Project: Integration in one dimension with Newton-Cotes quadratures

Tomasz Chwiej

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1 Introduction

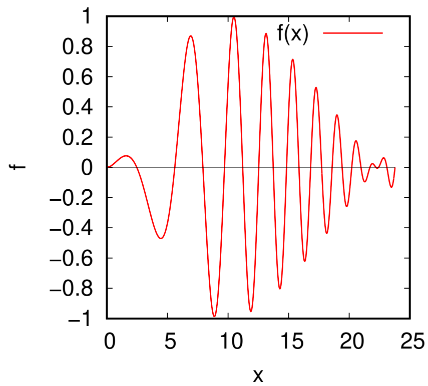

Figure 1: Function $f(x)$.

In project we will compute the one-dimensional integral

$$C = \int_a^b dx f(x) \tag{1}$$

and more explicitly

$$C = \int_0^{\sqrt{180\pi}} dx \sin\left(\frac{x^2}{10}\right) \cos\left(\sqrt{x}\right) = -0.1433994747827965 \tag{2}$$

Figure 1 shows this function in integration interval. At once we notice its strongly oscillating character and for this reason we may expect that any quadrature to be used in computation would require many nodes to densely sampling the interval and hence to provide numerically accurate result.
We use three Newton-Cotes quadratures with the same sequence of $\mathbf{n}$ equidistant integration nodes $\boldsymbol{x_i}$

$$n = 4 \cdot m + 1, \qquad (m - \text{integer}) \tag{3}$$

$$h = \frac{b-a}{n-1} \tag{4}$$

$$x_i = a + h \cdot i, \quad i = 0, 1, 2, \ldots, n-1 \tag{5}$$

$$f(x_i) \equiv f_i \tag{6}$$

These quadratures are

1. **trapezoidal**

$$C = \sum_{i=0}^{(n-1)-1} \frac{h}{2}(f_i + f_{i+1}) \tag{7}$$

2. **Simpson**

$$C = \sum_{i=0}^{\left(\frac{n-1}{2}\right)-1} \frac{h}{3}(f_{2i} + 4f_{2i+1} + f_{2i+2}) \tag{8}$$

3. **Milne**

$$C = \sum_{i=0}^{\left(\frac{n-1}{4}\right)-1} \frac{4h}{90}(7f_{4i} + 32f_{4i+1} + 12f_{4i+2} + 32f_{4i+3} + 7f_{4i+4}) \tag{9}$$

The computer implementation of these methods is straightforward.

Additionally we check the possibility of adaptive integration with Romberg method, it is defined by the following expressions

$$h_p = \frac{b-a}{2^p} \tag{10}$$

$$R_{0,0} = \frac{1}{2}(b-a)\left[f(a) + f(b)\right] \tag{11}$$

$$R_{p,0} = \frac{1}{2}R_{p-1,0} + h_p \sum_{i=1}^{2^{(p-1)}} f\left(a + (2i-1)h_p\right) \tag{12}$$

$$R_{p,m} = \frac{4^m R_{p,m-1} - R_{p-1,m-1}}{4^m - 1} \tag{13}$$

enabling us to form the array

$$\begin{array}{lllll} R_{0,0} & & & & \\ R_{1,0} & R_{1,1} & & & \\ R_{2,0} & R_{2,1} & R_{2,2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ R_{p,0} & R_{p,1} & R_{p,2} & \dots & R_{p,p} \end{array} \tag{14}$$

In array Eq.14 the first column collects the integral's values calculated for doubled number of nodes with respect to previous entry (Eq.12) while the diagonal elements contains values improved accordingly with Eq.13. Implementation of this algorithm requires a bit more effort, the code below would help solving this issue

```
double a, b, c
int i,p,m,p_max
double R[p_max][p_max]

R_{0,0} ← (b−a)/2 (f(a) + f(b))

for p=1 to p_max − 1 by 1 do
        h ← (b−a)/2^p
```

$c \leftarrow 0$

`for i=1 to` $2^{(p-1)}$ `by 1 do`

$c \leftarrow c + h \cdot f(a + h \cdot (2i-1))$

`end do`

$R_{p,0} \leftarrow \frac{1}{2} R_{p-1,0} + c$

`for m=1 to p by 1 do`

$R_{p,m} \leftarrow \frac{4^m R_{p,m-1} - R_{p-1,m-1}}{4^m - 1}$

`end do`

`end do`

2 Practical part

1. Write a computer program which calculates the value of the integral given Eq.2 with trapezoidal, Simpson and Milne methods. In calculations assume following parameters: $a = 0$, $b = \sqrt{180\pi}$ and an exact value $C_{exact} = -0.1433994747827965$. For $m = 10, 11, \ldots, 300$ and $n = 4m + 1$ calculate the integral and its error $|C - C_{exact}|$ for each method. Write data to file and prepare figure showing the dependence of integration error on the number of nodes. Use logharitmic scale on both axes.

2. Implement the Romberg adaptive integration scheme and perform calculations for $p_{max} = 16$. Calculate the error for first column $|R_{p,0} - C_{exact}|$ and for the diagonal $|R_{p,p} - C_{exact}|$. Write the data to a file and prepare the figure showing both errors in dependence on number of nodes. Use logharitmic scale on both axes.

3. At home prepare a report. Comment on the observed pace of convergence by comparing the results provided by four methods.

3 Example results

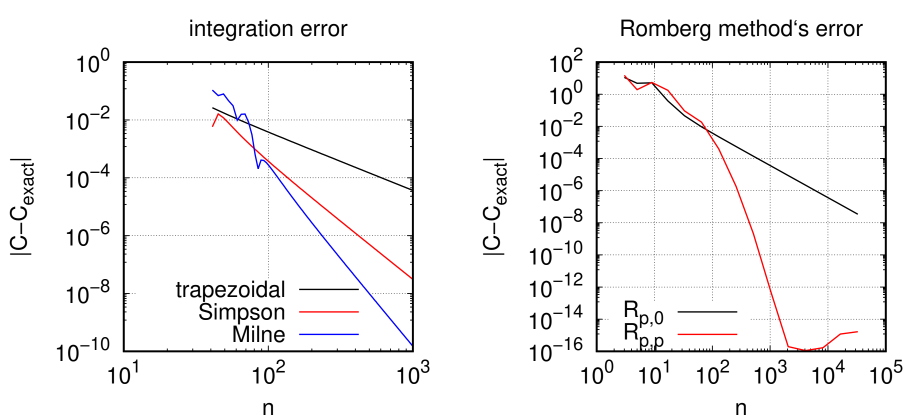

Figure 2: Integration error for trapezoidal, Simpson and Milne methods (left) and for the first column ($R_{p,0}$) and the diagonal ($R_{p,p}$) elements in the Romberg method (right).