On $(C_n; k)$ stable graphs

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Abstract

A graph G is called $(H; k)$-*vertex stable* if G contains a subgraph isomorphic to H ever after removing any k of its vertices; $\mathrm{stab}(H; k)$ denotes the minimum size among the sizes of all $(H; k)$-vertex stable graphs. In this paper we deal with $(C_n; k)$-vertex stable graphs with minimum size. For each n we prove that $\mathrm{stab}(C_n; 1)$ is one of only two possible values and we give the exact value for infinitely many n's. Furthermore we establish an upper and lower bound for $\mathrm{stab}(C_n; k)$ for $k \geq 2$.

1 Introduction

We deal with simple graphs without loops and multiple edges. We use the standard notation of graph theory, cf. [1].

Consider the following problem. Suppose that we have a net with a sensor placed in each vertex of the net. We assume that the sensors are cheap, however, the connections between them are costly. Furthermore, we require that for certain reasons a given configuration of sensors and connections between them must be assured. In fact, we require more. Some sensors may get damaged, hence, we want that even if some of them are spoiled, the special configuration of sensors and connections is still assured in the net. Clearly, we want to assure this configuration with minimal cost.

More formally, let H be any graph and k a non-negative integer. A graph G is called $(H; k)$-*vertex stable* if G contains a subgraph isomorphic to H ever after removing any k of its vertices. Then $\mathrm{stab}(H; k)$ denotes minimum size among the sizes of all $(H; k)$-vertex stable graphs.

Note that if H does not have isolated vertices then after adding to or removing from a $(H; k)$-vertex stable graph any number of isolated vertices we still have a $(H; k)$-vertex stable graph with the same size. Therefore, in the sequel we assume that no graph in question has isolated vertices.

The notion of $(H;k)$-vertex stable graphs was introduced in [2]. So far the exact value of $\mathrm{stab}(H;k)$ is known in the following cases: $\mathrm{stab}(K_{1,m};k) = m(k+1)$, $\mathrm{stab}(C_i;k) = i(k+1)$, $i = 3,4$, $\mathrm{stab}(K_4;k) = 5(k+1)$, see [2], and $\mathrm{stab}(K_5;k) = 7(k+1)$ for $k \geq 5$ [5], $\mathrm{stab}(K_n;k) = \binom{n+k}{2}$ for $n \geq 2k-2$ [6]. Furthermore, $\mathrm{stab}(K_{m,n};1) = mn+m+n$ if $n \geq m+2, m \geq 2$, see [4], and $\mathrm{stab}(K_{n,n+1};1) = (n+1)^2$ for $n \geq 2$, $\mathrm{stab}(K_{n,n};1) = n^2+2n$ for $n \geq 2$, see [3]. Moreover, in all the above examples vertex stable graph with minimum size are characterized.

In this paper we deal with $(C_n;k)$ vertex stable graphs with minimum size. For each $n \geq 5$ we prove that $\mathrm{stab}(C_n;1)$ is one of only two possible values. Furthermore, for infinitely many n's we determine the exact value $\mathrm{stab}(C_n;1)$. Namely, for $S := \{l^2+1, l^2+2, l^2+l+1-x, l^2+l+2, l^2+2l+1-y, (l+1)^2 : l \geq 2, x \in S_x, y \in S_y\}$ (for the definitions of sets S_x and S_y see (5), page 6, and (6), page 6, respectively) we have the following

Theorem 1 *If $n \in S$ then*

$$\mathrm{stab}(C_n;1) = n + \left\lceil 2\sqrt{n-1} \right\rceil .$$

Otherwise,

$$n + \left\lceil 2\sqrt{n-1} \right\rceil \leq \mathrm{stab}(C_n;1) \leq n + \left\lceil 2\sqrt{n-1} \right\rceil + 1.$$

We give also an upper and lower bound for $\mathrm{stab}(C_n;k)$ for $k \geq 2$ and sufficiently large n.

2 $(C_n;1)$ stability of graphs

Recall the following observation.

Proposition 2 ([2]) *Let δ_H be a minimal degree of a graph H. Then in any $(H;k)$-vertex stable graph G with minimum size, $\deg_G v \geq \delta_H$ for each vertex $v \in G$.*

Theorem 3 *Let $n \geq 5$ be an integer. Then*

$$\mathrm{stab}(C_n;1) \geq \begin{cases} n+2l & \text{if } n = l^2+1 \\ n+2l+1 & \text{if } n \in [l^2+2, l^2+l+1] \\ n+2l+2 & \text{if } n \in [l^2+l+2, l^2+2l+1]. \end{cases}$$

Proof. Let G be a $(C_n;1)$ stable graph with minimum size. Let $x_1, ..., x_m \in V(G)$ be the vertices of degree greater than or equal to 3 in G. We call x_i's branch vertices. By Proposition 2 all other vertices of G have degree 2. Hence, every branch vertex is joined with some branch vertices (possibly also with itself) by subdivided edges. Note that since G is minimal, every component of G contains a cycle of length n. Otherwise, a component would be redundant, which is a contradiction to the minimality of G. A cycle C contained in G that has exactly one vertex with degree greater than or equal to 3 in G (all remaining vertices of C have degree 2 in G) is called a *subdivided loop*. Note, that every subdivided loop which may appear in G has length n, too. Indeed, otherwise the vertices of degree 2 in G that belong to a subdivided loop would be redundant, which again is a contradiction to the minimality of G. Thus, G is connected and does not have any subdivided loop, because otherwise $||G|| \geq 2n$ which is greater then the lower bound from the theorem (in particular, by removing the vertex of degree ≥ 3 in G from a subdivided loop, we obtain at least two components: a path P_{n-1} and a component that contains a cycle C_n). Furthermore, $m \geq 3$ because otherwise by removing a branch vertex (or an arbitrary vertex if $m = 0$) from G we obtain an acyclic graph.

Let $A(x_i) \subset V(G)$ denote the set of all vertices of degree 2 which are on the subdivided edges adjacent to x_i. Note, that

$$\sum_{i=1}^{m} |A(x_i)| = 2(v - m). \tag{1}$$

Let $M = \max_i |A(x_i)| = |A(x_j)|$ for some $j \in \{1, ..., m\}$. Thus,

$$\begin{aligned} m \cdot M &\geq \sum_{i=1}^{m} |A(x_i)| \text{ , whence, by(1),} \\ M &\geq 2\frac{v - m}{m}. \end{aligned} \tag{2}$$

Note that since G is $(C_n; 1)$ stable, $G - x_j$ contains a cycle of length n. This cycle cannot contain any vertex from $A(x_j)$. Thus, $v - |A(x_j)| - 1 = v - M - 1 \geq n$. Hence,

$$v \geq (n-1)\frac{m}{m-2}. \tag{3}$$

Furthermore,

$$\begin{aligned} 2e &\geq 3m + 2(v - m) \text{ hence} \\ e &\geq m/2 + v \geq m/2 + (n-1)\frac{m}{m-2}. \end{aligned} \tag{4}$$

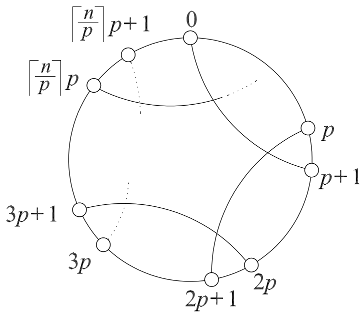
Let $f(x) := x/2 + (n-1)\frac{x}{x-2}$, $x > 2$. By simple computations, one can see that f has minimum in $x_0 = 2\sqrt{n-1} + 2$. Hence, $e \geq f(x_0) = n + 2\sqrt{n-1}$. Let l be an integer such that $l \leq \sqrt{n-1} < l + 1$. Thus, $n - 1 = l^2 + \alpha$, where $\alpha \in \{0, ..., 2l\}$. Therefore, if $n = l^2 + 1$, then $e \geq n + 2l$.

Let $n = l^2 + 1 + \alpha$, $\alpha \in [1, 2l]$. Note that f is decreasing for $x \leq x_0$ and increasing for $x \geq x_0$. Hence

$$e \geq \min\left\{f\left(\lfloor x_0 \rfloor\right), f\left(\lceil x_0 \rceil\right)\right\}.$$

Clearly $\lfloor x_0 \rfloor = 2 + \left\lfloor 2\sqrt{l^2 + \alpha} \right\rfloor < 2l + 4$ and $\lceil x_0 \rceil = 2 + \left\lceil 2\sqrt{l^2 + \alpha} \right\rceil > 2l + 2$. Therefore, since $\lceil x_0 \rceil - \lfloor x_0 \rfloor = 1$, $\{\lceil x_0 \rceil, \lfloor x_0 \rfloor\} \subset \{2l + 2, 2l + 3, 2l + 4\}$. It is easy to check that $f(2l + 2) = n + 2l + \frac{\alpha}{l}$, $f(2l + 4) = n + 2l + 1 + \frac{\alpha - l}{l+1}$. Furthermore, if we take $m = 2l + 3$ then one of vertices $\{x_1, ..., x_m\}$ has degree at least 4 because each graph has an even number of vertices of odd degree. Hence $e \geq f(2l + 3) + 1/2 = n + 2l + 1 + \frac{2\alpha - l}{2l+1}$. □

Figure 1:



Theorem 4 *Let $n \geq 3$ be an integer. Then*

$$\mathrm{stab}(C_n;1) \leq \begin{cases} n+2l+2 & \text{if } \ n \in [l^2+1, l^2+l] \\ n+2l+3 & \text{if } \ n \in [l^2+l+1, l^2+2l+1]. \end{cases}$$

Proof. Let us consider a cycle C_{n+p} with the vertex set $V = \{0,...,n+p-1\}$ and the edge set $E = \{(i,i+1);\ i = 0,...,n+p-1\}$ for some $0 < p$, with the numbers taken $\mathrm{mod}(n+p)$. Let G_p be a graph created from C_{n+p} by adding edges $(jp,(j+1)p+1)$ for $0 \leq j \leq \left\lceil \frac{n}{p} \right\rceil$, see Fig.1.

Note that G_p is $(C_n;1)$ stable. Suppose that we remove one vertex i from the cycle C_{n+p} for $i = 0,...,n+p$. By the symmetry of the graph we can assume that we remove one of the vertices from the set $S = \{p+1,...,2p\}$. Then there is a cycle C which contains all of vertices in C_{n+p} except p vertices in S with the vertex set $V_C = \{p, 2p+1, 2p+2,...,0,1,...,p-1\}$.
Moreover $||G_p|| = n+p+\left\lceil \frac{n}{p} \right\rceil+1$. Let $n = l^2+1+\alpha$ where $\alpha \in [0,2l]$. For $\alpha \in [0,2l-1]$, by taking $p = l$ we obtain that

$$||G_l|| = n+l+1+\left\lceil \frac{l^2+1+\alpha}{l} \right\rceil = n+2l+1+\left\lceil \frac{\alpha+1}{l} \right\rceil.$$

For $\alpha = 2l$, by taking $p = l+1$ we obtain that

$$||G_{l+1}|| = n+2l+3$$

which completes the proof. □
Theorems 3 and 4 imply the following corollary:

Corollary 5 *For $l \geq 2$*

$$\mathrm{stab}(C_n;1) \in \begin{cases} \{n+2l, n+2l+1, n+2l+2\} & \text{if } \ n = l^2+1 \\ \{n+2l+1, n+2l+2\} & \text{if } \ n \in [l^2+2, l^2+l] \\ \{n+2l+1, n+2l+2, n+2l+3\} & \text{if } \ n = l^2+l+1 \\ \{n+2l+2, n+2l+3\} & \text{if } \ n \in [l^2+l+2, l^2+2l+1]. \end{cases}$$

Our next aim is to determine the exact value of $\mathrm{stab}(C_n;1)$ for certain n's. Unfortunately, we were not able to do this for all n's.

The main ideas of the constructions are as follows.

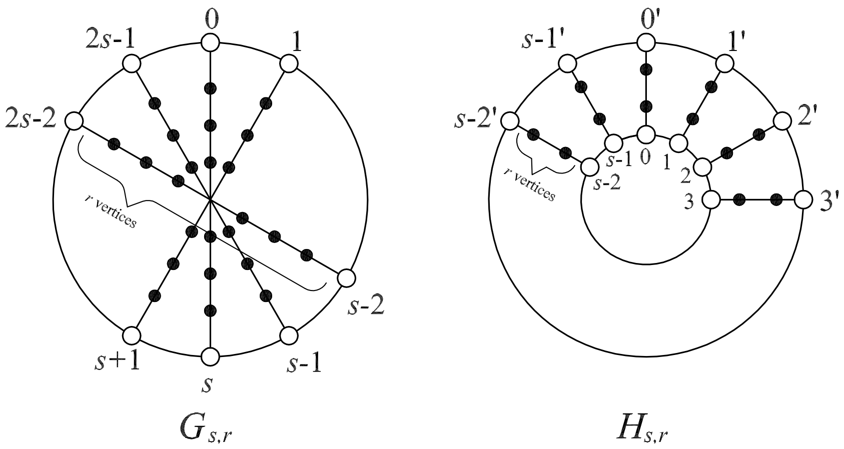
Construction A. Let us consider a cycle C_{2s}, where $s > 2$ is even, with the vertex set $V = \{0, ..., 2s-1\}$ and the edge set $E = \{(i, i+1);\ i = 0, ..., 2s-1\}$, where the numbers are taken $\mathrm{mod}(2s)$. Let $G_{(s,r)}$ be a graph obtained from C_{2s} in the following way. We join every two vertices $x, y \in V$ such that $|x - y| = s$ by adding s diagonals to C_{2s}. Next, we add r vertices on each diagonal this way that vertices $i, i+s \in V$ are joined by a path with order $r+2$ which is edge-disjoint with a cycle C_{2s} (see Fig.2). Let us denote such diagonals by D_i for $i = 0, ..., s-1$. Then $G_{(s,r)}$ has order $s(r+2)$ and size $s(r+3)$. Note that $G_{(s,r)}$ is $(C_{(r+1)(s-1)+s};1)$ stable. Indeed, suppose that we remove one vertex, say u, from $G_{(s,r)}$. By the symmetry of the graph we can assume that $u = 0$ or u is one of the vertices in D_0. Then the sequence $(1, D_1, s+1, s, s-1, D_{s-1}, 2s-1, 2s-2, D_{s-2}, s-2, s-3, D_{s-3}, 2s-3, 2s-4, D_{s-4}, ..., s+2, D_2, 2)$ represents the required cycle in $G_{(s,r)} - u$.

Construction B. Let us consider two copies of a cycle C_s, where $s > 2$ is odd and label consecutive vertices in one of these copies by 0 up to $s-1$ and by $0'$ up to $(s-1)'$ in the other. So, $(i, i+1)$ and $(i', (i+1)')$ is an edge, with the numbers taken $\mathrm{mod}(2s)$. Let $H_{(s,r)}$ be a graph obtained from these two copies of C_s in a following way. We join every vertex i and i' by an edge for $i = 0, ..., s-1$. Then, we add r vertices on each of s edges of type (ii') this way that vertices i, i' are joined by a path with order $r+2$ which is edge-disjoint with the two cycles C_s (see Fig.2). Let us denote a path vertex-disjoint with two copies of C_s which is between i and i' by D'_i for $i = 0, ..., s-1$. Then $H_{(s,r)}$ has order $s(r+2)$ and size $s(r+3)$. Note that $H_{(s,r)}$ is $(C_{(r+1)(s-1)+s};1)$ stable. Indeed, suppose that we remove one vertex, say u', from $H_{(s,r)}$. By the symmetry of the graph we can assume that $v = 0$ or v is one of the vertices in D_0. Then the sequence $\big(0', 1', D'_1, 1, 2, D'_2, 2', 3', D'_3, ..., s-1, D'_{s-1}, (s-1)'\big)$ represents the required cycle in $H_{(s,r)} - v$.

Construction C. Let $G'_{(s,r)}$ be a graph obtained from $G_{(s,r)}$ by adding one vertex between vertices with labels 0 and 1 and one vertex between vertices with labels s and $s+1$. Analogously, let $H'_{(s,r)}$ denote a graph obtained from $H_{(s,r)}$ by adding one vertex between vertices 0 and 1 and $0'$ and $1'$ in cycles C_s. Then $G'_{(s,r)}$ and $H'_{(s,r)}$ are $(C_{(r+1)(s-1)+s+1};1)$ stable. The required cycles are analogous to the ones from Construction A and Construction B.

- Let $n = l^2 + 1$. Then, for $s = l+1$, $r = l-2$, $G_{s,r}$ is a $(C_n;1)$ stable graph with minimum size for $l > 1$ odd and $H_{s,r}$ is a $(C_n;1)$ stable graph with minimum size for l even.
- Let $n = l^2 + 2$. Then, for $s = l+1$, $r = l-2$, $G'_{s,r}$ is a $(C_n;1)$ stable graph with minimum size for $l > 1$ odd and $H'_{s,r}$ is a $(C_n;1)$ stable graph with minimum size for l even.

Figure 2:



• Let $n = l^2 + l + 1 - x$, where

$$x \in S_x := \left\{ 0, 2, 2+4, ..., \sum_{i=0}^{q_0} 2i : \quad q_0 = \max\{q : \quad q(1+q) < l\} \right\}. \tag{5}$$

Observe that then there exists an odd integer c such that $c^2 = 1 + 4x$. Then, for $s = l + \frac{3-\sqrt{1+4x}}{2}$, $r = l + \frac{-3+\sqrt{1+4x}}{2}$ or $s = l + \frac{3+\sqrt{1+4x}}{2}$, $r = l + \frac{-3-\sqrt{1+4x}}{2}$, $G_{s,r}$ is a $(C_n; 1)$ stable graph with minimum size for s even and $H_{s,r}$ is a $(C_n; 1)$ stable graph with minimum size for $s > 1$ odd.

• Let $n = l^2 + l + 2$. Then, for $s = l + 1$, $r = l - 1$ or $s = l + 2$, $r = l - 2$, $G'_{s,r}$ is a $(C_n; 1)$ stable graph with minimum size for s even and $H'_{s,r}$ is a $(C_n; 1)$ stable graph with minimum size for $s > 1$ odd.

• Let $n = l^2 + 2l + 1 - y$, where

$$y \in S_y := \left\{ 0, 3, 3+5, ..., \sum_{i=0}^{q_0} (2i+1) : \quad q_0 = \max\{q : \quad q(q+2) < l\} \right\} \tag{6}$$

Observe that then there exists an integer d such that $d^2 = 1 + y$. Then, for $s = l + 2 - \sqrt{1+y}$, $r = l - 1 + \sqrt{1+y}$ or $s = l + 2 + \sqrt{1+y}$, $r = l - 1 - \sqrt{1+y}$, $G_{s,r}$ is a $(C_n; 1)$ stable graph with minimum size for s even and $H_{s,r}$ is a $(C_n; 1)$ stable graph with minimum size for $s > 1$ odd.

Combining the above constructions with Corollary 5 we obtain Theorem 1.

3 Extremal graphs

In this section we present some information about $(C_n; 1)$, $n \geq 5$, stable graphs with minimum size. Let $\tilde{G}$ denote a (multi,pseudo)graph arising from G by replacing all subdivided edges of G by edges and subdivided loops by loops.

Proposition 6 *Let $n = l^2+1$, $l \geq 3$. Let G be a $(C_n;1)$ stable graph with minimum size. Then the following statements hold.*

1. *$|G| = l^2 + l$, $||G|| = l^2 + 2l + 1$; furthermore, G has $2l + 2$ vertices of degree 3 and $l^2 - l - 2$ vertices of degree 2.*

2. *$\tilde{G}$ is a simple graph.*

3. *For each vertex $v \in \tilde{G}$ the graph $\tilde{G} - v$ is hamiltonian.*

4. *For each vertex $v \in \tilde{G}$ there exists an edge e_v incident to v such that e_v is involved in some hamiltonian cycle in every graph $\tilde{G}-x$ where x raises among all non-neighbors of v in $\tilde{G}$; moreover, if e_v is unique for v then in G it is subdivided by $l-2$ vertices.*

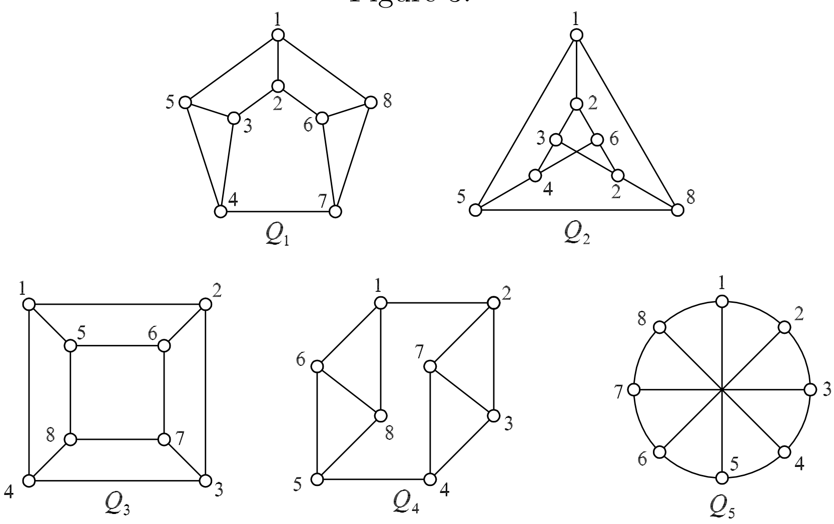
Proof. Note that from the proof of Theorem 3 it follows that in order to achieve the minimal size the inequalities (2), (3) and (4) must be equalities now. Thus, $|G| = l^2 + l$, $m = 2l+2$, $|A(x_i)| = l-2$ and $\deg_G(x_i) = 3$ for every $i = 1, ..., m$. Moreover, for every i, $|G - \{x_i\} - A(x_i)| = n$. Hence, a cycle C_n contains all vertices of $G - \{x_i\} - A(x_i)$. That means that for each i a graph $\tilde{G} - x_i$ is hamiltonian. Furthermore, the cycle contains all vertices from $A(v)$ of every non-neighbor v of x_i. Therefore, all edges incident to v in $\tilde{G}$ that are subdivided in G must be involved in a hamiltonian cycle in $\tilde{G} - x_i$. Since for each $v \in G$ at least one edge incident to v is subdivided, the third statement of the proposition holds.

Finally, by the same argument as in the proof of Theorem 3 we may exclude loops in $\tilde{G}$. Furthermore, $\tilde{G}$ does not have multiple edges. Indeed, otherwise we remove a vertex, say x, that is incident to a multiple edge, say xy. As a result we lose not only all vertices from $\{x\} \cup A(x)$ but also all vertices from $\{y\} \cup A(y)$, because $\deg_G(y) = 3$ (so it cannot be in any cycle in $G - x$). In such situation the number of vertices in G (by far) exceeds (3). □

Corollary 7 *$G_{(4,1)}$ is the only $(C_{10};1)$ stable graph with minimum size.*

Proof. Let G be a $(C_{10};1)$ stable graph with minimum size. By Proposition 6, G has 8 vertices of degree 3 and 4 vertices of degree 2. In particular $|\tilde{G}| = 8$. There are 5 connected cubic graphs $Q_1, ..., Q_5$ of order 8, see Fig. 3 [7]. It is easy to check that for $i = 2, 3, 4$, $Q_i - v_2$ is not hamiltonian. Moreover, $v_1 \in Q_1$ does not satisfy condition 3 of Proposition 6 which is seen by considering the graphs $Q_4 - v_3$, $Q_4 - v_4$ and $Q_4 - v_6$. Furthermore, note that the vertex $v_1 \in Q_5$ has a unique edge e_{v_1} satisfying condition 3 of Proposition 6, namely $e_{v_1} = v_1v_5$. This is seen by considering the graphs $Q_5 - v_i$, $i = 3, 4, 6, 7$. By symmetry, each vertex $v \in Q_5$ has the unique edge e_v. Therefore, by Proposition 6, $G = G_{(4,1)}$. □

Figure 3:



4 $(C_n; k)$ stability of graphs

Theorem 8 *Let $k \geq 2$ be a fixed integer. For each $0 < \epsilon \leq 1/(2k+4)$ there exists $n(\epsilon)$ such that if $n \geq n(\epsilon)$ then*

$$\mathrm{stab}(C_n; k) \geq n + 2\sqrt{(1-\epsilon)kn} - 1.$$

Proof. Let G be a $(C_n; k)$ stable graph with minimum size. We define x_i and $A(x_i)$, $i = 1, ..., m$, in the same way as in the proof of Theorem 3. For the same reasons as in the proof of Theorem 3 we assume that G is connected and does not have any subdivided loops. Furthermore we may assume that $m \geq k+2$. Let $M = \max_{1 \leq i_1 < ... < i_k \leq m} |A(x_{i_1}) \cup ... \cup A(x_{i_k})| = |A(x_{j_1}) \cup ... \cup A(x_{j_k})|$ for some $\{j_1, ..., j_k\} \subset \{1, ..., m\}$. Thus,

$$\begin{aligned} \binom{m}{k} \cdot M &\geq \sum_{i_1 < ... < i_k} |A(x_{i_1}) \cup ... \cup A(x_{i_k})| \\ &= \left[\binom{m-1}{k-1} + \binom{m-2}{k-1}\right](v-m), \end{aligned} \tag{7}$$

because vertices of degree 2 which lie on a fixed subdivided edge are counted $\binom{m-1}{k-1} + \binom{m-2}{k-1}$ times. It is not dificult to check that

$$M \geq \frac{(2m-k-1)k}{m(m-1)}(v-m).$$

Note that since G is $(C_n; k)$ stable, $G - \{x_{j_1}, ..., x_{j_k}\}$ contains a cycle of length n. This cycle cannot contain any vertex from $A(x_{j_1}) \cup ... \cup A(x_{j_k})$. Thus, $v - |A(x_{j_1}) \cup ... \cup A(x_{j_k})| - k = v - M - k \geq n$. Hence,

$$v \geq n\frac{m(m-1)}{m(m-1)-2mk+k^2+k} - \frac{km^2-k^2m}{m(m-1)-2mk+k^2+k}. \tag{8}$$

Furthermore,

$$\begin{aligned} 2e &\geq 3m + 2(v-m) \;\text{ hence} \\ e &\geq m/2 + n\frac{m(m-1)}{m(m-1)-2mk+k^2+k} - \frac{km^2-k^2m}{m(m-1)-2mk+k^2+k}. \end{aligned} \tag{9}$$

It is easy to check that if $m \geq \frac{k+1}{2\epsilon}$ and $\epsilon \leq \frac{1}{2k+4}$ then

$$e \geq \frac{m}{2} + n\frac{m-1}{m-1-(2-2\epsilon)k} - (k+1).$$

Let $f(x) = x/2 + n\frac{x-1}{x-1-(2-2\epsilon)k} - (k+1)$, $x \geq k+2$. By simple computations, one can see that f has minimum in $x_0 = 2\sqrt{(1-\epsilon)kn} + 2(1-\epsilon)k + 1$. This implies that $e \geq f(x_0) = n + 2\sqrt{(1-\epsilon)\,kn} - \epsilon k + 1/2 - 1 \geq n + 2\sqrt{(1-\epsilon)\,kn} - 1$. On the other hand, if $m < \frac{k+1}{2\epsilon}$ then for sufficiently large n we have $e > (1+c)n$ for a constant $c > 0$. Thus, for sufficiently large n, $e \geq n + 2\sqrt{(1-\epsilon)\,kn} - 1$, too. □

Theorem 9 *For each $k \geq 1$ and each $n \geq 3$*

$$\mathrm{stab}(C_n; k) \leq n + 2k\left\lceil \sqrt{n} \right\rceil + k^2.$$

Proof. Let us consider a cycle C_{n+kp} with the vertex set $V = \{0, ..., n+kp-1\}$ and the edge set $E = \{(i, i+1);\; i = 0, ..., n+kp-1\}$ for some $p > 0$, the numbers being taken mod$(n+kp)$. Let G_p^k be a graph created from C_{n+kp} by adding edges $\left\{(jp, (j+1)p+1), ..., (jp, (j+k)p+1) : j = 0, ..., (k-1) + \left\lceil \frac{n}{p} \right\rceil \right\}$. We will prove that G_p^k is $(C_n; k)$ stable. The proof is by induction on k. For $k = 1$ see the proof of Theorem 4. In general case, let $v \in [ip+1, (i+1)p]$ for some i. Then $G_p^k - \{ip+1, ..., (i+1)p\}$ contains a G_p^{k-1}. Hence, by the induction hypothesis, $G_p^k - \{ip+1, ..., (i+1)p\}$ is $(C_n; k-1)$ stable. This implies that G_p^k is $(C_n; k)$ stable, because v was chosen arbitrarily

Furthermore $||G_p^k|| = n + k^2 + k\left(p + \left\lceil \frac{n}{p} \right\rceil\right)$. Let $n = l^2 + 1 + \alpha$, where $\alpha \in [0, 2l]$. In the same way as in the proof of Theorem 4, by taking $p = l$ or $p = l+1$, we conclude that

$$||G_p^k|| \leq n + k^2 + k(2l+2) = n + k^2 + 2k\left\lceil \sqrt{n} \right\rceil.$$

□

5 Concluding remarks

We have presented the lower and upper bound for $\mathrm{stab}(C_n;k)$, $k \geq 1$. In case $k=1$ the bounds differ only by 1. Moreover, for infinitely many n's we have determined the exact value of $\mathrm{stab}(C_n;1)$. Note that from Theorem 1 it follows that the first unknown value is $\mathrm{stab}(C_{12};1) \in \{19,20\}$ and the next one $\mathrm{stab}(C_{15};1) \in \{23,24\}$. By an exhaustive computer search, we found exactly two $(C_{12};1)$ stable graphs with size 19 (one of them arises from Petersen graph by dividing four of five edges connecting the outer cycle with the inner one in the most popular picture of this graph). Hence, $\mathrm{stab}(C_{12};1)=19$. Therefore, in each case when we know the exact value of $\mathrm{stab}(C_n;1)$ it is equal to the lower bound from Theorem 1. We do not know if there is n with $\mathrm{stab}(C_n;1)$ equal to the upper bound from this theorem. We are closer to think that the answer is yes.

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References

[1] R. Diestel, *Graph Theory*, second ed., Springer-Verlag 2000.

[2] A. Dudek, A. Szymański, M. Zwonek, (H,k) *stable graphs with minimum size*, Discussiones Mathematicae Graph Theory 28(1) (2008) 137–149.

[3] A. Dudek, M. Zwonek, (H,k) *stable bipartite graphs with minimum size*, Discussiones Mathematicae Graph Theory, 29 (2009) 573–581.

[4] A. Dudek, A. Żak, *On vertex stability with regard to complete bipartite subgraphs*, Discussiones Mathematicae Graph Theory, 30 (2010) 663-669.

[5] J-L. Fouquet, H. Thuillier, J-M. Vanherpe and A.P. Wojda, On $(Kq;k)$ vertex stable graphs with minimum size, Discrete Math. (2011) article in press doi:10.1016/j.disc.2011.04.017.

[6] J-L. Fouquet, H. Thuillier, J-M. Vanherpe and A.P. Wojda, On $(Kq;k)$ stable graphs with small k, Research report RR-2011-01, LIFO University of Orleans, submitted.

[7] Weisstein, Eric W. "Cubic Graph." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/CubicGraph.html