On packing two graphs with bounded sum of sizes and maximum degree

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Abstract

A *packing* of graphs G_1 and G_2, both on n vertices, is a set $\{H_1, H_2\}$ such that $H_1 \cong G_1$, $H_2 \cong G_2$, and H_1 and H_2 are edge disjoint subgraphs of K_n. In 1978 Sauer and Spencer [Edge disjoint placement of graphs, *J. Combin. Theory Ser. B* 25 (1978), 295–302] proved that if $|E(G_1)| + |E(G_2)| < \frac{3}{2}n - 1$ then there is a packing of G_1 and G_2. Independently, Bollobás and Eldridge [Packing of graphs and applications to computational complexity, *J. Combin. Theory Ser. B* 25 (1978), 105–124] obtained a stronger result. Namely, they proved that if $|E(G_1)| + |E(G_2)| \leq 2n - 4$ then there is a packing of G_1 and G_2, provided that $\Delta(G_1) < n - 1$ and $\Delta(G_2) < n - 1$. In the paper we prove that for sufficiently large n, if $|E(G_1)| + |E(G_2)| + \max\{\Delta(G_1), \Delta(G_2)\} < \frac{5}{2}n - 2$ then there is a packing of G_1 and G_2. The bound is tight. Furthermore, we prove that if $|E(G_1)|+|E(G_2)|+\max\{\Delta(G_1), \Delta(G_2)\} \leq 3n-\alpha(n)$ where $\alpha(n) = o(n)$, then there is a packing of G_1 and G_2, provided that $\Delta(G_1) < n-1$ and $\Delta(G_2) < n - 1$. The bound is asymptotically tight.

1 Introduction

We deal with finite, simple graphs without loops or multiple edges. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$. The *order* of G is the number of vertices of G and is denoted by $|G|$. The *size* of G is the number of edges of G and is denoted by $||G||$. By $N_G(v)$ we denote the set of vertices adjacent to v in G. The *degree* (in G) of a vertex v is denoted by $d_G(v)$ and is equal to $|N_G(v)|$. A vertex v is called *total* if $d_G(v) = |G| - 1$. The *maximum degree* of G is denoted by $\Delta(G)$ and is equal to the maximum among degrees of all vertices of G. For a vertex set X, the set $N_G(X)$ denotes the *external neighborhood* of X in G, i.e.

$$N_G(X) = \{y \in V(G) \setminus X : y \text{ is adjacent to some } x \in X\}.$$

A *packing* of G and G' is a bijection $f : V(G') \to V(G)$ such that for every $u'v' \in E(G')$ we have $f(u')f(v') \notin E(G)$. If there is a packing of G and G' then we say that G and G' *pack*. The graph G'_f is defined by $V(G'_f) = V(G)$ and $E(G'_f) = \{f(u')f(v') : u'v' \in E(G')\}$.

Our starting point are two well known theorems concerning packing of two graphs with bounded sum of sizes.

Theorem 1 ([9]) *Let G and G' be two graphs of order n. If $||G|| + ||G'|| < \frac{3}{2}n - 1$ then G and G' pack.*

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The bound on the sum of sizes is tight because $G = S_{2n}$ (the star on $2n$ vertices) and $G' = nK_2$ (matching) obviously do not pack. However if one forbids total vertices in both graphs, then the bound on the sum of sizes can be significantly larger. Namely

Theorem 2 ([2]) *Let G and G' be two graphs of order n such that $\Delta(G) \leq n-2$ and $\Delta(G') \leq n-2$. If $||G|| + ||G'|| \leq 2n-4$ then G and G' pack.*

Again, the bound is tight because $G = S_n - e$ (the star on n vertices with one edge deleted) and $G' = C_n$ (the cycle) do not pack. It is obvious from the examples that a high maximum degree in one of the two graphs in question is the main reason that they do not pack. This is confirmed also by the following

Theorem 3 ([9]) *Let G and G' be two graphs of order n. If $2\Delta(G)\Delta(G') < n$ then G and G' pack.*

In the paper we consider analogues of Theorems 1 and 2 that arise by replacing the sum $||G||+||G'||$ by the sum $||G|| + ||G'|| + \max\{\Delta(G), \Delta(G')\}$. We obtain the following theorems.

Theorem 4 *Let G and G' be two graphs of order $n \geq 10^{10}$. If $||G||+||G'||+\max\{\Delta(G), \Delta(G')\} < \frac{5}{2}n - 2$, then G and G' pack.*

The bound is tight, which can be seen by taking (again) $G = S_{2n}$ and $G' = nK_2$.

Theorem 5 *Let G and G' be two graphs of order n such that $\Delta(G) \leq n-2$ and $\Delta(G') \leq n-2$. If $||G|| + ||G'|| + \max\{\Delta(G), \Delta(G')\} \leq 3n - 96n^{3/4} - 65$, then G and G' pack.*

The bound is asymptotically sharp, because the coefficient 3 cannot be increased. This follows from the example $G = S_n - e$ and $G' = C_n$. Furthermore, the bound cannot be essentially improved even with a stronger restriction on the maximum degree. Indeed, consider the following example from [12].

Example [12] Let $V_1, ..., V_{t-1}$ be pairwise disjoint subsets with $|V_i| = t$ for $i = 1, ..., t-2$ and $|V_{t-1}| = n - t(t-2)$. Furthermore, let $x \in V_{t-1}$. Let G be a graph with $V(G) = V_1 \cup \dots \cup V_{t-1}$ such that each V_i, $i = 1, ..., t-2$, induces a clique, V_{t-1} induce a star with center x and there are no other edges in G. In [12] it was proved that two copies of G do not pack provided that n is sufficiently large. Furthermore, $\Delta(G) = n - t(t-2) - 1$, $||G|| = \frac{t(t-1)}{2}(t-2) + n - t(t-2) - 1$. Hence, if $G' = G$ then $||G|| + ||G'|| + \max\{\Delta(G), \Delta(G')\} = 3n - 3 + t(t-2)(t-4)$.

Therefore, the coefficient 3 in Theorem 5 cannot be increased. Note that for $t = 3$, the bound on $||G|| + ||G'|| + \max\{\Delta(G), \Delta(G')\}$ is strongest.

We conjecture that

Conjecture 6 *Let G and G' be two graphs of order n such that $\Delta(G) \leq n-2$ and $\Delta(G') \leq n-2$. If $||G|| + ||G'|| + \max\{\Delta(G), \Delta(G')\} \leq 3n - 7$, then G and G' pack.*

In fact Theorem 4 is a simple consequence of Theorems 1 and 5. Indeed, if $\Delta(G) = n-1$ or $\Delta(G') = n-1$, then the assumption $||G|| + ||G'|| + \max\{\Delta(G), \Delta(G')\} < \frac{5}{2}n - 2$ implies that $||G|| + ||G'|| < \frac{3}{2}n - 1$. Thus G and G' pack by Theorem 1. On the other hand, if $\Delta(G) < n-1$, $\Delta(G') < n-1$ and $n \geq 10^{10}$, then $\frac{5}{2}n - 2 \leq 3n - 96n^{3/4} - 65$, so G and G' pack by Theorem 5.

For simplicity in the sequel let $\alpha(n) = 96n^{3/4} + 65$.

2 Preliminaries

In many places in the proofs we will use the following observation.

Proposition 7 *Let G and G' be graphs of order n such that $||G|| + ||G'|| + \max\{\Delta(G), \Delta(G')\} \leq 3n - \alpha(n)$. Let F and F' arises from G and G', respectively, by deleting m vertices from G and m vertices from G'. If the total number of edges covered (in G and G') by deleted vertices is at least $3m$, then $||F|| + ||F'|| + \max\{\Delta(F), \Delta(F')\} \leq 3n' - \alpha(n')$, where n' is the order of F and F'.*

Proof. Note that $\alpha(n)$ is increasing with respect to n. Thus,

$$||F|| + ||F'|| + \max\{\Delta(F), \Delta(F')\} \leq 3n - \alpha(n) - 3m = 3(n-m) - \alpha(n) \leq 3n' - \alpha(n').$$

$\square$

Lemma 8 *Let G and G' be graphs of the same order and $k \geq 1$ be any positive integer. If there is an independent set $U = \{u_1, ..., u_{2k}\} \subset V(G)$ and a set $U' = \{u'_1, ..., u'_{2k}\} \subset V(G')$ such that*

1. *$d_G(u_i) \leq k$ and $d_{G'}(u'_i) \leq k$ for $i = 1, ..., 2k$,*
2. *$N_G(u_i) \cap N_G(u_j) = \emptyset$ and $N_{G'}(u'_i) \cap N_{G'}(u'_j) = \emptyset$ for $i \neq j$ (i.e. vertices of U and U' have pairwise disjoint neighborhoods in G and G', respectively),*
3. *there is a packing of $G - U$ and $G' - U'$,*

then there exists a packing of G and G'.

Proof. Let $F := G - U$, $F' = G' - U'$ and let f_1 be a packing of F and F'. Below we show that we can find a packing f of G and G' by extending f_1.
For any $v' \in V(F')$ let us define $f(v') := f_1(v')$. Then let us consider a bipartite graph B with partite sets U and U'. For $i, j \in \{1, ..., 2k\}$ the vertices $v'_i \in U'$, $v_j \in U$ are joined by an edge in B if and only if $f_1(N_{G'}(v'_i)) \cap N_G(v_j) = \emptyset$. So, if v'_i, v_j are joined by an edge in B we can put $f(v'_i) = v_j$.
Since $d_{G'}(v'_i) \leq k$ for $i \in \{1, ..., 2k\}$ and v_i have disjoint neighborhoods in G, $d_B(v'_i) \geq 2k - k = k$. Similarly, $d_B(v_j) \geq 2k - k = k$. Let $S' \subset U'$. If $|S'| \leq k$ then obviously $|N_B(S')| \geq |S'|$. Notice that if $|S'| > k$, then $N_B(S') = U$. Indeed, otherwise let $v_j \in U$ be a vertex of B which has no neighbor in S'. Thus $d_B(v_j) \leq |U'| - |S'| \leq 2k - (k+1) = k - 1$, a contradiction. Hence, in any case $|S'| \leq |N(S')|$. Thus, by Hall's theorem there is a matching M in B. Therefore we can define $f(v'_i) = v_j$ for $i, j \in \{1, ..., 2k\}$ such that v'_i, v_j are incident with the same edge in M. $\square$

An *embedding* of G is a permutation σ of its vertices such that for every $uv \in E(G)$ we have $\sigma(u)\sigma(v) \notin E(G)$. In other words an embedding of G is a packing of two copies of G. If $\sigma(x) \neq x$ for every vertex $x \in V(G)$, then we say that G is *fixed-point-free* embeddable. Later we will need the following

Theorem 9 ([10]) *Every graph of order n and size at most $n - 2$ is fixed-point-free embeddable.*

A *starry tree* is a graph H such that (1) $V(H)$ can be partitioned into four sets V_1, V_2, V_3 and $\{x\}$ that each induce a tree, (2) there is at least one edge incident to x, (3) all edges not belonging to the trees induced by V_1, V_2 and V_3 are incident to x and (4) there are not edges between x and $V_2 \cup V_3$. A vertex x we call *a middle vertex* of H. Note that a starry tree is always disconnected.

The following lemma was proved in [12]

Lemma 10 ([12]) *Every starry tree admits an embedding such that its middle vertex is the image of one of its neighbors.*

For completeness we repeat the proof from [12].
Proof. Let H be a starry tree and let the tree components induced by V_1, V_2 and V_3 be denoted by T_1, T_2 and T_3, respectively. The proof is by induction on $|V_1| + |V_2| + |V_3|$. If $|V_1| + |V_2| + |V_3| = 3$, then the existence of an embedding as required is obvious. Assume that $|V_1| + |V_2| + |V_3| \geq 4$. We distinguish two cases:

Case 1. There exists a leaf l in T_1 such that the middle vertex x is adjacent to l.
Let $u \in V(T_2)$, $v \in V(T_3)$ be vertices such that $T_2 - u$ as well as $T_3 - v$ either is disconnected or has at most one vertex. Thus, by Theorem 9 (or trivially in the latter situation), there is an embedding σ_2 of $T_2 - \{u\}$ and there is an embedding σ_3 of $T_3 - \{v\}$.
Suppose first that $|T_1| = 1$ with $V(T_1) = \{l\}$. Then, the product $(l, x, v, u)\sigma_2\sigma_3$ is an embedding

as required of H. Suppose next that $|T_1| = 2$ with $V(T_1) = \{l, l'\}$. Then, $(l, x, u)(l', v)\sigma_2\sigma_3$ is an embedding as required of H. So we may assume that $|T_1| \geq 3$. Let l' be the neighbor of l in T_1. Let $y \in V(T_1)$ such that $T_1 - \{l, l', y\}$ either is disconnected or has at most one vertex. Thus, by Theorem 9 (or trivially in the latter situation), there is an embedding σ_1 of $T_1 - \{l, l', y\}$. Then the product $(l, x, u, y)(l', v)\sigma_1\sigma_2\sigma_3$ is an embedding as required of H.

Case 2. No leaf of T_1 is adjacent to x.
Let L be the set of the leaves of T_1, $L = \{l_1, ..., l_s\}$. Note, that $|T_1| \geq 3$. Suppose first that all the leaves of T_1 have a common neighbor y. Since H is a starry tree (so there is at least one edge incident to x) and the leaves of T_1 are not joined with x, xy is an edge of H. Let $H' = H - L$. Clearly, H' is a starry tree. Thus, by the induction hypothesis there is an embedding as required σ' of H'. Furthermore, since y is the only neighbor of x in H', we have $\sigma'(y) = x$. In particular y is not a fixed point of σ'. Thus the product $(l_1)...(l_s)\sigma'$ (i.e. $l_1, ..., l_s$ are fixed points) is an embedding as required of H.

So we may assume that there are $l_1, l_2 \in L$ with neighbors (in T_1) y_1, y_2, respectively, such that $y_1 \neq y_2$, $y_1 \neq l_2$ and $y_2 \neq l_1$ (recall that $|T_1| \geq 3$ in this case). Let $H'' = H - \{l_1, l_2\}$. Clearly, H'' is a starry tree. Hence, by the induction hypothesis, there is an embedding as required σ'' of H''. Then $(l_1, l_2)\sigma''$ is an embedding as required of H if $\sigma''(y_1) = y_1$ or $\sigma''(y_2) = y_2$. Otherwise, $(l_1)(l_2)\sigma''$ is an embedding as required of H. □

Let H be a graph and let $S \subset V(H)$. We call a connected component F of $H - S$ a *minimal component* (with respect to S) if F is a tree and every vertex of F has at most one neighbor in S. Let M_1,...,M_p denote all the minimal components of $H - S$. We define $R(S) = H - S - V(M_1) - ... - V(M_p)$. Furthermore, let $r(S)$ denote the sum of the number of edges of $R(S)$ and the number of all vertices in $R(S)$ which are joined with S by at least two edges. Clearly,

$$r(S) \geq ||R(S)||. \tag{1}$$

Moreover,

$$r(S) \geq |R(S)|. \tag{2}$$

Indeed, since $R(S)$ does not contain minimal components, every component of $R(S)$ which is a tree contains a vertex joined with S by at least two edges. On the other hand, every component of $R(S)$ which is not a tree has at least as many edges as vertices.
Finally,

Lemma 11 *If $||H|| - |N_H(S)| \leq |H - S| - c$ then $H - S$ has at least $c + r(S) - |R(S)|$ minimal components.*

Proof. Note that $r(S)$ counts all edges in $R(S)$ and some edges between $R(S)$ and S which are not counted in $|N_H(S)|$ because $|N_H(S)|$ counts only the neighbors and ignores the number of connections. Furthermore, there are exactly $|H| - |S| - |R(S)| - p$ edges in $\bigcup_{i=1}^{p} M_i$. Hence, by the assumption on $||H||$, we have

$$|N_H(S)| + |H - S| - c \geq ||H|| \geq (|N_H(S)| + r(S)) + (|H| - |S| - p - |R(S)|).$$

Therefore, $p \geq c + r(S) - |R(S)|$. □

3 Some properties of (eventual) minimum counterexamples

Throughout this section we assume that n is the minimum number for which there exist graphs G and G', both of order n with $\Delta(G) < n-1$, $\Delta(G') < n-1$ and $||G|| + ||G'|| + \max\{\Delta(G), \Delta(G')\} \leq 3n - \alpha(n)$, which do not pack. By V_i, V'_i we denote the sets of the vertices of degree i in G and G', respectively. Furthermore, $v_i := |V_i|$ and $v'_i := |V'_i|$.

Lemma 12 $\Delta(G) + \Delta(G') \geq 13034$

Proof. Note that if $n \leq 96^4$ then $3n - \alpha(n) \leq 2n - 4$. Thus G and G' pack by Theorem 2. Hence, we may assume that $n > 96^4$. Furthermore, if $2\Delta\Delta' < n$, then G and G' pack by Theorem 3. Thus, $\Delta\Delta' \geq n/2 \geq 96^4/2$. Since $\Delta + \Delta'$ is minimum when $\Delta = \Delta'$ (with the assumption that $\Delta\Delta' = const$), the claim follows. □

Lemma 13 *$\Delta(G) \leq n - \alpha(n) + 4$ and $\Delta(G') \leq n - \alpha(n) + 4$.*

Proof. Otherwise $||G|| + ||G'|| \leq 2n - 4$ and so G and G' pack by Theorem 2. □

Lemma 14 *$v_0 \leq 1$ and $v'_0 \leq 1$.*

Proof. Suppose $v_0 \geq 2$ and let x, y be isolated vertices of G. Furthermore, let $x' \in V(G')$ with $d_{G'}(x') = \Delta'$. If $d_{G'}(x') \geq 3$ then, by Proposition 7 and by the minimality assumption, there is a packing g of $G - x$ and $G - x'$. Clearly, a bijection f_1 such that $f_1(x') = x$ and $f_1(v') = g(v')$ for every $v' \in V(G') \setminus \{x'\}$ is a packing of G and G'.

So we may assume that $1 \leq \Delta' \leq 2$. Then, by Lemma 12, $\Delta > 4$. Let $z \in V(G)$ with $d_G(z) = \Delta > 4$. Let $\{y', z'\}$ includes the neighbors of x'. By Proposition 7 and by the minimality assumption, there is a packing h of $G - \{x, y, z\}$ and $G - \{x', y', z'\}$. Hence, a bijection f_2 such that $f_2(x') = z$, $f_2(y') = y$, $f_2(z') = x$ and $f_2(v') = h(v')$ for every $v' \in V(G') \setminus \{x', y', z'\}$ is a packing of G and G'.

Similar arguments hold in case when $v'_0 \geq 2$. □

Lemma 15 *$v_1 \leq 61$ or $v'_1 \leq 61$ or $|N_G(V_1)| = 1$ or $|N_{G'}(V'_1)| = 1$.*

Proof. Suppose for a contradiction, that $|N_G(V_1)| \geq 2$, $|V_1| \geq 62$, $|N_{G'}(V'_1)| \geq 2$ and $|V'_1| \geq 62$. By Lemma 12 we may assume that G contains a vertex x with $\deg x \geq 6000$ or G' contains a vertex x' with $\deg x' \geq 6000$. Let $x_1, x_2 \in V_1$ and y_1, y_2 with $y_1 \neq y_2$, be the neighbors of x_1 and x_2 respectively. Furthermore, Let $x'_1, x'_2 \in V'_1$ and y'_1, y'_2 with $y'_1 \neq y'_2$, be the neighbors of x'_1 and x'_2 respectively. Since, $|N_G(V_1)| \geq 2$ and $|V_1| > 2$, we may also assume that $y_1 \neq x_2$. Similarly, $y'_1 \neq x'_2$.

Note that the sum of edges covered by y_1 and y_2 in G, and by y'_1 and y'_2 in G', is at most 12. Indeed, otherwise by the minimality assumption there is a packing g of $H := G - \{x_1, x_2, y_1, y_2\}$ and $H' := G' - \{x'_1, x'_2, y'_1, y'_2\}$. Then a bijection f such that $f(x'_1) = y_1$, $f(x'_2) = y_2$, $f(y'_1) = x_2$, $f(y'_2) = x_1$ and $f(v) = g(v)$ for $v \in V(H')$ is a packing of G and G'.

Therefore, we may assume that $d_G(y_1) + d_G(y_2) \leq 10$, $d_{G'}(y'_1) + d_{G'}(y'_2) \leq 10$, $d_G(y_1) + d_{G'}(y'_2) \leq 10$ and $d_{G'}(y'_1) + d_G(y_2) \leq 10$. In particular x is not a neighbor of any vertex from V_1, or x' is not a neighbor of any vertex from V'_1. Suppose, without loss of generality, that $d_G(y_1) = \max\{d_G(y_1), d_G(y_2), d_{G'}(y'_1), d_{G'}(y'_2)\}$. Since $d_G(y_1) + d_G(y_2) \leq 10$, y_2 has at most $10 - d_G(y_1)$ neighbors in V_1. Thus, every vertex other than y_1 of G has at most $10 - d_G(y_1)$ neighbors in V_1. Moreover, since $d_G(y_1) + d_{G'}(y'_2) \leq 10$, y'_2 has at most $10 - d_G(y_1)$ neighbors in V'_1. Analogously, every vertex of G' has at most $10 - d_G(y_1)$ neighbors in V'_1.

Let $W_1 \subset V_1$ be the set of all vertices of degree 1 which are at distance equal to 1 or 2 from y_1. Let $W'_1 \subset V'_1$ be the set of all vertices of degree 1 which are at distance equal to 1 or 2 from y'_1. Let $X_1 = V_1 \setminus W_1$ and $X'_1 = V'_1 \setminus W'_1$. Thus, $|W_1| \leq (d_G(y_1) - 1)(10 - d_G(y_1)) + 1$. Hence, $|X_1| \geq |V_1| - (d_G(y_1) - 1)(10 - d_G(y_1)) - 1$. Since every vertex other than y_1 of G has at most $10 - d_G(y_1)$ neighbors in V_1, we have

$$|N_G(X_1)| \geq \frac{|V_1| - (d_G(y_1) - 1)(10 - d_G(y_1)) - 1}{10 - d_G(y_1)}.$$

Therefore, if $|V_1| \geq (d_G(y_1) - 1)(10 - d_G(y_1)) + 1 + 2(d_G(y_1) - 1)(10 - d_G(y_1)) + 1$ then $|N_G(X_1)| \geq d_G(y_1)$, so we can find an independent set $U \subset V_1$ of $2d_G(y_1) - 1$ vertices of degree 1 that have different neighbors and are at distance at least 3 from y_1. Analogously, if $|V_1'| \geq (d_{G'}(y_1') - 1)(10 - d_G(y_1)) + 1 + 2(d_G(y_1) - 1)(10 - d_G(y_1)) + 1$ then $|N(X_1')| \geq 2d_G(y_1) - 1$, so we can find an independent set $U' \subset V_1'$ of $2d_G(y_1) - 1$ vertices of degree 1 that have different neighbors and are at distance at least 3 from y_1'. It is easy to check that the above statement is true if $|V_1| \geq 62$ and $|V_1'| \geq 62$ since the largest number of vertices of degree 1 is needed when $d_G(y_1) \in \{5, 6\}$.

Let $z = x$ if $x \neq y_1$, or z be an arbitrary vertex from $V \setminus (U \cup \{x_1, y_1\})$ otherwise. Similarly, let $z' = x'$ if $x' \neq y_1'$, or z' be an arbitrary vertex from $V' \setminus (U' \cup \{x_1', y_1'\})$ otherwise. Recall that $\{z, z'\} \neq \{y_1, y_1'\}$. Consider now graphs $F := G - (U \cup \{z, x_1, y_1\})$ and $F' := G' - (U' \cup \{z', x_1', y_1'\})$. By the minimality assumption, there is a packing h_2 of F and F'. Furthermore, h_3 such that $h_3(z') = x_1$, $h_3(x_1') = z$ and $h_3(u') = h_2(u')$ for each $u' \in V(G') \setminus (U' \cup \{y_1'\})$ is a packing of $G - (U \cup \{y_1\})$ and $G' - (U' \cup \{y_1'\})$. Then, by Lemma 8, there is a packing of G and G', a contradiction. □

4 Proof of Theorem 5

Proof. The proof is by a contradiction. We assume that n is the minimum number such that there exist G and G' both of order n with $\Delta(G) < n - 1$, $\Delta(G') < n - 1$ and $||G|| + ||G'|| + \max\{\Delta(G), \Delta(G')\} < 3n - \alpha(n)$, which do not pack.

Let $V = V(G)$, $V' = V(G')$, $\Delta = \Delta(G)$ and $\Delta' = \Delta(G')$. For any bijection $f : V' \to V$, in order to refer to the edge sets $E(G)$, $E(G'_f)$ and $E(G) \cap E(G'_f)$ more easily, we color the edges in $E(G)$ red and the edges in $E(G'_f)$ blue, so that the edges in $E(G) \cap E(G'_f)$ have been colored both blue and red, making them purple. If a vertex u in $V(G)$ has a neighbor opposite a red (blue) edge, we call it a *red* (*blue*) *neighbor* of u. Also, for any vertex u, let R_u denote the set of red neighbors of u and B_u, the set of blue neighbors of u. Note that R_u and B_u need not be disjoint. Given $u, v \in V(G)$, an $\{u, v\}$-swap is a new bijection $g' := g \circ f$ from $V(G')$ to $V(G)$ where $g(u) = v$, $g(v) = u$ and $g(w) = w$ for all remaining vertices in $V(G)$.

By Lemma 15, without loss of generality we may assume that $|N_{G'}(V_1')| = 1$ or $|V_1'| \leq 61$. Let $S \subset V$ be a maximal independent set of vertices that have pairwise disjoint neighborhoods and such that

$$1 \leq d_G(v) \leq n^{1/4} \text{ for each } v \in S.$$

Furthermore, let $S' \subset V'$ be a maximal independent set of vertices that have pairwise disjoint neighborhoods and such that

$$2 \leq d_{G'}(v') \leq n^{1/4} \text{ for each } v' \in S'.$$

We will show that there exist a set A that satisfies

$$\begin{aligned} A &\subset V \\ |A| &\leq |N_G(S)| \\ |N_G(A)| &\geq n - 6n^{3/4} - |N_G(S)|. \end{aligned} \tag{3}$$

Indeed, by the maximality of S every vertex $v \in V \setminus N_G(S)$ with $1 \leq d_G(v) \leq n^{1/4}$ has a neighbor in $N_G(S)$. Let $A = N_G(S)$. Since $v_0 \leq 1$ and there are (by far) less than $6n^{3/4}$ vertices of degree greater than $n^{1/4}$ in G (as well as in G'), (3) follows.

Similar, but a bit more complicated calculations are made for G' as well. Namely there is a set A' that satisfies

$$\begin{aligned} A' &\subset V' \\ |A'| &\leq |N_{G'}(S')| + 1 \\ |N_{G'}(A')| &\geq n - 6n^{3/4} - 61 - |N_{G'}(S')|. \end{aligned} \tag{4}$$

Indeed, by the maximality of S' every vertex $v' \in V' \setminus N_{G'}(S')$ with $2 \leq d_{G'}(v') \leq n^{1/4}$ has a neighbor in $N_{G'}(S')$. Let $A' = N_{G'}(S')$ if $|N_{G'}(V_1')| \geq 2$ or $A' = N_{G'}(S') \cup \{y'\}$ if $N_{G'}(V_1') = \{y'\}$. Since $v_0' \leq 1$ and there are less than $6n^{3/4}$ vertices of degree greater than $n^{1/4}$ in G', and at most 61 vertices of degree 1 in G' if $|N_{G'}(V_1')| \geq 2$, (4) follows.

From now on we no longer use the assumption that $|N_{G'}(V_1')| = 1$ or $|V_1'| \leq 61$. It was used only to construct the set A' and in the sequal we will use only properties (4). It will be important later. The proof falls into three cases depending on how big the sets S and S' are.

Suppose first that $|S| \geq 14n^{1/2}$. If $|S'| \geq 2n^{1/4}$ then it is possible to choose sets $U \subset V$ and $U' \subset V'$ that satisfy conditions of Proposition 7 and Lemma 8. Indeed, for U and U' we may take any $2\lfloor n^{1/4} \rfloor$-subsets of S and S', respectively. By the definition of S and S', vertices from U and U' cover in common at least $3|U|$ edges of G and G'. Thus, by Proposition 7 and by the minimality assumption, $G - U$ and $G' - U'$ pack. Hence, by Lemma 8 the graphs G and G' pack which is a contradiction. Thus, $|S'| < 2n^{1/4}$ and so $|N_{G'}(S')| < 2n^{1/2}$. Therefore, by (4) we have

$$\begin{aligned} |A'| &< 2n^{1/2} + 1 \\ |N_{G'}(A')| &\geq n - 6n^{3/4} - 61 - 2n^{1/2} - 1. \end{aligned} \tag{5}$$

We construct a set $B' \subset V'$ by adding to A' all vertices $u' \in V' \setminus A'$ with $d_{G'}(u') \geq n^{1/2}$. The number of such vertices is (by far) less than $6n^{1/2} - 1$. Thus,

$$\begin{aligned} |B'| &< 2n^{1/2} + 1 + 6n^{1/2} - 1 = 8n^{1/2} \\ |N_{G'}(B')| &\geq n - 6n^{3/4} - 62 - 2n^{1/2}. \end{aligned}$$

Finally, we construct a set $C' \subset V'$ by removing from B' all vertices u' with $d_{G'}(u') \leq 2n^{1/4}$. Hence, we obtain $C' \subset V'$ such that

$$\begin{aligned} &|C'| < 8n^{1/2} \\ &|N_{G'}(C')| \geq |N_{G'}(B')| - 2n^{1/4}|A'| \geq n - 11n^{3/4}, \\ &d_{G'}(u') > 2n^{1/4} \text{ if } u' \in C' \text{ and } d_{G'}(u') < n^{1/2} \text{ if } u' \notin C'. \end{aligned} \tag{6}$$

We modify also a set S. Recall that S is independent, $|S| \geq 14n^{1/2}$ and vertices from S have pairwise disjoint neighborhoods in G. Since at most $6n^{1/2}$ vertices of G may have degree greater than or equal to $n^{1/2}$, at most $6n^{1/2}$ vertices from S may have a neighbor of degree greater than or equal to $n^{1/2}$. We remove such vertices from S (note that we still have at least $8n^{1/2} > |C'|$ vertices that remain). Additionally we remove certain number of vertices in order to obtain a set of cardinality equal to $|C'|$. Concluding, we have obtained a set $C \subset V$ such that

$$\begin{aligned} &|C| = |C'| =: t < 8n^{1/2} \\ &C \text{ is independent in } G \text{ and } N_G(u) \cap N_G(v) = \emptyset \text{ for } u, v \in C, u \neq v \\ &d_G(u) \leq n^{1/4} \text{ if } u \in C \text{ and } d_G(u) < n^{1/2} \text{ if } u \in N_G(C). \end{aligned} \tag{7}$$

Let $C = \{c_1, ..., c_t\}$ and $C' = \{c_1', ..., c_t'\}$. We distinguish three possibilities

a) $||G|| \leq n - \Delta' - n^{1/2} - |C'|$. Then G has at least $\Delta' + n^{1/2} + |C'|$ components that are trees. Since $v_0 \leq 1$ (and $|C'| \geq 1$), we may assume that G has s components $T_1, ..., T_s$ that are non-trivial trees (i.e. trees with at least two vertices) where $s \geq \Delta' + n^{1/2}$ and $s \geq |C'|$. Let $L = \{l_1, ..., l_s\}$ be a set of pendant vertices of G such that $l_i \in T_i$. Let $B = \{l_1, ..., l_t\}$ (recall that $t = |C'| \leq s$) be any t-element subset of L. Every bijection $g : V' \to V$ satysfying $g(C') = B$ and such that $g|_{V(G') \setminus C'}$ is a packing of $G - B$ and $G' - C'$, is called B-admissible. First we show that every t-element subset B of L has a B-admissible bijection. By the assumption and (6), we have

$$\begin{aligned} ||G - B|| + ||G' - C'|| &\leq 3n - \alpha(n) - \max\{\Delta, \Delta'\} - |N_{G'}(C')| \\ &< 3n - \alpha(n) - (n - 11n^{3/4}) \\ &< 2n - 16n^{1/2} - 4 \leq 2(n - t) - 4, \end{aligned} \tag{8}$$

because $t = |C'| \le 8n^{1/2}$. Furthermore, since $|G-B| = |G'-C'| \ge n-8n^{1/2}$, by Lemma 13 neither $G-B$ nor $G'-C'$ has a total vertex. Hence, by Theorem 2, $G-B$ and $G'-C'$ pack for any such B. Let f_B be a packing of $G-B$ and $G'-C'$. Thus, any extension of f_B is B-admissible. Given $B \subset L$, by g_B we denote a B-admissible bijection that has the minimum number of purple edges. Let $B = \{b_1, ..., b_t\}$ be a t-element subset of L for which g_B has the minimum number of purple edges. We will show that g_B is a packing of G and G'. Suppose, for the contrary, that $g_B(c')g_B(x')$ is a purple edge. Since g_B is a packing of $G-B$ and $G'-C'$, $g_B(x') \in B$ or $g_B(c') \in B$. We may assume that $g_B(c') =: b \in B$. Let $x = g_B(x')$. Since B is independent in G and xb is a purple edge, $x \in N_G(B)$. Thus, x is the only red neighbor of b (clearly, x is also a blue neighbor of b). We will show that there exists $z \in L$ such that $B_z \cap R_b = \emptyset$ and $R_z \cap B_b = \emptyset$. Indeed, recall that $d_{G'}(u') < n^{1/2}$ if $u' \notin C'$. In particular x has fewer than $n^{1/2}$ blue neighbors. Moreover, $|R_b| = 1$. Thus there are fewer than $n^{1/2}$ vertices $z \in L$ such that $R_b \cap B_z \neq \emptyset$. Furthermore, since b_i have pairwise different red neighbors, there are at most Δ' vertices $z \in L$ such that $B_b \cap R_z \neq \emptyset$. Since, $|L| \ge \Delta' + n^{1/2}$, we can find a vertex $z \in L$ such that $B_z \cap R_b = \emptyset$ and $R_z \cap B_b = \emptyset$. Let $Z = B \setminus \{b\} \cup \{z\}$ if $z \notin B$ or $Z = B$, otherwise. Note that an (b, z)-swap is Z-admissible and has fewer purple edges than g_B, which contradicts the choice of B or the definition of g_B.

b) $n - \Delta' - n^{1/2} - |C'| < ||G|| \le n - n^{3/4}$. Then G has at least $n^{3/4}$ components that are trees. Thus there is a tree with at most $n^{1/4}$ vertices. Let T be such a tree. Furthermore by (6), and by the assumption on $||G||$, we have

$$\begin{aligned} ||G'|| - |N_{G'}(C')| &\le 3n - \alpha(n) - \max\{\Delta, \Delta'\} - ||G|| - (n - 11n^{3/4}) \\ &< 2n - \max\{\Delta, \Delta'\} + 11n^{3/4} - \alpha(n) - n + \Delta' + n^{1/2} + |C'| \\ &< n + 11n^{3/4} - \alpha(n) + 9n^{1/2} < n - 9n^{1/2} \le |G' - C'| - n^{1/4}. \end{aligned}$$

Thus, by Lemma 11 and (2), $G' - C'$ contains at least $n^{1/4}$ minimal components. Let M'_1,...,$M'_{p'}$ denote the minimal components of $G' - C'$, with $p' \ge n^{1/4} > |T|$. Let $L' = \{l'_1, ..., l'_{|T|-1}\}$ be leaves from pairwise different components M'_i, $i = 1, ..., |T|-1$. Recall that all vertices in C' have degrees greater than $2n^{1/4}$ in G'. Furthermore, by (6) and Lemma 13, we have $|C'| \ge 2$. Since each vertex from L' has at most one neighbor in C' (by the definition of minimal component), there is $c' \in C'$ which is connected with at most $\frac{|T|-1}{2}$ vertices from L'. Note, that $d_{G'}(c')+||T|| > 2n^{1/4}+|T|-1 \ge 3|T|-1$. Thus, by Proposition 7 and by the minimality assumption, there is a packing of $G-V(T)$ and $G' - (\{c'\} \cup L')$. Let f_1 be the packing. Furthermore, since $||G'[\{c'\} \cup L']|| < |T|/2$ and $||T|| = |T| - 1$, by Theorem 1, there is a packing, say f_2, of T and $G'[\{c'\} \cup L']$. It is easily seen now that g such that $g(u') = f_1(u')$ for $u \in V' \setminus (\{c'\} \cup L')$ and $g(u') = f_2(u')$ for $u' \in (\{c'\} \cup L')$ is a packing of G and G'.

c) $||G|| > n - n^{3/4}$. Then, by (6) and by the assumption on $||G||$, we have

$$\begin{aligned} ||G'|| - |N_{G'}(C')| &< 3n - \alpha(n) - \max\{\Delta, \Delta'\} - ||G|| - (n - 11n^{3/4}) \\ &\le n - \alpha(n) - \max\{\Delta, \Delta'\} + 12n^{3/4} \\ &\le n - 8n^{1/2} - 18n^{3/4} - \max\{\Delta, \Delta'\} \le |G' - C'| - 18n^{3/4} - \max\{\Delta, \Delta'\}. \end{aligned}$$

Thus, by Lemma 11 and (2), $G' - C'$ contains at least $18n^{3/4} + \max\{\Delta, \Delta'\}$ minimal components. Let M'_1,...,$M'_{p'}$ be the minimal components of $G' - C'$ with

$$p' \ge 18n^{3/4} + \max\{\Delta, \Delta'\}. \tag{9}$$

Suppose first that at most $18n^{3/4}$ minimal components consist of only one vertex. Thus we have at least $\max\{\Delta, \Delta'\}$ minimal components that are of order greater than or equal 2. Hence each of these components has at least two leaves. We create a set $L' = \{l'_1, ..., l'_{q'}\}$ by choosing exactly two leaves from every minimal component of order greater than or equal to 2, and by choosing the one vertex from each minimal component of order 1. Note that

$$|L'| = q' \ge 18n^{3/4} + 2\max\{\Delta, \Delta'\} \ge 18n^{3/4} + \Delta + \Delta'. \tag{10}$$

Let $s = |N_G(C)|$. Given a s-element subset X' of L', a bijection $f : V' \to V$ is called X'-admissible if $f(X') = N_G(C)$ and for every purple edge xy we have $x \in N_G(C)$ and $y \notin C$. First we will show that there exists a set X' for which we can find an X'-admissible bijection. Indeed, recall that by (7),

$$s = |N_G(C)| = \sum_{i=1}^{t} d_G(c_i) < 8n^{3/4}. \tag{11}$$

We consecutively assign s different minimal components to vertices from C' in such a way that every vertex c'_i obtains $d_G(c_i)$ minimal components disjoint with c'_i. This is possible because, by (9), $p' > \Delta' + n^{1/4}|C'|$ and $d_G(c_i) \le n^{1/4}$. Without loss of generality we assume that M'_i, $i = 1, ..., s$, are assigned minimal components and $X' = \{x'_1, ..., x'_s\}$ with $x'_i \in M'_i \cap L'$.
Let $F = G - (C \cup N_G(C))$ and $F' = G' - (C' \cup X')$. We will show that F and F' satisfy the assumptions of Theorem 2. Since $|X'| = |N_G(C)|$, by (6), (7) and (11) , we have

$$|F| = |F'| \ge n - 8n^{1/2} - 8n^{3/4} \ge n - 9n^{3/4}, \tag{12}$$

This and Lemma 13 imply in particular that neither F nor F' has a total vertex. Furthermore, by (6),

$$\begin{aligned} ||F|| + ||F'|| &\le 3n - \alpha(n) - |N_{G'}(C')| \\ &\le 3n - \alpha(n) - (n - 11n^{3/4}) \le 2n - 18n^{3/4} - 4 \le 2|F| - 4. \end{aligned} \tag{13}$$

Hence, by Theorem 2, there is a packing h of F and F'. Let $N_G(C) = \{x_1, ..., x_s\}$ where the first $d_G(c_1)$ elements are in $N_G(c_1)$, the next $d_G(c_2)$ are in $N_G(c_2)$ and so on (by (7), $N_G(c_i)$ are pairwise disjoint). We define an X'-admissible bijection f in the following way: $f(c'_i) = c_i$, $f(x'_i) = x_i$ and $f(w') = h(w')$ for $w' \in V(F')$. Since C is an independent set of G and there are no edges between c'_i and the minimal components assigned to c'_i, this is indeed an X'-admissible bijection.

Given an s-element subset X' of L' let $g_{X'}$ (if exists) denote a X'-admissible bijection which has minimum number of purple edges. Let X' be an s-element subset of L' for which $g_{X'}$ exists and has minimum number of purple edges. We will show that $g_{X'}$ is a packing of G and G'. Suppose, for the contrary, that ux is a purple edge. Since $g_{X'}$ is X'-admissible, $u \in N_G(C)$ and $x \notin C$ (or $x \in N_G(C)$ and $u \notin C$, which is analogous). We will show that there is an $z' \in L'$ such that $R(z) \cap B(u) = \emptyset$ and $B(z) \cap R(u) = \emptyset$ where $z = g_{X'}(z')$. By (7) there are at most $n^{1/2}$ red edges incident to u and exactly one of them is incident with $C = g_{X'}(C')$. Thus, there are at most $2(n^{1/2} - 1) + \Delta'$ vertices $z \in g_{X'}(L')$ such that $R_u \cap B_z \neq \emptyset$. (this is because at most two l'_i may belong to a fixed minimal component of $G' - C'$). Furthermore, there are at most two blue edges incident to u (recall that u is an image of some l'_i which is a pendant vertex in $G' - C'$, and since l'_i belongs to a minimal component of $G' - C'$, it is joined with C' by at most one edge). So, one edge incident to u has the second end in the same blue minimal component as u, and the second (if exists) is incident with C. Since vertices of C have degrees at most $n^{1/4}$ there are at most $n^{1/4} + \Delta$ vertices $z \in g_{X'}(L')$ such that $B_u \cap R_z \neq \emptyset$. Therefore, since by (10) we have $|L'| > \Delta + \Delta' + 2n^{1/2} + n^{1/4}$, there is a vertex $z \in g_{X'}(L')$ such that $R_u \cap B_z = \emptyset$ and $B_u \cap R_z = \emptyset$. Let $u', z' \in L'$ be such that $g_{X'}(u') = u$ and $g_{X'}(z') = z$. Let $Z' = X' \setminus \{u'\} \cup \{z'\}$ if $z' \notin X'$ or $Z' = X'$ otherwise. Note that (u, z)-swap is a Z'-admissible bijection. Moreover, it has fewer purple edges than $g_{X'}$ which is a contradiction with the choice of X' or the definition of $g_{X'}$. Thus $g_{X'}$ is a packing of G and G'.

Suppose now that more than $18n^{3/4}$ minimal components consist of only one vertex. Without loss of generality we assume that $|M'_1| = |M'_2| = ... = |M'_{q''}| = 1$ where $q'' = \lceil 18n^{3/4} \rceil$. Assume that $c'_t \in C'$ has the largest number of neighbors in $\mathcal{M}' := \bigcup_{i=1}^{q''} M'_i$. Thus if $i \le t-1$ then c'_i has at most $9n^{3/4}$ neighbors in $\mathcal{M}'$. Again we consecutively assign s different minimal components to vertices from C' in such a way that every vertex c'_i obtains $d_G(c_i)$ minimal components disjoint with c'_i. However this time we require that if $i \le t-1$ then the minimal components assigned to c'_i

are from $\mathcal{M}'$. This is possible since the number of minimal components assigned to all c'_i is equal to $s \leq 8n^{3/4}$ (see (11)), $|\mathcal{M}'| \geq 18n^{3/4}$, $d_G(c_i) \leq n^{1/4}$ and each c'_i with $i \leq t-1$ has at most $9n^{3/4}$ neighbors in $\mathcal{M}'$. Moreover, since by (9), $p' \geq tn^{1/4} + \max\{\Delta, \Delta'\}$, we can assign further $d_G(c_t)$ arbitrary (but disjoint with c'_t) minimal components to c'_t. Let $X' = \{x'_1, ..., x'_s\}$ with x'_i belonging to corresponding different minimal components assigned to c'_i, $i = 1, ..., t$. Furthermore, let Y be a set of all vertices of G that are at distance 2 from c_t and do not belong to $N_G(C)$, i.e.

$$Y = N_G\left(N_G(c_t)\right) \setminus N_G(C).$$

Let $Y = \{y_1, ..., y_r\}$. Note that, by (7), $r \leq n^{3/4}$. Finally, let $Y' = \{y'_1, ..., y'_r\}$ such that $\{y'_i\} = M'_{s+i}$, and $N_G(C) =: X = \{x_1, ..., x_s\}$.
This time $F := G - (C \cup X \cup Y)$ and $F' := G' - (C' \cup X' \cup Y')$. Thus, as before

$$\begin{aligned} |F| = |F'| &\geq n - 8n^{1/2} - 8n^{3/4} - n^{3/4} \geq n - 10n^{3/4} \text{ and} \\ ||F|| + ||F'|| &\leq 3n - \alpha(n) - |N'_G(C')| \leq 2|F| - 4. \end{aligned}$$

Moreover, by Lemma 13, neither F nor F' has a total vertex. Therefore, F and F' satisfy the assumptions of Theorem 2. Thus, there exist a packing, say h, of F and F'. Define a bijection $f : V \to V'$ in the following way: $f(c'_i) = c_i$, $f(x'_i) = x_i$, $f(y'_i) = y_i$ and $f(w') = h(w')$ for $w' \in V(F')$. We claim that f is a packing of G and G'. By the choice of x'_i, there are no purple edges incident to C. Furthermore, since there are no blue edges contained in $(X \cup Y)$ there are no purple edges contained in $(C \cup X \cup Y)$ neither. Clearly, there are no purple edges contained in $V(F)$ as well. Hence, the only purple edges may appear between $X \cup Y$ and $V(F)$. However, the only blue edges between $X \cup Y$ and $V(F)$ are incident with $N_G(c_t)$. Moreover, all red edges incident to $N_G(c_t)$ are contained in $(N_G(c_t) \cup Y) \subset (X \cup Y)$. Thus, there are no purple edges, and so f is a packing of G and G'. Thus in any case when $|S| \geq 14n^{1/2}$ we obtained a packing of G and G', a contradiction.

The case when $|S'| \geq 14n^{1/2}$ is analogous. Note that A has all the properties of A', cf. (3) and (4). Since the assumption that $|N_{G'}(V'_1)| = 1$ or $|V'_1| \leq 61$ was not used after A' had been constructed, we may proceed as before and obtain a packing of G and G', and thus a contradiction.

So we may assume that $|S| < 14n^{1/2}$ and $|S'| < 14n^{1/2}$. Thus, $|N_G(S)| < 14n^{3/4}$ and $N_{G'}(S') < 14n^{3/4}$. By (3) and (4) we have $A \subset V(G)$ and $A' \subset V(G')$ with

$$\begin{aligned} &|A'| = |A| \leq 14n^{3/4} + 1 \\ &|N_G(A)| + |N_{G'}(A')| \geq 2n - 40n^{3/4} - 61 \end{aligned} \tag{14}$$

(we may assume that $|A| = |A'|$ because otherwise we may increase the smaller set by adding an adequate number of arbitrary vertices without spoiling the properties of A and A').

Let $f : V' \to V$ be a bijection satisfying $f(A') = A$. Define G''_f (in short G'') to be a graph with $V(G''_f) = V(G)$ and $E(G''_f) = E(G) \cup E(G'_f)$. Furthermore, let $c_f = |E(G) \cap E(G'_f)|$ be the number of purple edges in G''_f. Let $\Delta'' = \Delta(G''_f)$ and $A'' = A$.

Clearly,

$$\begin{aligned} &\Delta'' \leq \Delta + \Delta', \\ &||G''|| = ||G|| + ||G'|| - c_f, \\ &|A''| = |A| \leq 14n^{3/4} + 1, \\ &|N_{G''}(A'')| \geq |N_G(A)| + |N_{G'}(A')| - c_f \geq 2n - 40n^{3/4} - 61 - c_f \end{aligned} \tag{15}$$

Let $a \in A''$ and $x, y \in V \setminus A''$, $x \neq y$. If $x \in (R_a \setminus B_a)$ and $y \in (B_a \setminus R_a)$ then we call the path xay a *critical red-blue path*. Let $f : V' \to V$ be a bijection satisfying $f(A') = A$ with the minimum number of critical blue-red paths. Note that

$$\begin{aligned} ||G''|| - |N_{G''}(A'')| &\leq ||G|| + ||G'|| - c_f - (|N_G(A)| - |N_{G'}(A')| - c_f) \\ &\leq 3n - \alpha(n) - \max\{\Delta(G), \Delta(G')\} - (2n - 40n^{3/4} - 61) \\ &< n - 56n^{3/4} - \max\{\Delta, \Delta'\} \leq |G'' - A''| - 3|A''| - \max\{\Delta, \Delta'\}. \end{aligned}$$

Thus, by Lemma 11, $G'' - A''$ contains at least $3|A''| + \max\{\Delta, \Delta'\} + r(A'') - |R(A'')|$ minimal components. Let M_1'',...,$M_{p''}''$ be minimal components of $G'' - A''$ with

$$p'' \geq 3|A''| + \max\{\Delta, \Delta'\} + r(A'') - |R(A'')|. \tag{16}$$

Let $\mathcal{M}'' = \bigcup_{i=1}^{p''} M_i''$. Observe that there is no critical blue-red path xay such that $x, y \in V(\mathcal{M}'')$. Indeed, otherwise the $\{x, y\}$-swap has fewer critical blue-red paths than f, because each vertex of $\mathcal{M}''$ has at most one neighbor in A'' (by the definition of minimal components). This is a contradiction with the choice of f. Therefore, every $a \in A''$ has at most $\max\{\Delta, \Delta'\}$ neighbors in $\mathcal{M}''$.

In the next part of the proof we will show that G'' is embeddable. This fragment follows almost exactly the lines of the parts c), d) and e) of the proof of the main result in [12] pp. 16–17. Note that this is sufficient to prove that G and G' pack. Indeed, we take a subgraph isomorphic to G from one copy of G'' and a subgraph isomorphic to G' from the other one.

We assign to a vertex u'' of A'' a minimal component which is connected with u''. Let l'' be the maximum number of minimal components assigned to vertices of A'' in this way. If $l'' < |A''|$, then we assign an arbitrary minimal component to every remaining vertex of A''. Let $\mathcal{N}''$ be the set of minimal components not yet assigned. Now, we assign $2|A''|$ different minimal components to vertices from A'' in such a way that every vertex $u'' \in A''$ has two minimal components in $\mathcal{N}''$ disjoint with u''. This is possible because $|\mathcal{N}''| \geq \max\{\Delta, \Delta'\} + 2|A''|$ and u'' has at most $\max\{\Delta, \Delta'\}$ neighbors in the set of minimal components. So, we have constructed l'' starry trees with middle vertices in A''. Note, that l'' is the maximum number of starry trees with middle vertices in A''. Let L'' be a set of such starry trees with $|L''| = l''$.

Without loss of generality we may assume that we have assigned $M_1'', ..., M_{3|A''|}''$. Let $G_1'' := G''[A'' \cup V(M_1'') \cup ... \cup V(M_{3|A''|}'')]$ and $G_2'' := G'' - V(G_1'')$. Note that all edges of G'' between $V(G_1'')$ and $V(G_2'')$ are incident to A''. Below we will show that there exists an embedding of G_1'' such that every vertex from A'' is the image of its neighbor outside of A''.

Suppose first that $l'' = |A''|$. Then we pack every starry tree J_i'' in such a way that the middle vertex is the image of one of its neighbors in the same starry tree (the required embedding exists by Lemma 10). Let σ_i'' be the required embedding of J_i''. We claim that the product $\sigma'' = \sigma_1''....\sigma_{|A''|}''$ is an embedding of G_1'' as well. Since σ_i'' is an embedding of J_i'', only edges between different starry trees may spoil the embedding of G_1''. Furthermore, every middle vertex is mapped on a non-middle vertex. Since there are no edges between M_i'' and M_j'' for $i \neq j$, the edges between middle vertices do not spoil the embedding. It remains to check the edges of the form xy where x is the middle vertex of some starry tree and y is a non-middle vertex of another starry tree. However, since the middle vertex of each starry tree is the image of one of its neighbors in the same starry tree and this neighbor has no other neighbors outside its minimal component, these edges also do not spoil the embedding.

Suppose now, that $l'' < |A''|$. Again, we pack every starry tree in such a way that the middle vertex is the image of one of its neighbors. Moreover, since L is maximal, each remaining vertex of A'' has no neighbors in each of the remaining minimal components (otherwise, we would have an extra starry tree). Hence, by Theorem 9, each of the remaining vertices from A'' together with three non-trivial minimal components (not involved in any starry tree) can be packed without fixed points. We claim that the product of these embeddings is a proper embedding of G_1''. Suppose for a contradiction that the image of an edge e_1'' in G_1'' coincides with some other edge e_2'' in G_1''. Using the previous argument, e_2'' must join a vertex $z'' \in U''$ which is not in any starry tree from L'' with a non-middle vertex of some starry tree H. Moreover, e_1'' must join the middle vertex of H with some minimal component which is not in any starry tree from L''. However, now we can exchange the two minimal components that contain one of the endvertices of the edges e_1'' and e_2''. This way we obtain more than l'' starry trees and we get a contradiction. Hence G_1'' is embeddable.

Now we prove that G''_2 is embeddable as well. By (1) and (16) we have

$$\begin{aligned} ||G''_2|| &= ||R(A'') \cup M''_{3|A''|+1} \cup \cdots \cup M''_{p''}|| = ||R(A'')|| + |M''_{3|A''|+1}| + \cdots + |M''_{p''}| - (p'' - 3|A''|) \\ &\le ||R(A'')|| + |M''_{3|A''|+1}| + \cdots + |M''_{p''}| - (r(A'') - |R(A'')| + \max\{\Delta, \Delta'\}) \\ &\le |R(A'')| + |M''_{3|A''|+1}| + \cdots + |M''_{p''}| - 2 = |R(A'') \cup M''_{3|A''|+1} \cup \cdots \cup M''_{p''}| - 2 = |G''_2| - 2. \end{aligned}$$

Thus, by Theorem 2, G'' is embeddable.

Let σ', σ'' denote embeddings of G''_1 and G''_2, respectively. Then $\sigma = \sigma'\sigma''$ is an embedding of G''. Suppose for a contradiction that the image of an edge xy in G'' coincides with some other edge $\sigma(x)\sigma(y)$ in G''. Then $x, \sigma(x) \in V(G''_1)$ and $y, \sigma(y) \in V(G''_2)$. By construction of G''_1 and G''_2 we have that x and $\sigma(x)$ belong to A''. Then we get a contradiction, since the image of every vertex in A'' is not in A''. As we mentioned earlier, the embedding of G'' contradicts the assumption that G and G' do not pack. Hence we deduce no counterexample to Theorem 5 exists. $\square$

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