# Dissections of polygons into convex polygons

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**Abstract**

In the paper we present purely combinatorial conditions that allow us to recognize the topological equivalence (or non-equivalence) of two given dissections. Using a computer program based on this result, we are able to generate a set which contains all topologically non-equivalent dissections of a $p_0$-gon into convex $p_i$-gons, $i = 1, ..., n$, where $n$, $p_0, ..., p_n$ are integers such that $n \geq 2$, $p_i \geq 3$. By analyzing generated structures, we are able to find all (up to similarity) dissections of a given type. Since the number of topologically non-equivalent dissections is huge even if the number of parts is small, it is necessary to find additional combinatorial conditions depending on the type of sought dissections, which will allow us to exclude the majority of generated structures. We present such conditions for some special dissections of a triangle into triangles. Finally we prove two new results concerning perfect dissections of a triangle into similar triangles.

**Keywords:** dissection, topologically equivalent, convex polygon, similar triangles
**2000 MSC:** 05B45, 59C20

## 1 Introduction

A *dissection* of a planar polygon $P_0$ is a finite set of pairwise internally disjoint polygons $P_i$, $i = 1, ..., n$, whose union is $P_0$. Polygon $P_0$ is called a *divided polygon* and polygons $P_1, ..., P_n$ are called *tiles*. A *P-dissection* of $P_0$ is a dissection of a polygon $P_0$ in which all tiles are similar to a given polygon $P$. In what follows we consider only dissections into convex polygons. Two dissections are said to be *congruent* if one of them can be made to coincide with the other or its reflection by a rigid motion of the plane. Two dissections are said to be *equal* if one of them can be changed in scale so as to be congruent to the other.

The purpose of this paper is to present a computer aided method which will allow us to find all non-equal dissections with some given properties.

The first step in our method is to find all possible but essentially different ways of dividing a $p_0$-gon into $p_i$-gons for given integers $p_0, p_1, ..., p_n$, $p_i \geq 3$, $n \geq 2$ (see Fig. 2 for all possible ways of dividing a triangle into four triangles). This step is performed with the help of our computer program. The algorithm is based on Theorem 1 (Section 3) in which we present purely combinatorial conditions that allow us to determine whether or not two dissections are topologically equivalent.

The second step is also performed with the help of a computer. Since the number of distinct dissections is huge even if the number of tiles is relatively small (for example there are 20198 topologically non-equivalent dissections of a triangle into 7 triangles [7, 10]) then it is necessary to find strong criteria depending on the type of sought dissections, which will allow us to eliminate the majority of generated structures. Such criteria are presented in Section 4 in which we consider dissections of a triangle into similar triangles.

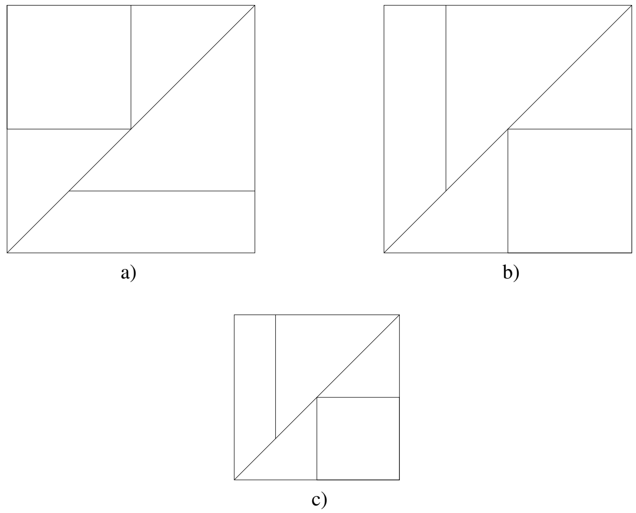


Fig. 1. Equal dissections, but only a) and b) are congruent

The last step is made without a computer. At this stage we deal with a small number of dissections, hence, we can analyze them in detail.

In Section 5 we present two applications of the described method. We prove that there are exactly two non-right triangles $T$ that have a perfect (i.e. all tiles are pairwise incongruent) $T$-dissection into 7 tiles (it is known that there are infinitely many non-right triangles $T$ having a perfect $T$-dissection into $n$, $n = 6, 8$ or $n \geq 10$, tiles (cf. [3]), and that there are no non-right triangles $T$ having a perfect $T$-dissection into less than six tiles [13]). We also prove that there are exactly two non-right triangles $\triangle$ such that an equilateral triangle has a perfect $\triangle$-dissection into 7 tiles (it is known [14] that 7 is the smallest possible number of tiles in such dissection).

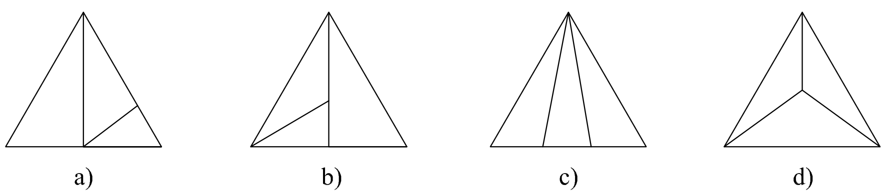


Fig. 2. Different ways of dividing a triangle into 3 triangles.

## 2 Notation

A graph $G$ is a pair of sets, $V(G)$ and $E(G)$ (in short $V$ and $E$), where $E(G)$ is a set of 2-elements subsets of $V(G)$. The elements of the set $V(G)$ are called *vertices* of $G$, the elements of the set $E(G)$ are the *edges* of $G$. An edge $\{x, y\}$ is usually written as $xy$ or $yx$. A vertex $x$ is *incident* with an edge $e$ if $x \in e$. A vertex $x$ is a *neighbor* of a vertex $y$ in $G$ if $xy \in E(G)$. The *degree* of a vertex $x$ in $G$, $\deg_G x$ (in short $\deg x$), is the number of neighbors of $x$ in $G$. If $U$ is any set of vertices of $G$, we write $G - U$ for a graph obtained from $G$ by deleting all the vertices in $U$ and their incident edges. Two graphs $G$ and $G'$ are *isomorphic* if there exists a bijection $\iota : V(G) \to V(G')$ with $xy \in E(G) \Leftrightarrow \iota(x)\iota(y) \in E(G')$ for all $x, y \in V(G)$. Such a map $\iota$ is called an *isomorphism.* A *path* is a non-empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, ..., x_k\}$, $E = \{x_0x_1, x_1x_2, ..., x_{k-1}x_k\}$; if $k \geq 2$ then the graph $\{V, E \cup \{x_kx_0\}\}$ is called a *cycle*. A non-empty graph $G$ is called *connected* if any two of its vertices are linked by a path in $G$. $G$ is called $k$-*connected* if $|V| > k$ and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$.

The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices form an edge. A graph which is drawn in the plane in such a way that no two edges meet in a point other than a common end is called *plane*. An abstract graph that can be drawn in this way is called *planar*. For plane graph $G$, the regions of a set $\mathbb{R}^2 \setminus G$ are the *faces* of $G$. Exactly one face is unbounded and is called the *outer* face.

In what follows $n$, $p_0, ..., p_n$ are positive integers such that $n \geq 2$, $p_i \geq 3$ for $i = 0, ..., n$. Consider a dissection of a $p_0$-gon $P_0$ into $p_i$-gons $P_i$. Every dissection appoints a plane graph, called its *dissection-graph*. Its vertices are corners of polygons $P_0, ..., P_n$, its edges are segments joining two vertices and containing no other vertex, its bounded faces are interiors of polygons $P_1, ..., P_n$. The unbounded face equals $\mathbb{R}^2 \setminus P_0$. The cycles of a dissection-graph $G$ bounding the faces of $G$ are said to be *facial*. We distinguish two sets among boundary vertices of a dissection-graph

$V_2$—set of corners of $P_0$,
$V_b$—set of vertices lying on the boundary of $P_0$, but different from its corners.

Let $v_2$ and $v_b$ denote the cardinalities of the above sets. Thus, $v_2 = p_0$.
Similarly let us distinguish two sets among internal vertices

$V_4$—set of vertices each of which is an internal point of a side of some tile,
$V_3$—set of vertices (lying in the interior of $P_0$) each of which is a corner of every tile it belongs to.

Let $v_4$ and $v_3$ denote the cardinalities of those sets.

Recall the following known way of counting vertices of a dissection graph. Note that the sum of angles of tiles at every vertex from $V_3$ is equal to $2\pi$ and the sum of angles of tiles at every vertex from $V_b \cup V_4$ is equal to $\pi$. Moreover, the sum of angles at the corners of $P_0$ is equal to $(p_0 - 2)\pi$. Thus, summing angles of $n$ tiles we obtain

$$\sum_{i=1}^{n}(p_i - 2) \cdot \pi = (p_0 - 2)\pi + v_3 \cdot 2\pi + (v_b + v_4) \cdot \pi.$$

Thus

$$2v_3 + v_4 + v_b = \sum_{i=1}^{n}(p_i - 2) - p_0 + 2. \tag{1}$$

Using Euler's formula $|E| = n + |V| - 1 = n + v_2 + v_3 + v_4 + v_b - 1$ and formula (1) we obtain

$$|E| = 1 - v_3 + \sum_{i=1}^{n}(p_i - 1). \tag{2}$$

# 3 Topologically non-equivalent dissections

Recall the following definition. Let $G = (V, E)$ and $G' = (V', E')$ be plane graphs and let $F$ and $F'$ be the faces of $G$ and $G'$, respectively. Suppose that $G$ and $G'$ are isomorphic as abstract graphs and consider any abstract isomorphism $\iota : V \longrightarrow V'$ between $G$ and $G'$. We call $\iota$ a *topological isomorphism* if there exists a homeomorphism $h$ from the plane $\mathbb{R}^2$ to itself that induces $\iota$ on $V \cup E$ (that means $h$ agrees with $\iota$ on $V$, and maps every plane edge $xy \in G$ onto the plane edge $\iota(x)\iota(y) \in G'$). We call graphs $G$ and $G'$ *topologically equivalent* if there exists a topological isomorphism between $G$ and $G'$. Note that different dissections can have topologically equivalent graphs, see Fig. 2 a) and c)). Therefore, let us introduce the following

**Definition 1** *Let $P_0 = \bigcup_{i=1}^{n} P_i$ and $P_0' = \bigcup_{i=1}^{n} P_i'$ be two dissections of polygons $P_0$ and $P_0'$ into $n$ polygons denoted by $P_i$, $P_i'$, respectively. We call those two dissections topologically equivalent if*

1. *their dissection-graphs are topologically equivalent, with $h$ being a corresponding homeomorphism of the plane, and*
2. *for every polygon $P_i$, $i = 0, ..., n$, a vertex $y$ is a corner of $P_i$ if and only if $h(y)$ is a corner of $h[P_i]$.*

Clearly, $h$ maps the faces of $G$ onto the faces of $G'$ and preserves the adjacency of vertices, edges and faces. Thus, the correspondence $P_i \mapsto h[P_i]$ is a bijection between sets of tiles of two topologically equivalent dissections. Moreover, $h[P_0] = P_0'$. Hence, the following proposition is easy to see.

**Proposition 1** *Let $P_0 = \bigcup_{i=1}^{n} P_i$ and $P_0' = \bigcup_{i=1}^{n} P_i'$ be two dissections of polygons $P_0$ and $P_0'$, and $h$ a corresponding homeomorphism of the plane. Let $G$ and $G'$ be their dissection-graphs. Then the two dissections are equal if and only if they are topologically equivalent and for any two edges $e, f \in E(G)$*

- *the lengths of $e$, $f$, $h[e]$ and $h[f]$ satisfy $|e|/|h[e]| = |f|/|h[f]|$,*
- *the angle between $e$ and $f$ is equal to the angle between $h[e]$ and $h[f]$.*

$\square$

Therefore, we can obtain all non-equal dissections by examining topologically non-equivalent ones. However, Definition 1 is inconvenient and cannot be applied to a computer program. Our next aim is to find a more convenient (for the use of a computer) combinatorial characterization of topologically equivalent dissections.

Let $G$ be a graph of a dissection of a $p_0$-gon $P_0$ into polygons. Then $\tilde{G}$ denotes a plane graph which arises from $G$ by adding a new vertex $x$ on the outer face of $G$ together with $p_0$ arcs which connect $x$ with every corner of $P_0$. We use the same notation for the planar graph isomorphic to each plane graph $\tilde{G}$.

**Lemma 1** *Let $G$ be a dissection-graph and $\{x_1, ..., x_k\}$ be a subset of $V(G)$. If $x_1, ..., x_k$ are collinear then the planar graph $\tilde{G} - \{x_1, ..., x_k\}$ is connected.*

Proof. In the proof we use the following definition. Let $A$ be a plane graph and let $y$ be a vertex of $A$. We call $y$ a *strongly-convex* vertex of $A$ if all inner points of all edges of $A$, which are incident to $y$, lie on some open half-space determined by a line containing $y$. Since tiles are convex polygons, the only possible strongly-convex vertices in $\tilde{G}$ are corners of $P_0$ and $x$ ($x$ being an additional vertex of $\tilde{G}$).
Assume that vertices $x_1, ..., x_k$ separate $\tilde{G}$. Let $S$ be a component of $\tilde{G} - \{x_1, ..., x_k\}$ which does not contain $x$ . Thus, $S$ does not contain any strongly-convex vertex of $\tilde{G}$. Consider a subgraph $H$ of a graph $\tilde{G}$ induced by vertices $x_1, ..., x_k$ and all vertices of $S$, $V(S)$. Note that $H$ contains all edges of $\tilde{G}$ which are incident to $V(S)$. Hence, vertices $x_1, ..., x_k$ and $V(S)$ are not collinear because the degree of each vertex of $S$ in $H$ is greater than or equal to 3. Thus, the convex hull of the vertices of $H$ is a convex non-degenerate polygon $Q$. The corners of $Q$ are some vertices of $H$ (extreme points of $H$). Note that at most two vertices from $x_1, ..., x_k$ can be corners of $Q$. Thus at least one vertex of $S$ is a corner of $Q$. Let $s$ be such a vertex. Moreover, since every edge of $H$ is an edge of $G$, all the edges of $H$ are line segments. Thus, $Q$ contains a subgraph $H$. Hence $s$ is a strongly-convex vertex of $H$. Since $H$ contains all edges of $\tilde{G}$ which are incident to $V(S)$, then $s$ is also a strongly-convex vertex of $\tilde{G}$, a contradiction with previous observations. $\square$

**Corollary 1** *Let $G$ be a dissection-graph. Then $\tilde{G}$ is 3-connected.*

Proof. Suppose $x_1$ and $x_2$ separate $\tilde{G}$. Clearly, $G$ is 2-connected, hence $x_1 \neq x$ and $x_2 \neq x$, ($x$ being an additional vertex of $\tilde{G}$). Thus, $x_1$ and $x_2$ are vertices of $G$. Hence, by Lemma 1 they do not separate $\tilde{G}$, a contradiction. □

**Theorem 1** *Let $P_0 = \bigcup_{i=1}^n P_i$ and $P'_0 = \bigcup_{i=1}^n P'_i$ be two dissections of polygons $P_0$ and $P'_0$ into $n$ polygons denoted by $P_i$, $P'_i$, respectively. Let respectively $G$ and $G'$ be their dissection-graphs. Furthermore, let $X_i$, $X'_i$, $i = 0, ..., n$, denote the sets of corners of every polygon $P_i$ and of every polygon $P'_i$, respectively. Then, these two dissections are topologically equivalent if and only if*

1. *$G$ and $G'$ are isomorphic as abstract graphs, and*
2. *there exists a graph isomorphism $\iota : V(G) \to V(G')$ such that $\iota[X_0] = X'_0$ and $\{\iota[X_i] : i = 1, ...n\} = \{X'_i : i = 1, ..., n\}$.*

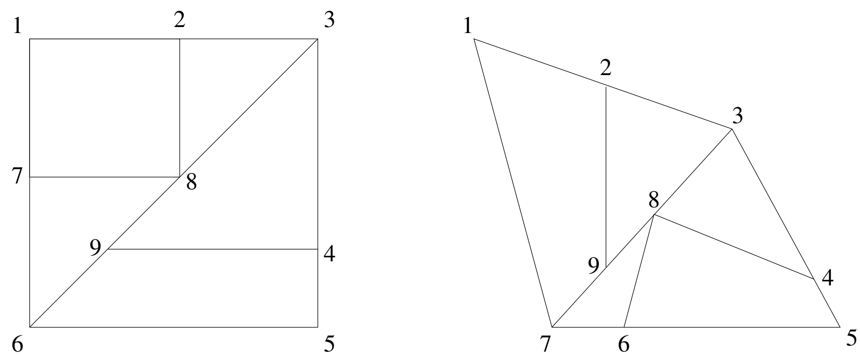


Fig. 3. Topologically equivalent dissections; $\iota = \binom{123456789}{543217689}$.

Proof. Obviously, topologically equivalent dissections satisfy both items of the theorem. We will show that if two dissections satisfy both condition then they are topologically equivalent. We show first that their dissection-graphs are topologically equivalent. Let $\tilde{G}$, $\tilde{G}'$ be plane graphs resulting from $G$ and $G'$ by adding vertices $x$ and $x'$ on the outer face of $G$ and $G'$, respectively, together with suitable arcs. Furthermore, let $\tilde{\iota} : V\left(\tilde{G}\right) \to V\left(\tilde{G}'\right)$ be a graph isomorphism which is a natural extension of $\iota$, namely $\tilde{\iota}|_{V(G)} = \iota$ and $\tilde{\iota}(x) = x'$. Since $\tilde{G}$ and $\tilde{G}'$ are 3-connected then, by Whitney's known theorem [12], they are uniquely embeddable in the sphere. That means that their facial cycles are uniquely determined and that any graph isomorphism of $\tilde{G}$ and $\tilde{G}'$ which map the outer cycle of $\tilde{G}$ to the outer cycle of $\tilde{G}'$, can be extended to a homeomorphism of whole plane. Thus, $C$ is a facial cycle of $\tilde{G}$ if and only if $\tilde{\iota}[C]$ is a facial cycle of $\tilde{G}'$. Let $C = xyv_1...v_kzx$ be a facial cycle of $\tilde{G}$. Then $y, z \in X_0$ and $v_1, ..., v_k \in V(G) \setminus X_0$. Moreover, $\tilde{\iota}(x)\tilde{\iota}(y)\tilde{\iota}(v_1)...\tilde{\iota}(v_k)\tilde{\iota}(z)\tilde{\iota}(x)$ is a facial cycle of $\tilde{G}'$. Hence, $\tilde{\iota}(y), \tilde{\iota}(z) \in X'_0$ and $\{\tilde{\iota}(v_1), ..., \tilde{\iota}(v_k)\} \cap X'_0 = \emptyset$. Therefore, we can redraw (if necessary) the arcs from $x$ and $x'$ to the corners of $P_0$ and $P'_0$ in such a way that $xyv_1...v_kzx$ is the outer cycle of $\tilde{G}$, and $\tilde{\iota}(x)\tilde{\iota}(y)\tilde{\iota}(v_1)...\tilde{\iota}(v_k)\tilde{\iota}(z)\tilde{\iota}(x)$ is the outer cycle of $\tilde{G}'$. Hence, there exists a homeomorphism $h$ of the plane which induces $\tilde{\iota}$ on $\tilde{G}$. Thus, $h$ induces also $\iota$ on $G$. Thus, $G$ and $G'$ are topologically equivalent graphs.
The second item in Definition 1 arises from the fact that $h[P_i] \in \{P'_1, ..., P'_n\}$, $i = 1, ..., n$, and $h(y) = \iota(y)$ if $y \in V(G)$. □

Therefore, we have obtained a combinatorial characterization of topologically equivalent dissections, which can be applied to a computer algorithm.

Given positive integers $n$, $p_0, ..., p_n$, we generate the set $\Gamma(p_0; p_1, ..., p_n)$ of all possible triples

$$(H, Y_0, \{Y_1, ..., Y_n\}),$$

where $H$ is a graph and $Y_i \subset V(H)$, $i = 0, ..., n$. At that point, we consider two triples $(H, Y_0, \{Y_1, ..., Y_n\})$ and $(H', Y'_0, \{Y'_1, ..., Y'_n\})$ being the same, if they satisfy items 1 and 2 of Theorem 1 (with $G = H$, $G' = H'$, $X_i = Y_i$ and $X'_i = Y'_i$, $i = 0, ..., n$). Furthermore, we require that the triples satisfy the following conditions resulting from their geometrical interpretation.

1. for every $i = 0, ..., n$, the set $Y_i = \{v_{i,1}, ..., v_{i,p_i}\}$ is a set of $p_i$ distinct vertices of the graph $H$,

2. $H$ is planar and 2-connected,

3.
   - $|E(H)| \leq 1 + \sum_{i=1}^{n}(p_i - 1)$
   - $|E(H)| - |V(H)| = n - 1$,

4. the graph $\tilde{H} := (V(H) \cup \{x\}, E(H) \cup \{(x, v_{0,1}), ..., (x, v_{0,p_0})\})$ is planar and 3-connected,

5. for every $i \in \{1, ..., n\}$ vertices $v_{i,1}, ..., v_{i,p_i}$ belong to the same and not containing $x$ facial cycle of $\tilde{H}$ (facial cycles of $\tilde{H}$ are, uniquely determined because of Whitney's theorem and Lemma 1).

6.
   - every vertex $u \in V(H)$ is an element of at most as many sets $Y_i$, $i \in \{0, ..., n\}$, as is its degree $\deg_H u$ in $H$, and at least as many sets $V_i$ as is its degree minus 1, $\deg_H u - 1$;
   - if $u \in Y_0$, then $u$ is an element of exactly as many sets $Y_i$, $i \in \{0, ..., n\}$ as is the degree of $u$ in $H$.

All above conditions are satisfied by every dissection (of a $p_0$-gon into convex $p_i$-gons, $i = 1, ..., n$) for which $H$ is isomorphic to its dissection-graph, $Y_0$ corresponds to the set $X_0$, and sets $Y_i$ correspond to the sets $X_i$. Condition 2 is obvious. Condition 3 follows from the Euler formula and formula (1). Condition 4 is a consequence of Corollary 1, and conditions 1, 5 and 6 follows from the geometrical interpretation of sets $Y_i$. Thus, by Theorem 1, it follows that the set $\Gamma(p_0; p_1, ..., p_n)$ contains all topologically non-equivalent dissections of a $p_0$-gon into convex $p_i$-gons, $i = 1, ..., n$. It should be pointed out, however, that $\Gamma(p_0; p_1, ..., p_n)$ may contain triples that do not correspond to any dissection of a $p_0$-gon into $p_i$-gons, $i = 1, ..., n$. For example, there are 180 topologically non-equivalent dissections [7, 10] of a triangle into 5 triangles, while the number of triples in a corresponding set $\Gamma$ is equal to 181. Conditions 1-6 are not sufficient to guarantee the existence of a dissection corresponding to a given triple from the set $\Gamma$. It would be an interesting problem to find such conditions even in the case of dissections of a triangle into triangles.

We give some explanation about the application of our algorithm. The generation of all elements satisfying conditions 1-4 is performed with the help of the program Plantri [6]. Using Plantri we generate all non-isomorphic 3-connected planar graphs of size less than or equal to $1+\sum_{i=1}^{n}(p_i-1)+p_0$ and satisfying Euler formula with the number of inner faces equal to $n+p_0-1$. We next take only those generated graphs that contain at least one vertex of degree equal to $p_0$. From each of those graphs, we remove, in all possible ways, a vertex of degree $p_0$ (together with adjacent edges). In this way we obtain all pairs $(H, Y_0)$ ($Y_0$ being a set of neighbors of a removed vertex) satisfying conditions 1-4. Note that at this stage some of these pairs may repeat. We next find for each pair $(H, Y_0)$ the faces of the corresponding graph $\tilde{H}$. Since $\tilde{H}$ is 3-connected its facial cycles are uniquely determined. Moreover, the facial cycles of $\tilde{H}$ are exactly its induced and non-separating cycles (that means such cycles $C$ which do not contain a chord and with the property that $\tilde{H} - V(C)$ is connected), see [1] p. 89. Let $C_1, ..., C_n$ denote those facial cycles of $\tilde{H}$ that do not contain the special vertex $x$. We consider every permutation $\sigma$ of the set $1, ..., n$ and choose, in all possible ways, $p_{\sigma(i)}$ vertices from each facial cycle $C_i$. We keep only those choices which satisfy condition 6. The equivalence of two triples is verified by examining every bijection between $V(H)$ and $V(H')$.

## 4 $\triangle$-dissections

In this section we present some combinatorial criteria that have to be satisfied in every dissection of a triangle into similar triangles. In what follows, $T$ denotes a divided triangle and $\triangle$ denotes a triangle which is a common shape of all tiles. The following deep theorem of Laczkovich [5] will be useful in proving some of our further results.

**Theorem 2 ([5])** *Suppose that the triangle $T$ can be dissected into finitely many triangles with angles $\alpha, \beta, \gamma$. If $\alpha, \beta, \gamma$ are rational multiples of $\pi$, then, with a suitable permutation of $\alpha, \beta, \gamma$, one of the following statements is true:*

1. *the angles of $T$ are $\alpha$, $\beta$, $\gamma$,*
2. *$\gamma = \pi/2$ and the angles of $T$ are $\alpha$, $\alpha$, $2\beta$,*
3. *$(\alpha, \beta, \gamma)$ equals one of the triples*
$$\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3}\right), \quad \left(\frac{\pi}{3}, \frac{\pi}{12}, \frac{7\pi}{12}\right), \quad \left(\frac{\pi}{3}, \frac{\pi}{30}, \frac{19\pi}{30}\right), \quad \left(\frac{\pi}{3}, \frac{7\pi}{30}, \frac{13\pi}{30}\right),$$
*and $T$ is equilateral,*
4. *$(\alpha, \beta, \gamma) = (\pi/6, \pi/6, 2\pi/3)$ and the angles of $T$ are $\pi/6$, $\pi/3$, $\pi/2$,*
5. *$(\alpha, \beta, \gamma) = (\pi/10, 3\pi/10, 6\pi/10)$ and the angles of $T$ are $3\pi/10$, $3\pi/10$, $4\pi/10$,*
6. *$(\alpha, \beta, \gamma) = (\pi/10, 2\pi/10, 7\pi/10)$ and the angles of $T$ are $\pi/10$, $\pi/10$, $8\pi/10$,*
7. *$(\alpha, \beta, \gamma)$ equals one of the triples*
$$\left(\frac{\pi}{8}, \frac{\pi}{4}, \frac{5\pi}{8}\right), \quad \left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}\right), \quad \left(\frac{\pi}{12}, \frac{\pi}{4}, \frac{2\pi}{3}\right),$$
*and $T$ is the isosceles right triangle.*

In the case when all the $p_i$, $i = 0, ..., n$, in formulas (1) and (2), are equal to 3, we have

$$2v_3 + v_4 + v_b = n - 1, \tag{3}$$

and

$$|E(G)| = 2n + 1 - v_3. \tag{4}$$

We will use also the following known proposition which is very simple but is also very efficient.

**Proposition 2 ([13])** *Every $\triangle$-dissection of a polygon $P_0$ into non-right triangles satisfies the following conditions*

1. *if $x \in V_2$ then $\deg x \geq 2$,*
2. *if $x \in V_3$ then $\deg x \geq 3$,*
3. *if $x \in V_4$ or $x \in V_b$ then $\deg x \geq 4$.*

Table 1 presents the number of topologically non-equivalent dissections of a triangle into $n$ triangles [7, 10] for $2 \leq n \leq 7$, and the number of those dissections which satisfy Proposition 2. The set of dissections satisfying Proposition 2 was obtained using a computer program described in the previous section. Independently, the same set of dissections was found by Vicher who used his own computer program. We have checked that both sets are the same. A complete list of dissections satisfying Proposition 2 can be found in [11].

| the number of tiles $n$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| the number of dissections | 1 | 4 | 23 | 180 | 1806 | 20198 |
| the number of dissections satisfying Proposition 2 | 0 | 1 | 2 | 7 | 39 | 224 |

Table 1.

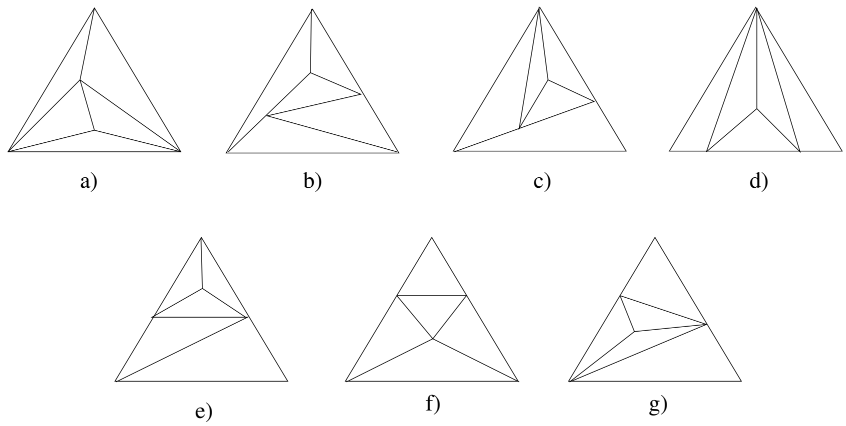


Fig. 4. Dissections into 5 tiles, satisfying Proposition 2.

**Proposition 3** *Consider a dissection of a triangle $T_0$ into $n$ triangles $T_i$, $i = 1, \dots, n$. If the degree of every vertex $x \in V_b \cup V_4$ is greater than or equal to 4, then*

$$\sum_{x\in V_2} \deg x + \sum_{x\in V_3} \deg x \le 6 + 6v_3 \tag{5}$$

*with equality if and only if for all $x \in V_b \cup V_4$, $\deg x = 4$.*

Proof. Using (3) and (4), we obtain

$$4n + 2 - 2v_3 = 2e = \sum_{x\in V_2} \deg x + \sum_{x\in V_3} \deg x + \sum_{x\in V_b\cup V_4} \deg x \ge$$

$$\sum_{x\in V_2} \deg x + \sum_{x\in V_3} \deg x + 4(v_b + v_4) = \sum_{x\in V_2} \deg x + \sum_{x\in V_3} \deg x + 4n - 4 - 8v_3.$$

Thus,

$$\sum_{x\in V_2} \deg x + \sum_{x\in V_3} \deg x \le 6 + 6v_3$$

with equality if and only if for all $x \in V_b \cup V_4$, $\deg x = 4$. □

In the following theorem we present next, very easy to check, conditions that must be satisfied in every $\triangle$-dissection of $T$.

**Theorem 3** *Let $\triangle$ be a non-right triangle and assume that in a $\triangle$-dissection of a triangle $T$, $\sum_{x\in V_2} \deg x \ge 7$. Then one of the following statements is true*

- *for each vertex $y \in V_3$ with $\deg y \le 5$ the following formula holds*

$$\deg y + \sum_{x\in V_2} \deg x = 12, \tag{6}$$

- *$\sum_{x\in V_2} \deg x = 7$ and $T$ is an isosceles right triangle.*

Proof. Let $A, B, C$ be angles of $T$ and $\alpha, \beta, \gamma$ angles of $\triangle$. Note that each angle in the dissection is a linear combination of $\alpha$, $\beta$ and $\gamma$ with non-negative integer coefficients. In particular $\pi = A+B+C = p\alpha + q\beta + r\gamma$, $p, q, r \geq 0$. Since $\deg A + \deg B + \deg C \geq 7$ then $p+q+r \geq 4$. Hence, at least one of coefficients $p, q, r$ is equal to 0. Without loss of generality we assume that $r = 0$.

Assume now that not all of the angles $\alpha, \beta, \gamma$ are rational multiples of $\pi$. Let $y \in V_3$ with $\deg y \leq 5$. Thus,

$$\begin{bmatrix} 1 & 1 & 1 \\ p & q & 0 \\ p' & q' & r' \end{bmatrix} \begin{bmatrix} \alpha/\pi \\ \beta/\pi \\ \gamma/\pi \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

where $p, q, p', q', r'$ are non-negative integers such that $p+q \geq 4$ and $3 \leq p'+q'+r' \leq 5$. Because not all of the angles $\alpha, \beta, \gamma$ are rational multiples of $\pi$ then the determinant

$$\det \begin{bmatrix} 1 & 1 & 1 \\ p & q & 0 \\ p' & q' & r' \end{bmatrix} = 0.$$

Moreover, if there is a solution then also

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & q & 0 \\ 2 & q' & r' \end{bmatrix} = 0.$$

Thus,

$$\begin{cases} p(r'-q') = q(r'-p') \\ r'-q' = q(r'-2). \end{cases} \tag{7}$$

Note that because $\deg y \leq 5$ and $\triangle$ is non-right, a distribution of $\alpha, \beta, \gamma$ around $y$ can be 5-0-0, 4-1-0, 3-2-0, 3-1-0 or 3-0-0 (with a suitable permutation of $\alpha, \beta, \gamma$). In particular, two positive coefficients among $p', q', r'$ are always different and at least one of $p', q', r'$ is equal to 0. If $p' = 0$ then $r' \neq q'$. Thus, by (7), $r' \neq 2$ and

$$p = \frac{r'}{r'-2} \quad \text{and} \quad q = \frac{r'-q'}{r'-2}.$$

Thus $r' = 0$, $r' = 3$ or $r' = 4$. It is easy to see that if $r' = 3$ then $p+q+r+p'+q'+r' = 9$, hence (6) holds. If $r' = 4$ then $q' = 1$, but now $q = 3/2$, a contradiction. If $r' = 0$ then $q' = 5$ or $q' = 3$, hence $q$ is not an integer, a contradiction. In the case when $q' = 0$ we argue analogously. Finally, suppose that $p' \neq 0$, $q' \neq 0$ and $r' = 0$. Then by (7), $q' = 2q$ and $p' = 2p$ which is not possible because then $p'+q' = 2(p+q) \geq 8$, a contradiction. Thus we have completed the proof in the case when not all of the angles $\alpha, \beta, \gamma$ are rational multiples of $\pi$.

To prove our theorem when $\alpha, \beta, \gamma$ are all rational multiples of $\pi$ we will carefully follow Laczkovich's proof of Theorem 2, see [4] pp. 90–92. Let $\alpha = a\pi/n$, $\beta = b\pi/n$ and $\gamma = c\pi/n$, where $a, b, c, n$ are positive integers such that $a+b+c = n$. In [5] Laczkovich proved that for every integer $k$ prime to $2n$, the existance of $\triangle$-dissection of $T$ implies the existance of $\triangle'$-dissection of $T'$ where the angles of $\triangle'$ are either $\alpha' = \{ka/n\}\pi, \beta' = \{kb/n\}\pi, \gamma' = \{kc/n\}\pi$ or $\alpha' = (1-\{ka/n\})\pi, \beta' = (1-\{kb/n\})\pi$, $\gamma' = (1-\{kc/n\})\pi$, where $\{x\}$ denotes the fractional part of the real number $x$. He calls the $\triangle'$-dissection of $T'$ the conjugate tiling belonging to $k$. Recall that $\pi = A+B+C = p\alpha + q\beta$, where $p, q$ are non-negative integers and $p+q \geq 4$. If $p = q = 2$ then $\gamma = \pi/2$, a contradiction. Thus we may assume that $p \geq 3$ and $p \geq q$. Laczkovich next proved that the only possible values of $\alpha$ are $\pi/3$, $\pi/4$, $\pi/5$, $\pi/6$, $\pi/10$ and $3\pi/10$. Then he showed that the only possibilities which can arise are

1. $\alpha = \pi/4$, $p+q = 4$ and $T$ is an isosceles right triangle,
2. $(\alpha, \beta, \gamma) = (\pi/10, 7\pi/10, 2\pi/10)$, $p = 3$ and $q = 1$,

3. $(\alpha, \beta, \gamma) = (3\pi/10, \pi/10, 6\pi/10)$, $p = 3$ and $q = 1$,

4. $\alpha = \pi/6$.

Case 1 leads to the second statement of our theorem. In cases 2 and 3 the only non-negative integers $p', q', r'$ such that $p'\alpha + q'\beta + r'\gamma = 2\pi$ and $p' + q' + r' \leq 5$ are $p' = 0$ , $q' = 2$, $r' = 3$. Hence (6) holds.
Assume that case 4 holds. If, for example, $p = 5$ and $q = 2$ then $\beta = \pi/12$. Considering the conjugate tiling belonging to $k = 5$ we have either

$$5\left\{\frac{5}{6}\right\} + 2\left\{\frac{1}{12}\right\} = 1 \text{ or } 5\left(1 - \left\{\frac{5}{6}\right\}\right) + 2\left(1 - \left\{\frac{1}{12}\right\}\right) = 1$$

(in the conjugate tiling $p\alpha' + q\beta' = A' + B' + C' = \pi$, where $A', B', C'$ denote the angles of $T'$). Since neither of these equalities hold, the case when $p = 5$ and $q = 2$ is impossible. Analogously, using the conjugate tilings belonging to either $k = 5$ or $k = 7$, we obtain that $(p, q)$ is equal to one of the following pairs: $(6, 0)$, $(5, 1)$, $(4, 1)$, $(4, 2)$, $(3, 1)$ or $(3, 3)$. In cases when $p + q < 6$, $\triangle$ is a right triangle, a contradiction. Hence, $p+q = 6$. In cases when $q > 0$, $\triangle$ has angles $\pi/6, \pi/6, 2\pi/3$. Thus, if there exists a vertex $y \in V_3$ with $\deg y \leq 5$ then necessarily $\deg y = 3$. Hence (6) remains true. Finally, if $p = 6$ and $q = 0$ then by Theorem 2, $\triangle$ has angles $\pi/6, \pi/6, 2\pi/3$ and formula (6) holds, or $\triangle$ is similar to $T$. Moreover, all angles of $T$ are multiples of $\pi/6$. Thus, if $\triangle$ is similar to $T$ then all angles of $\triangle$ are multiples of $\pi/6$. Since $\triangle$ is non-right, the angles of $\triangle$ are $\pi/6, \pi/6, 2\pi/3$ or $\pi/3, \pi/3, \pi/3$. Thus again for each vertex $y \in V_3$ with $\deg y \leq 5$ we have that $\deg y = 3$. □

**Corollary 2** *Let $\triangle$ be a non-right triangle. Then in every $\triangle$-dissection of a triangle $T$*

$$\sum_{x \in V_2} \deg x \leq 9. \tag{8}$$

Proof. By Propositions 2 and 3, $\sum_{x \in V_2} \deg x = 6$ or there exists a vertex $y \in V_3$ with $3 \leq \deg y \leq 5$. Thus, by Theorem 3 $\sum_{x \in V_2} \deg x \leq 9$. □

**Corollary 3** *Let $T$ be a non-right triangle. Then, in every $T$-dissection of $T$ at least two corners of $T$ have degree equal to* 2.

Proof. Let $A$, $B$ and $C$ be corners of $T$ and let $T = \bigcup_{i=1}^{n} T_i$. Suppose that $\deg A \geq 3$ and $\deg B \geq 3$. Thus, by Proposition 3, the dissection contains a vertex $y \in V_3$ with $\deg y \leq 5$. Hence, by Theorem 3, $\deg y = 12 - (\deg A + \deg B + \deg C)$. Hence $\deg y + \deg B \leq 7$. Let $\triangle A'BC'$ be a triangle similar to $T$ such that $A$ lies in the middle of the side $A'B$, $C$ lies in the middle of the side $C'B$. Let $D$ be the middle of the side $A'C'$. Let us draw two segments connecting $D$ with $A$ and $B$. Thus, we have obtained a $T$-dissection of a triangle $\triangle A'BC'$ into $n + 3$ triangles similar to it. Since $\deg A' + \deg B + \deg C' \geq 7$, by Theorem 3, $\deg y = 12 - (\deg A' + \deg B + \deg C')$. However, $\deg A' = \deg C' = 2$, hence, $\deg y + \deg B = 8$, a contradiction. □

# 5 Perfect dissections

In this section we present two applications of our method. A $\triangle$-dissection is *perfect* if all tiles are pairwise incongruent. Tutte [8] proved that there is no perfect dissection of an equilateral triangle into smaller equilateral triangles. On the other hand an equilateral triangle can be easily dissected in a perfect way into any greater than or equal to 3 number of right triangles. Thus, the question is: are there perfect dissections of an equilateral triangle into non-right triangles? If yes, what is the minimum number of tiles in such dissection? In [14] it is proved that there is no perfect dissection of an equilateral triangle into less than 7 similar non-right triangles. On the other hand an example of such dissection into 7 tiles is given.

**Theorem 4** *An equilateral triangle $T$ has a perfect $\triangle$-dissection into 7 tiles if and only if $\triangle$ is one (up to similarity) of the following two triangles*

1. *$\triangle 1$ have angles $\alpha_1, \beta_1, \gamma_1$ satisfying $\gamma_1 = 2\pi/3$, $\alpha_1 + \beta_1 = \pi/3$ and*

$$\sin^2 \beta_1 \left(\sin^2 \alpha_1 + 3/4\right) \left(\sin^3 \beta_1 - \sin^3 \alpha_1\right) = (3/4) \sin^5 \alpha_1, \tag{9}$$

2. *$\triangle 2$ have angles $\alpha_2, \beta_2, \gamma_2$ satisfying $\gamma_2 = 2\pi/3$, $\alpha_2 + \beta_2 = \pi/3$ and*

$$\left(\sin^2 \beta_2 + 3/4\right)^2 \sin^3 \alpha_2 = \sin^7 \beta_2. \tag{10}$$

*Moreover, there are exactly two non-equal perfect $\triangle$-dissections of $T$ into 7 tiles, see Fig. 5 b), c).*

Before we prove this theorem, we will prove some Lemmas.

**Lemma 2** *In every $\triangle$-dissection of an equilateral triangle $T$ into non-right triangles, the degrees of the corners of $T$ are 2, 2, 2 or 3, 3, 3.*

Proof. Let $A$, $B$ and $C$ be the corners of $T$. Suppose that degrees of the corners of $T$ are neither $2, 2, 2$ nor $3, 3, 3$. Then, by Corollary 2 one of the corners of $T$, say $A$, has the degree equal to 2, and another one, say $B$, has the degree greater than or equal to 3. Thus, one of the angles of $\triangle$, say $\alpha$, is equal to $\pi/3$, and another one, say $\beta$, is less than $\pi/3$. If $\deg B = 3$ or $\deg C = 3$ then $\beta = \pi/6$, a contradiction with the assumption that $\triangle$ is non-right. Hence, $\deg B = 4$ or 5 and $\deg C = 2$. Thus, the angles of $\triangle$ are $\pi/9, \pi/3, 5\pi/9$ or $\pi/12, \pi/3, 7\pi/12$, a contradiction with Theorem 2. □

**Lemma 3** *In every perfect $\triangle$-dissection of an equilateral triangle $T$ into non-right triangles $v_b \geq 2$ and at least two sides of $T$ are divided by vertices from the set $V_b$.*

Proof. Suppose that two sides of $T$ do not contain any vertex from $V_b$. Since $T$ is equilateral, these two sides have the same length. Moreover, each of them is the largest side of some tile. Thus, two tiles in the dissection are congruent, a contradiction. □

**Lemma 4 ([14])** *In every perfect $\triangle$-dissection of an equilateral triangle $T$ into non-right triangles $v_3 > 0$.*

For completeness we repeat the proof from [14].
Proof. Let $\alpha$, $\beta$, $\gamma$ be angles of $\triangle$. Suppose $v_3 = 0$. Then, due to Proposition 3, for all $x \in V_2$, $\deg x = 2$ and for all $x \in V_b \cup V_4$, $\deg x = 4$. Thus, one of the angles of $\triangle$, say $\beta$, is equal to $\pi/3$. If $\alpha = \pi/3$ or $\gamma = \pi/3$ then $\triangle$ is equilateral, hence, the dissection is not perfect (see [8]). Assume $\alpha \neq \pi/3$ and $\gamma \neq \pi/3$. Since each angle of $T$ is equal to $\pi/3 = \beta$, there exists a vertex $y \in V_b \cup V_4$ that compensates the total number of angles $\beta$ in the dissection. Thus, there exists a vertex $y \in V_b \cup V_4$ such that no angle at it equals $\beta$. Hence, $p\alpha + q\gamma = \pi$ with $p + q = 3$. We can assume that $p \geq q$. Thus, $2\alpha + \gamma = \pi$ or $3\alpha = \pi$. However, in both cases $\triangle$ is equilateral. Thus, the dissection is not perfect. □

Consider an arbitrary dissection of a polygon $P$ into triangles $\triangle_1, ..., \triangle_n$. If there exists a subset $i_1, ..., i_k$, $2 \leq k \leq n-1$, such that $\bigcup_{j=1}^{k} \triangle_{i_j}$ is a triangle, then we write that the dissection contains a *$k$-subdissection.* In [13] it was proved that there is no perfect $\triangle$-dissection of a triangle into less than 5 non-right triangles. Therefore,

**Lemma 5** *Consider a $\triangle$-dissection of a triangle $T$ into non-right triangles. If the dissection contains a 3-subdissection or a 4-subdissection then it is not perfect.* □

Proof of Theorem 4. Let $E_n$ be a set of those elements of $\Gamma(3;\overbrace{3,...,3}^{n})$ which satisfy Proposition 2, Theorem 3 and Lemmas 2-5. By applying our computer program we obtained that $E_n = \emptyset$ for $n \leq 6$, and $E_7$ is a set of 3 dissections presented in Fig. 5. Sets $E_n$ for $n \leq 7$ can be obtained independently from a list of dissections in [11], where elements of $E_7$ have numbers 115, 116 and 119. Let $\alpha, \beta, \gamma$, $\alpha \leq \gamma$ and $\beta \leq \gamma$, be angles of $\triangle$. A distribution of $\alpha, \beta, \gamma$ in the dissections in question must satisfy the following rules: i) every tile has exactly one angle $\alpha$, one angle $\beta$ and one angle $\gamma$, ii) all three angles around a vertex $x \in V_3$ with $\deg x = 3$ equal $\gamma$ (hence $\gamma = 2\pi/3$), iii) at every corner of $T$ there is exactly one angle $\alpha$ and one angle $\beta$ (otherwise $\alpha = \beta$ and we can replace one by the other), iv) no two tiles have a common '$\delta,\epsilon$-side', $\delta, \epsilon \in \{\alpha, \beta, \gamma\}$, v) at every vertex $x \in V_4 \cup V_b$ with $\deg x = 4$, there is one angle $\alpha$, one angle $\beta$ and one angle $\gamma$ (otherwise $\alpha = \pi/3$ or $\beta = \pi/3$ which is not possible because $\gamma = 2\pi/3$). One can check that, up to permutation of $\alpha, \beta, \gamma$, in each dissection in question there is only one distribution of angles, that satisfy the above rules, see Fig. 5. However, in Fig. 5 a) the marked lines are equal sides of an isosceles trapezoid. Thus, the two thick-line tiles are congruent, hence, this dissection is not perfect.

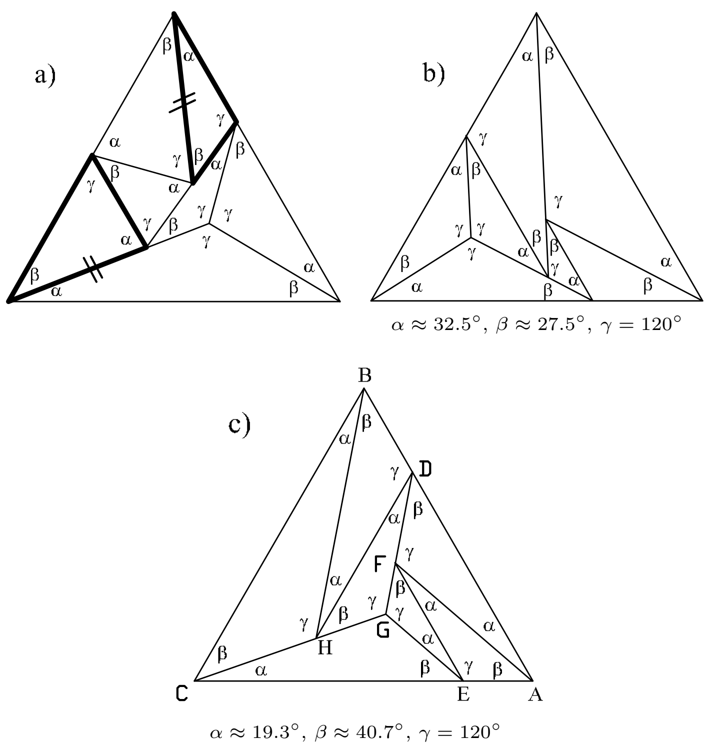


Fig. 5. Illustrating the proof of Theorem 4

Let us examine the dissection from Fig. 5 c). We can assume that $|EF| = 1$. Then, by the law of sines, $|EG| = \frac{\sin\beta}{\sin\gamma}$, $|GF| = \frac{\sin\alpha}{\sin\gamma}$ and $|AF| = \frac{\sin\gamma}{\sin\beta}$. Furthermore,

$$|FD| = \frac{\sin\alpha}{\sin\beta}|AF| = \frac{\sin\alpha\sin\gamma}{\sin^2\beta},$$

$$|DH| = \frac{\sin\gamma}{\sin\beta}\left(|GF| + |FD|\right) = \frac{\sin\alpha(\sin^2\beta + \sin^2\gamma)}{\sin^3\beta},$$

$$|GH| = \frac{\sin\alpha}{\sin\gamma}|DH| = \frac{\sin^2\alpha(\sin^2\beta + \sin^2\gamma)}{\sin\gamma\sin^3\beta},$$

$$|BH| = \frac{\sin\gamma}{\sin\beta}|DH| = \frac{\sin\alpha\sin\gamma(\sin^2\beta + \sin^2\gamma)}{\sin^4\beta},$$

$$|CG| = \frac{\sin\beta}{\sin\alpha}|EG| = \frac{\sin^2\beta}{\sin\alpha\sin\gamma}.$$

Now,

$$\frac{\sin\alpha}{\sin\beta}|BH| = |CH| = |CG| - |GH|$$

which leads to formula (10). On the other hand $\gamma = 2\pi/3$ hence $\alpha + \beta = \pi/3$. Hence, as one can check, formula (10) has only one solution: $\alpha \simeq 32.5°, \beta \simeq 27.5°, \gamma = 120°$. Thus, there is at most one, and in fact exactly one, perfect $\triangle$-dissection of an equilateral triangle which is topologically equivalent to the dissection from Fig. 5 c). Using similar ideas, one can prove that there is exactly one such dissection which is topologically equivalent to the dissection from Fig. 5 b). In the latter case the angles of $\triangle$ satisfy formula (9) which has exactly one solution: $\alpha \simeq 19.3°, \beta \simeq 40.7°, \gamma = 120°$. By Proposition 1 these are the only non-equal dissections with required properties. $\square$

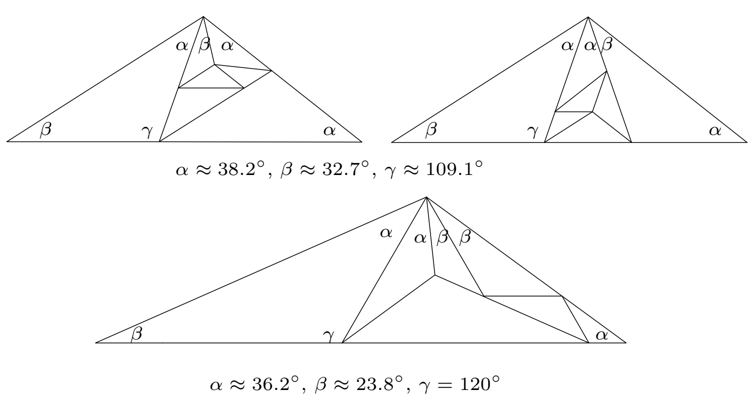


Fig. 6. Perfect T-dissections into 7 tiles

It is known [13] that the number of tiles in any perfect $T$-dissection of a non-right triangle $T$ is greater than or equal to 6. On the other hand, almost every triangle $T$ has a perfect $T$-dissection into $n$ parts, $n = 6, 8$ or $n \geq 10$, see [3].

**Theorem 5** *A non-right triangle $T$ has a perfect $T$-dissection into 7 tiles if and only if $T$ is one (up to similarity) of the following two triangles*

1. *$T1$ has angles $\alpha_1, \beta_1, \gamma_1$ satisfying*

$$\alpha_1 = 2\gamma_1 - \pi, \beta_1 = 2\pi - 3\gamma_1 = \pi \mathit{and} \sin^4\alpha_1 = \sin^3\beta_1\sin\gamma_1, \tag{11}$$

*or*

*2. T2 has angles $\alpha_2, \beta_2, \gamma_2$ satisfying*

$$\gamma_2 = 2\pi/3, \alpha_2 + \beta_2 = \pi/3 \mathit{and} \sin^5 \alpha_2 = \sin^3 \beta_2 (\sin^2 \alpha_2 + 3/4). \tag{12}$$

Proof. Let $\alpha, \beta, \gamma$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$ be angles of $T$. The suitable set $\Gamma$ of structures satisfying Proposition 2, Theorem 3, Corollary 3 and Lemma 5 has 8 elements pictured in Figs. 7 and 8. These elements can be found independently in [11], where they have numbers 106, 109, 111, 112, 165, 187, 192 and 220. Since there is no perfect $\triangle$-dissection of an equilateral triangle into five non-right triangles, we can exclude dissections from Fig. 7 a), b), c). Moreover, there are exactly two non-similar triangles that have a perfect $\triangle$-dissection into five non-right triangles, see [13]. However, only one of them can be dissected in a way which is topologically equivalent to the 5-subdissection in Fig. 7 d), e). Thus, only one non-right triangle $T$ has $T$-dissection which is topologically equivalent to dissection from Fig. 7 d) or e). The angles of this triangle satisfy condition (11), cf. [13, p.304], $\alpha \simeq 32.7°, \beta \simeq 38.2°, \gamma \simeq 109.1°$ . Note that these are not all non-equal dissections of this type. Indeed, we can obtain two new dissections by a reflection of the 5-subdissection about its largest height. Moreover, next different dissections arise by an analogous reflection of an isosceles trapezoid contained in the 5-subdissection.
Note that the rules i), ii), iv) and v) from the proof of Theorem 4 have to be satisfied in dissections from Figs. 8 a), b), too. Moreover, the rule iii) can be replaced now by the following rule iii') the angle of $T$ divided into four angles consists of two angles $\alpha$ and two angles $\beta$ (because $\gamma = 2\pi/3 = 2\alpha + 2\beta$). One can check that, up to permutation of $\alpha, \beta, \gamma$, for each dissection in question there is only one distribution of angles $\alpha, \beta, \gamma$ that satisfy the above rules, see Figs. 8 a), b). However, dissection a) is not perfect because the marked edges are equal sides of an isosceles trapezoid, hence thick-line triangles are congruent.
In Fig. 8 c) two angles at vertex $C$ are equal. Indeed, otherwise $\alpha + \beta = \gamma$, hence $\gamma = \pi/2$ and $T$ is a right triangle, a contradiction. Thus we can assume that $\angle C = 2\alpha$. Note that one of the remaining angles of $T$ is equal $\alpha$ because otherwise the sum of angles of $T$ equals $2\alpha+\beta+\gamma = \pi+\alpha$, a contradiction. Hence, because of symmetry, we can assume that $\angle A = \alpha$. In this case the rules i) and iv) have to be satisfied, too. Thus, around the inner vertex there is one angle $\beta$ and one angle $\gamma$. Moreover, angles $\alpha$ do not appear around the inner vertex, because otherwise the sum of two angles of $T$ equals $\pi$ which is not possible in a non-right triangle. Hence, up to permutation of $\alpha, \beta, \gamma$ their distribution in Fig. 8 c) is uniquely determined. However, the dissection is not perfect because the marked edges are equal sides of an isosceles trapezoid, hence thick-line triangles are congruent

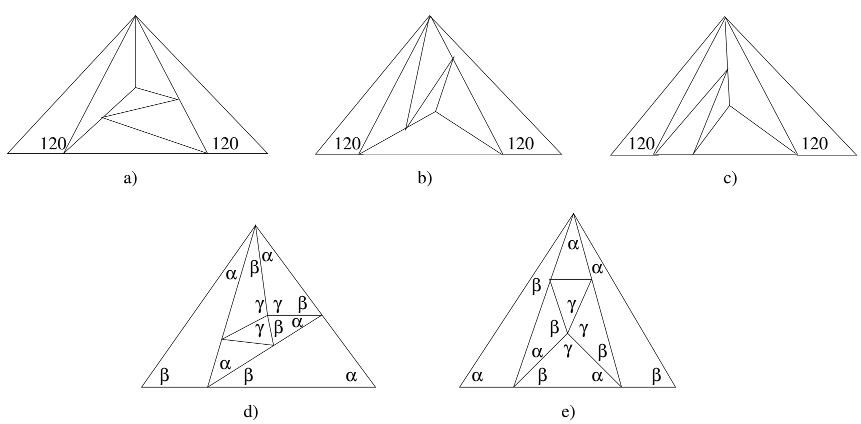


Fig. 7. Illustrating the proof of Theorem 5

Therefore, it remains to examine Fig. 8 b). Without loss of generality we can assume that $|FH| = 1$.

Then $|EF| = \frac{\sin\beta}{\sin\alpha}$, $|EH| = \frac{\sin\gamma}{\sin\alpha}$ and $|CH| = \frac{\sin\alpha}{\sin\beta}$. Consequently

$$
\begin{aligned}
|CG| &= \frac{\sin\alpha}{\sin\gamma}|CH| = \frac{\sin^2\alpha}{\sin\beta\sin\gamma}, \\
|GH| &= \frac{\sin\beta}{\sin\gamma}|CH| = \frac{\sin\alpha}{\sin\gamma}, \\
|CD| &= \frac{\sin\gamma}{\sin\beta}|CG| = \frac{\sin^2\alpha}{\sin^2\beta}, \\
|DG| &= \frac{\sin\alpha}{\sin\beta}|CG| = \frac{\sin^3\alpha}{\sin^2\beta\sin\gamma.}
\end{aligned}
$$

Therefore, in triangle $DEG$

$$\frac{|DG|}{\sin\beta} = \frac{|GH| + |EH|}{\sin\alpha}.$$

Hence $\alpha, \beta, \gamma$ satisfy condition (12). One can check that (12) has exactly one solution: $\alpha \simeq 23.8°, \beta \simeq 36.2°, \gamma = 120°$. $\square$

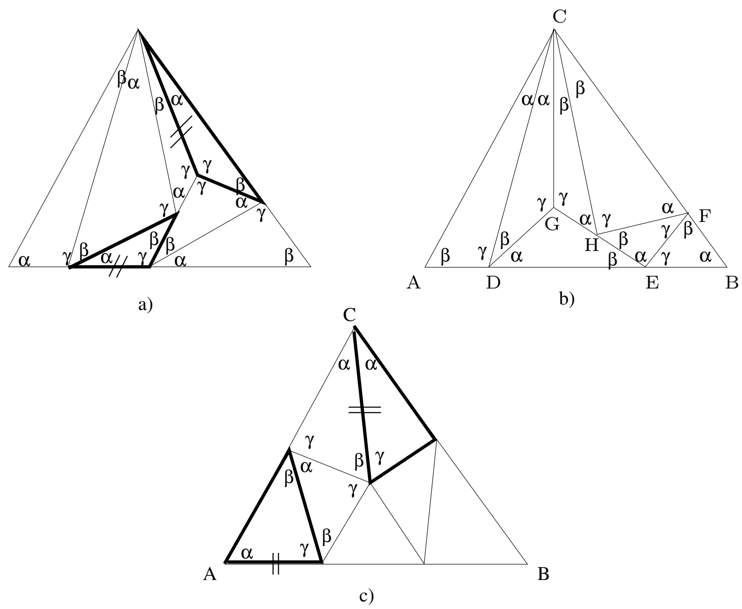


Fig. 8. Illustrating the proof of Theorem 5

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