Then $(l_1, l_2)\sigma''$ is an embedding as required of H if $\sigma''(y_1) = y_1$ or $\sigma''(y_2) = y_2$. Otherwise, $(l_1)(l_2)\sigma''$ is an embedding as required of H. □

Lemma 11 *Let G be a graph with minimum order n such that G is a non-embeddable graph with $||G|| + \Delta(G) \leq 2n - \alpha(n)$. Then G has no isolated vertices.*

Proof. Suppose for a contradiction, that y is an isolated vertex of G. By Lemma 9, there is $x \in G$ with $d_G(x) \geq 38$. Let $G' = G - \{x, y\}$. By Proposition 7, $||G'|| + \Delta(G') \leq 2|G'| - \alpha(|G'|)$. Thus, by the minimality assumption there is an embedding σ' of G'. Then $(xy)\sigma'$ is an embedding of G, a contradiction. □

Lemma 12 *Let G be a graph with minimum order n such that G is a non-embeddable graph with $||G|| + \Delta(G) \leq 2n - \alpha(n)$. If two vertices of G of degree 1 have mutually different neighbors, then G has at most 20 vertices of degree 1.*

Proof. Let V_1 denote the set of all vertices of G with degree 1. Suppose for a contradiction, that $|N(V_1)| \geq 2$ and $|V_1| > 20$. By Lemma 9 we may assume that G contains a vertex x with $d_G(x) \geq 38$. Let $x_1, x_2 \in V_1$ and y_1, y_2 with $y_1 \neq y_2$, be the neighbors of x_1 and x_2 respectively.

Note that y_1 and y_2 cover at most 7 edges. Indeed, otherwise $G' := G - \{x_1, x_2, y_1, y_2\}$ arises from G by deleting 4 vertices and at least 8 edges. Hence, $||G'|| + \Delta(G') \leq 2|G'| - \alpha(|G'|)$, by Proposition 7. Thus, by the minimality assumption, there is an embedding σ' of G'. Then, $(x_1, y_1, x_2, y_2)\,\sigma'$ is an embedding of G. On the other hand, if $d_G(y_1) = 1$, then $G'' := G - \{x, x_1, y_1\}$ also satisfies $||G''|| + \Delta(G'') \leq 2|G''| - \alpha(|G''|)$ by Proposition 7. Hence, by the minimality assumption there is an embedding σ'' of G''. Then $(x, x_1, y_1)\,\sigma''$ is an embedding of G. The same argument holds if $d_G(y_2) = 1$.

Therefore, we may assume that $2 \leq d_G(y_1) \leq 6$ and $2 \leq d_G(y_2) \leq 6$, G has no isolated edges and x is not a neighbor of any vertex from V_1. Moreover, $d_G(y_1) + d_G(y_2) \leq 8$ if y_1y_2 is an edge of G, and $d_G(y_1) + d_G(y_2) \leq 7$ otherwise. In particular, y_2 has at most $7 - d_G(y_1)$ neighbors in V_1. Analogously, every vertex other than y_1 of G has at most $7 - d_G(y_1)$ neighbors in V_1. Let $V_1' \subset V_1$ be the set of all vertices of degree 1 which are at distance equal to 1 or 2 from

y_1. Let $V_1'' = V_1 \setminus V_1'$. Thus, $|V_1'| \leq (d_G(y_1) - 1)(7 - d_G(y_1)) + 1$. Hence,

$$|V_1''| \geq |V_1| - (d_G(y_1) - 1)(7 - d_G(y_1)) - 1. \tag{1}$$

Since every vertex other than y_1 of G has at most $7 - d_G(y_1)$ neighbors in V_1 and vertices from V_1 have no neighbors in V_1, we have

$$|N(V_1'')| \geq \frac{|V_1''|}{7 - d_G(y_1)} \geq \frac{|V_1| - (d_G(y_1) - 1)(7 - d_G(y_1)) - 1}{7 - d_G(y_1)}, \tag{2}$$

by (1). Note that if $|V_1| \geq 20$ then $|V_1| \geq 2\,(d_G(y_1) - 1)\,(7 - d_G(y_1)) + 2$, since the largest number of vertices of degree 1 is needed when $d_G(y_1) = 4$. Therefore, if $|V_1| \geq 20$ then $|N(V_1'')| \geq d_G(y_1)$, by (2). Let $\{z_1, ..., z_q\} \subset N(V_1'')$ with $q = d_G(y_1)$ and let $l_i \in V_1''$ be a neighbor of z_i, $i = 1, ..., q$. Clearly, $W := \{l_1, ..., l_q\}$ is an independent set of G, $W \subset V_1$ and vertices of W have different neighbors. Moreover, since $W \subset V_1''$, we have that $W \cup \{y_1\}$ is independent and $N_G(l_i) \cap N_G(y_1) = \emptyset$ for every $i = 1, ..., q$.

Consider now a graph $G''' := G - (W \cup \{x, x_1, y_1\})$. Note that in order to obtain G''' we remove from G, $d_G(y_1) + 3$ vertices and at least $d_G(y_1) + (d_G(y_1) + d_G(x) - 1) \geq 2(d_G(y_1) + 3)$ edges. Therefore, by Proposition 7, $||G'''|| + \Delta(G''') \leq 2|G'''| - \alpha(|G'''|)$. Hence, by the minimality assumption, there is an embedding σ''' of G'''. Furthermore, $(x, x_1)\sigma'''$ is an embedding of $G - (W \cup \{y_1\})$. Then, by Lemma 4, there is an embedding of G, a contradiction. $\square$

4 Proof of Theorem 2

Proof. . Assume that G is a counterexample to Theorem 2 with minimum order n. By Lemma 8, $n \geq 2745$ and, by Lemma 9, $\Delta(G) \geq 38$. Moreover, by Lemma 11, G has no isolated vertices. Let $k = \lfloor n^{1/3} \rfloor$.

Part a). We first prove that G has a small set U of vertices that covers almost n edges. Let S denote a maximal independent set of G such that

i) if $v \in S$ then $2 \leq d_G(v) \leq k$,

ii) if $u, v \in S$ then $N_G(u) \cap N_G(v) = \emptyset$.

By Proposition 7, $||G - S|| + \Delta(G - S) \leq 2|G - S| - \alpha(|G - S|)$. Thus, if $S \neq \emptyset$, then, by the minimality assumption, $G - S$ is embeddable. Hence,

$$|S| \leq 2k - 1. \tag{3}$$

because otherwise G is embeddable by Lemma 4 (with $l = k$), Clearly, (3) holds also if $S = \emptyset$. Thus,

$$|N(S)| \leq 2k^2 - k \leq 2n^{2/3} - n^{1/3} < 2n^{2/3}. \tag{4}$$

Let $V_j := \{v \in V(G) \setminus N(S) : d_G(v) = j\}$. By the definition of S, every vertex from $V_2 \cup \cdots \cup V_k$ has a neighbor in $N(S)$. Furthermore, the number

n_k of vertices of degree greater than k does not exceed $4n^{2/3}$ because $2||G|| = \sum_{v\in V(G)} d_G(v) < 4n$. Therefore and since $|V_0| = 0$, we have

$$|N(N(S))| \geq |V_2 \cup \cdots \cup V_k| \geq n - |V_1| - n_k - |N(S)| \geq n - |V_1| - 4n^{2/3} - |N(S)|. \tag{5}$$

Suppose first that all the vertices of degree 1 have a common neighbor x. In this case we define $U = N(S) \cup \{x\}$. By (3), (4) and (5) we have that $|U| \leq 2n^{2/3} - n^{1/3} + 1$ and $|N(U)| \geq n - 6n^{2/3}$. On the other hand, if two vertices of degree 1 have different neighbors, then we define $U = N(S)$. Note that $U \neq \emptyset$. Indeed, otherwise

$$4n - 28n^{2/3} - 40 \geq 2||G|| = \sum_{u\in V(G)} d_G(u) \geq 20 + (n-20)n^{1/3} \tag{6}$$

because in this case there are at most 20 vertices of degree 1, see Lemma 12. However, for $n \geq 2745$ inequality (6) is false. Therefore, by (5) and (4) and since $|V_1| \leq 20$, we have that $|U| \leq 2n^{2/3} - n^{1/3}$ and $|N(U)| \geq n - 20 - 6n^{2/3}$. Hence in each case, G contains a set U such that

$$\begin{aligned} |U| &\leq 2n^{2/3} - n^{1/3} + 1 \\ |N(U)| &\geq n - 6n^{2/3} - 20. \end{aligned} \tag{7}$$

Part b). Now, we will prove that $G - U$ contains a forest with sufficiently many components that have an important additional property. Let T_1,...,T_p denote connected components of $G-U$ which are trees such that each vertex of T_i is adjacent to at most one vertex in U. We call these components *minimal components* of $G-U$. Let $R := G - U - V(T_1) - \cdots - V(T_p)$. Let r denote the sum of the size of R and the number of all vertices in R which are joined (in G) with U by at least two edges. Since R does not contain minimal components, every component of R which is a tree contains a vertex joined with U by at least two edges. On the other hand, every component of R which is not a tree has at least as many edges as vertices. Hence,

$$r \geq |R|. \tag{8}$$

Moreover, r counts all edges in R and some edges between R and $N(S)$ which are not counted in inequality (7), because this inequality counts only the number of vertices in $N(U)$ and ignores the number of connections.

Note that there are exactly $n - |U| - |R| - p$ edges in $\bigcup_{i=1}^{p} T_i$. Below we show that p is greater than or equal to $3|U| + \Delta - |R| + r + 2$. By the assumption and by (7), we have

$$\begin{aligned} 2n - 14n^{2/3} - 20 - \Delta \geq ||G|| &\geq |N(U)| + (n - |U| - p - |R|) + r \\ &\geq 2n - 8n^{2/3} + n^{1/3} - 21 - p - |R| + r. \end{aligned}$$

Thus

$$p \geq 6n^{2/3} + n^{1/3} - 1 - |R| + r + \Delta \geq 3|U| + \Delta - |R| + r + 2, \tag{9}$$

because n is sufficiently large.

We will now partition $V(G)$ into two sets each of which induces an embeddable subgraph. First, we assign to as many as possible vertices of U different minimal components in such a way that if a minimal component T_j is assigned to a vertex $u \in U$ then there is at least one edge (in G) joining T_j and u. Let l be the maximum number of minimal components assigned to vertices of U in this way and let $\mathcal{M} = \{M_1, ..., M_l\}$ be the set of these minimal components. If $l < |U|$, then we assign an arbitrary minimal component to every remaining vertex of U. Let $\mathcal{M}'$ be the set of minimal components not yet assigned. Now, we assign $2|U|$ different minimal components to vertices from U in such a way that every vertex $u \in U$ has two minimal components in $\mathcal{M}'$ such that there is no edge (in G) between u and these two minimal components. This is possible because $|\mathcal{M}'| \geq \Delta + 2|U|$. So, we have constructed l starry trees with middle vertices in U. Note, that l is the maximum number of starry trees with middle vertices in U. Without loss of generality we may assume that we have assigned $T_1, ..., T_{3|U|}$.

Part c). Let $G' := G[U \cup V(T_1) \cup \cdots \cup V(T_{3|U|})]$ and $G'' := G - V(G')$. Below we will show that there exists an embedding of G' such that every vertex from U is mapped outside of U.

Suppose first that $l = |U|$. Then we pack every starry tree in such a way that the middle vertex is the image of one of its neighbors in the same starry tree (the required embedding exists by Lemma 10). Let σ_i be the required embedding of M_i. We claim that the product $\sigma = \sigma_1 \cdots \sigma_{|U|}$ is an embedding of G' as well. Since σ_i is an embedding of M_i, only edges between different starry trees may spoil the embedding of G'. Furthermore, every middle vertex is mapped on a non-middle vertex. Since there are no edges between T_i and T_j for $i \neq j$, the edges between middle vertices do not spoil the embedding. It remains to check the edges of the form xy where x is the middle vertex of some starry tree and y is a non-middle vertex of another starry tree. However, since the middle vertex of each starry tree is the image of one of its neighbors in the same starry tree and this neighbor has no other neighbors outside its minimal component, these edges also do not spoil the embedding.

Suppose now, that $l < |U|$. Again, we pack every starry tree in such a way that the middle vertex is the image of one of its neighbors. Moreover, since $\mathcal{M}$ is maximal, each remaining vertex of U has no neighbors in each of the remaining minimal components (otherwise, we would have an extra starry tree). Hence, by Theorem 6, each of the remaining vertices from U together with three non-trivial minimal components (not involved in any starry tree) can be embedd without fixed points. We claim that the product of these embeddings is a proper embedding of G'. Suppose for a contradiction that the image of an edge e in G' coincides with some other edge e' in G'. Using the previous argument, e' must join a vertex $z \in U$ which is not in any starry tree from $\mathcal{M}$ with a non-middle vertex of some starry tree M_j. Moreover, e must join the middle vertex of M_j with some minimal component which is not in any starry tree from $\mathcal{M}$. However, now we can exchange the two minimal components that contain one of the endvertices of the edges e and e'. This way we obtain more than l starry trees and we get a contradiction. Hence G' is embeddable.

Part d). Here we prove that G'' is embeddable as well. Recall that $r \geq ||R||$. Furthermore, by (9) we have

$$\begin{aligned}||G''|| &= ||R \cup T_{3|U|+1} \cup \cdots \cup T_p|| = ||R|| + |T_{3|U|+1}| + \cdots + |T_p| - (p - 3|U|) \\ &\leq ||R|| + |T_{3|U|+1}| + \cdots + |T_p| - (r - |R| + \Delta) - 2 \\ &\leq |R| + |T_{3|U|+1}| + \cdots + |T_p| - 2 = |R \cup T_{3|U|+1} \cup \cdots \cup T_p| - 2 = |G''| - 2.\end{aligned}$$

Thus, by Theorem 1, G'' is embeddable.

Part e). Let σ', σ'' denote embeddings of G' and G'', respectively. Then $\sigma = \sigma'\sigma''$ is an embedding of G. Suppose for a contradiction that the image of an edge xy in G coincides with some other edge $\sigma(x)\sigma(y)$ in G. Then $x, \sigma(x) \in V(G')$ and $y, \sigma(y) \in V(G'')$. By construction of G' and G'' we have that x and $\sigma(x)$ belong to U. Then we get a contradiction, since the image of every vertex in U is not in U. The embedding σ contradicts the assumption that G was non-embeddable, so we deduce no counterexample to Theorem 2 exists. □

5 Concluding remarks

Let D be a digraph with a vertex set $V(D)$ and an arc set $A(D)$. For a vertex x of $V(D)$, let $N^+(x) = \{y \in V(D) : xy \in A(D)\}$ and $N^-(x) = \{y \in V(D) : yx \in A(D)\}$. Then $d^+(x) = |N^+(x)|$ is the out degree of x and $d^-(x) = |N^-(x)|$ is the in degree of x. The degree of a vertex x, denoted by $d(x)$, is defined by $d(x) = d^+(x) + d^-(x)$. If xy and yx belong to $A(D)$, then we say that x and y are joined by a pair of *symmetric arcs*.

Similarly as in case of graphs, we say that D is *embeddable* (in its complement) if there is a permutation σ of $V(D)$ such that if xy is an an arc of D, then $\sigma(x)\sigma(y)$ is not an arc of D.

If a digraph D has only symmetric arcs, then by Theorem 1, D is embeddable if $||D|| \leq 2n - 4$. This leads to the following conjecture.

Conjecture 13 ([1]) *Let D be a digraph of order n. If D has at most $2n - 4$ arcs, then D is embeddable.*

Conjecture 13 is asymptotically true, see [7].

Theorem 14 ([7]) *Let D be a digraph of order n. If D has at most $2n - 10n^{2/3} - 7$ arcs, then D is packable.*

As a corollary of Theorem 2 we obtain that Conjecture 13 is true for digraphs that have sufficiently many symmetric arcs. Let $d^*(x) = |N^+(x) \cup N^-(x)|$ and let $\Delta^* = \max\{d^*(x) : x \in V(D)\}$.

Corollary 15 *Let D be a digraph of order n and size m with $m \leq 2n - 4$. If the number of pairs of symmetric arcs of D is at least $\Delta^* + 14n^{2/3} + 16$ then D is embeddable.*

Proof. Let s denote the number of pairs of symmetric arcs in D. Construct a graph $G(D)$ by replacing every arc or every pair of symmetric arcs of D by an edge with the same endvertices. Note that $||G(D)|| = m - s$ and $\Delta(G(D)) = \Delta^*$. By the assumption on n and on s we have

$$||G(D)|| + \Delta(G(D)) = m - s + \Delta^* \leq 2n - 4 - (14n^{2/3} + 16 + \Delta^*) + \Delta^* = 2n - 14n^{2/3} - 20$$

Thus, by Theorem 2, G is embeddable. Therefore, D is embeddable as well. □

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