Decomposition of complete bipartite graphs into open trails

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Abstract

It has been showed in [4] that any bipartite graph $K_{a,b}$, where a, b are even is decomposable into closed trails of prescribed even lengths. In this article we consider the corresponding question for open trails. We prove a necessary and sufficient condition for graphs $K_{a,b}$ to be decomposable into edge-disjoint open trails of positive lengths (less than ab) whenever these lengths sum up to the size of the graph $K_{a,b}$. Let $K'_{a,a} := K_{a,a} - I_a$ for any 1-factor I_a. We also prove that $K'_{a,a}$ for odd a can be decomposed in a similar manner.

1 Introduction

Consider a simple graph G whose size we denote by $e(G)$. Write $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph G.

We say that a graph G is *Eulerian* if there exists a closed trail through every edge of G. Here and subsequently, a trail T of length n we identify with a sequence $(v_1, v_2, \ldots, v_{n+1})$ of vertices of T such that $v_i v_{i+1}$ are distinct edges of T for $i = 1, 2, \ldots, n$. Notice that we do not require the v_i to be distinct. A trail T is closed if $v_1 = v_{n+1}$ and T is open if $v_1 \neq v_{n+1}$.
However, closed trail will be regarded as an Eulerian graph of size n. A graph G is said to be *even* if the degrees of all its vertices are even. By

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Euler's theorem, a connected even graph is Eulerian (i.e. contains a closed trail passing through all its edges exactly once).

A sequence of positive integers $\tau = (t_1, t_2, \ldots, t_p)$ is called *admissible for a graph* G if it adds up to $e(G)$ and for each $i \in \{1, ..., p\}$ there exists an open trail of length t_i in G. Let $\tau = (t_1, t_2, \ldots, t_p)$ be an admissible sequence for G. If G is edge-disjointly decomposable into open trails $T_1, T_2, \ldots, T_p$ of lengths $t_1, t_2, \ldots, t_p$ respectively, then τ is called *realizable in* G and the sequence $(T_1, T_2, \ldots, T_p)$ is said to be *a* G*-realization of* τ or *a realization of* τ *in* G.

Let $K_{a,b}$ be the complete bipartite graph with two sets of vertices A and B such that $|A| = a$ and $|B| = b$. In our paper we prove a necessary and sufficient condition for graphs $K_{a,b}$ to be decomposable into edge-disjoint open trails of positive lengths $t_1, t_2, ...t_p$ for any admissible sequence $\tau = (t_1, t_2, ...t_p)$.

Such problems were first investigated by P.N. Balister.

Theorem 1 ([1]) *Let* $L = \sum_{i=1}^{p} t_i$, $t_i \geqslant 3$, *with* $L = \binom{n}{2}$ *when* n *is odd and* $\binom{n}{2} - \frac{n}{2} - 2 \leqslant L \leqslant \binom{n}{2} - \frac{n}{2}$ *when* n *is even. Then we can write some subgraph of* K_n *as an edge union of circuits of lengths* $t_1, \ldots, t_p$.

Theorem 2 ([2]) *The following conditions are both necessary and sufficient for packing* $\bigcup_{i=1}^{p} P_{l_i}$ *into* K_n *with endpoints mapped to distinct vertices:*
$L = \binom{n}{2}$ *or* $L \leqslant \binom{n}{2} - 3$ *if* $r = 0$,
$L \leqslant \binom{n}{2} - \frac{n}{2}$ *if* $r > 0$ *and* r *(or* n*) is even,*
$L \leqslant \binom{n}{2} - p$ *if* r *(or* n*) is odd:*
where $n = 2p + r$ *and* $L = \sum_{i=1}^{p} l_i$. *In particular,* $L \leqslant \binom{n-1}{2}$ *is always sufficient.*

A motivation and applications of Theorems 1 and 2 can be found in problems concerning vertex-distinguishing proper edge-coloring of graphs.

A similar theorem for the closed trails has been proved in [4] by M. Horňák and M. Woźniak.

Theorem 3 ([4]) *If* a, b *are positive even integers, then if* $\sum_{i=1}^{p} t_i = ab$ *and there is a closed trail of length* t_i *in* $K_{a,b}$ *(for all* $i \in \{1, \ldots, p\}$*), then* $K_{a,b}$ *can be (edge-disjointly) decomposed into closed trails* $T_1, T_2, \ldots, T_p$ *of lengths* $t_1, t_2, \ldots, t_p$ *respectively.*

This problem is also solved by S. Cichacz for directed bipartite graphs and bipartite multigraphs, see [3].

Let $K_{a,a}$ be a complete bipartite graph and let I_a denote a 1-factor in $K_{a,a}$. We denote by $K'_{a,a}$ a graph $K_{a,a} - I_a$.

2 Decomposition of bipartite graphs into open trails

There is no loss of generality in assuming that $a \leq b$.

Let us observe that in any complete bipartite graph $K_{a,b}$ different from $K_{1,1}$ and $K_{2,b}$ for odd b does not exist an open trail of length ab. Hence, $p \geq 2$ for each admissible sequence $\tau = (t_1, ..., t_p)$ for each graph $K_{a,b}$ different from $K_{1,1}$ and $K_{2,b}$ for any odd b.

Theorem 4 *For each complete bipartite graph $K_{a,b}$ and for each admissible sequence $\tau = (t_1, ..., t_p)$ for $K_{a,b}$ there exists a realization of τ in $K_{a,b}$ if and only if one of the following conditions holds:*

1^0 $a = 1$ or

2^0 a and b are both even.

Let $A := \{x_1, ..., x_a\}$ and $B := \{v_1, ..., v_b\}$.

Necessity. We show that if $a > 1$ and a or b is odd then there exists an admissible sequence τ for $K_{a,b}$ such that there is no realization for τ in $K_{a,b}$. We divide this proof into several parts:

A. Let us assume that $a = 2$ and b is odd. It can be easily seen that there exists an open trail of length two in $K_{2,b}$ and because of Euler's theorem there exists an open trail of length $(2b-2)$ in $K_{2,b}$. Hence $\tau := (2, 2b-2)$ is an admissible sequence for $K_{2,b}$ but τ is not realizable in $K_{2,b}$.

B. Let $a \geq 3$ and $b \geq 3$. Assume first that a is odd while b is even. Thus, $d(x_i)$ is even for any $i \in \{1, ..., a\}$ and $d(v_j)$ is odd for any $j \in \{1, ..., b\}$. Let G_1 be a subgraph of $K_{a,b}$ induced by the set of vertices $\{x_1, v_1, v_2, \ldots, v_{b-1}\}$ (see fig. 1). Let $G' := K_{a,b} - E(G_1)$. Observe that the only two vertices in G' of odd degree are x_1 and v_b. Thus, in $K_{a,b}$ there exists an open trail of length $(ab - b + 1)$. Moreover, there exists an open trail of length $(b-1)$, but a sequence $\tau := (b-1, ab-b+1)$ is not realizable in $K_{a,b}$ (because if T_1 denotes an open trail of length $(b-1)$ in $K_{a,b}$, then in $K_{a,b} - E(T_1)$ there are at least

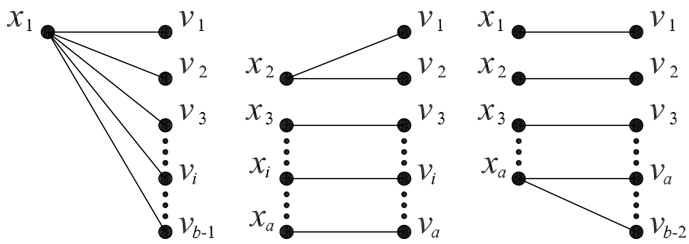

Figure 1: Subgraphs G_1,G_2 and G_3.

four vertices of odd degree). Analogously we show that such sequence τ is not realizable in $K_{a,b}$ for a even and b odd.

C. Let $a \geq 3$ and $b \geq 3$ be both odd. Let us consider two subcases:

a) $a = b$. Let G_2 be a subgraph of $K_{a,a}$ with the vertex set $V(G_2) = \{x_2, \ldots, x_a, v_1, \ldots, v_a\}$ and the edge set $E(G_2) = \{x_2v_1, x_2v_2, x_3v_3, \ldots, x_iv_i, \ldots, x_av_a\}$ (see fig. 1). In $K_{a,a} - E(G_2)$ there exist only two vertices of odd degree, namely x_1 and x_2. Hence in $K_{a,a}$ is an open trail of length $(a^2 - a)$. There is also an open trail of length a in $K_{a,a}$. But the sequence $\tau := (a, a^2 - a)$ is not realizable in $K_{a,a}$.

b) $a < b$. Let G_3 be a subgraph of $K_{a,b}$ with $V(G_3) = \{x_1, \ldots, x_a, v_1, \ldots, v_{b-2}\}$ and with $E(G_3) = \{x_1v_1, x_2v_2, x_3v_3, \ldots, x_av_a, x_av_{a+1}, \ldots, x_av_{b-3}, x_av_{b-2}\}$. Observe that $d_{G_3}(x_1) = \ldots = d_{G_3}(x_{a-1}) = d_{G_3}(v_1) = \ldots = d_{G_3}(v_{b-2}) = 1$ and $d_{G_3}(x_a) = b - a - 1$ (see fig. 1). Hence, in $K_{a,b} - E(G_3)$ the only two vertices of odd degree are v_{b-1} and v_b. Notice that we allow $a = b - 2$. This implies that there exists an open trail of length $(ab - b + 2)$ in $K_{a,b}$. Obviously, in $K_{a,b}$ exists an open trail of length $(b - 2)$. However, an edge-disjoint decomposition of $K_{a,b}$ into open trails of lengths $(b - 2)$ and $(ab - b + 2)$ does not exist.

Sufficiency. Assume first that $a = 1$. It can be easily seen that $K_{1,b}$ is arbitrarily decomposable into open trails of length one and two.

From now on, let us assume that G is any complete bipartite graph $K_{a,b}$ such that a and b are even. Let $\tau = (t_1, ..., t_p)$ be a sequence of positive integers such that $\sum_{i=1}^{p} t_i = ab$ and $p \geq 2$. We show that there exists a τ-realization in $K_{a,b}$. We consider the following cases:

A. Let us suppose that t_i is even for any $i \in \{1, ..., p\}$.

Case I. Assume now that t_i is not an even multiplicity of b for any $i \in$

$\{1, ..., p\}$. Consider a sequence

$$V := (v_1, x_1, v_2, x_2, v_3, x_1, ..., x_1, v_b, x_2,$$

$$v_1, x_3, v_2, x_4, v_3, x_3, ..., x_3 v_b, x_4, ..., v_1, x_{a-1}, ..., x_{a-1}, v_b, x_a, v_1).$$

Clearly, this sequence of vertices creates an Eulerian trail in $K_{a,b}$. We show

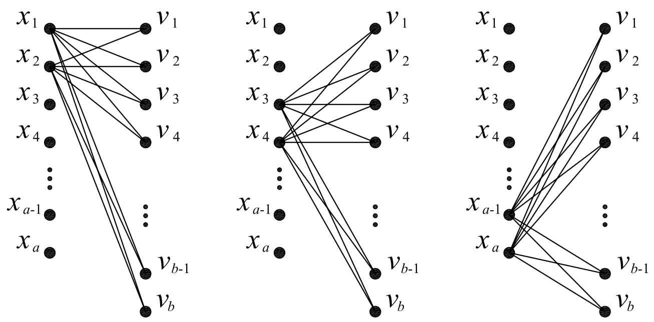

Figure 2: Sequence V.

that we can part V into subsequences V_1,...,V_p such that for any $i \in \{1, ..., p\}$ the set of vertices in V_i describes an open trail T_i in $K_{a,b}$ of length t_i and T_1,...,T_p are edge-disjoint subgraphs of $K_{a,b}$ (see fig 2).

Let us start at the following observation: let $W = (w_1, ..., w_k) \subset V$ be a subsequence of consecutive elements of V such that $w_1 = w_k = v_i$ for some $i \in \{1, ..., b\}$. The set of vertices in W creates a closed trail in $K_{a,b}$ of length $m \cdot b$ for some even $m \leq a$.

We will define subsequences V_1,...,V_p of V. Let V_1 contain $(t_1 + 1)$ first elements of V so it starts at v_1 and its next elements are the consecutive elements of V up to $(t_1 + 1)$-th element. Let us denote this element by v^2. Observe that it belongs to B (obviously, it is different than v_1). Let V_2 start at v^2 and let it contain next $(t_2 + 1)$ elements of the sequence V. We denote the last element of V_2 by v^3 so $V_2 = (v^2, ..., v^3)$. In a similar way we can define the rest of subsequences V_3,...,V_p. The last element of sequence V_i we will denote by v^{i+1}. It is easy to see that $v^i \in B$ for any $i \in \{2, ..., p\}$. Thus, a sequence V_i contains consecutive elements of V, starts at some vertex in B and finishes at another for each $i \in \{1, ..., p\}$. Hence, because of the above observation for any $i \in \{1, ..., p\}$ the set of vertices of V_i describes an open trail T_i of length t_i in G. Moreover, T_1,...,T_p are edge-disjoint subgraphs of G.

Case II. Let $t_1 = m_1 \cdot b$,...,$t_l = m_l \cdot b$ for some $l \in \{1, ..., p\}$ and for some even integers $m_1, ..., m_l$. Suppose first that $l \geq 2$ and let $m := m_1 + ... + m_l$. Then, consider sequences:

$$V' := (x_m, v_1, x_1, v_2, x_2, v_3, x_1, ..., x_1, v_b, x_2,$$

$$v_1, x_3, v_2, x_4, v_3, x_3, ..., x_3 v_b, x_4, ..., v_1, x_{m-1}, ..., x_{m-1}, v_b, x_m)$$

and

$$V'' := (v_1, x_{m+1}, v_2, x_{m+2}, v_3, x_{m+1}, ..., x_{m+1}, v_b, x_{m+2},$$

$$..., v_1, x_{a-1}, ..., x_{a-1}, v_b, x_a, v_1).$$

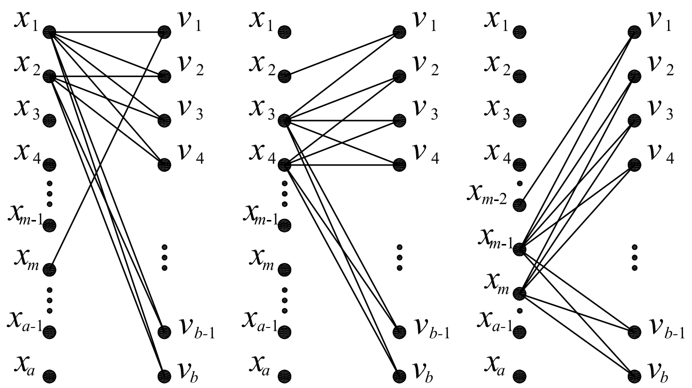

Figure 3: Sequence V'.

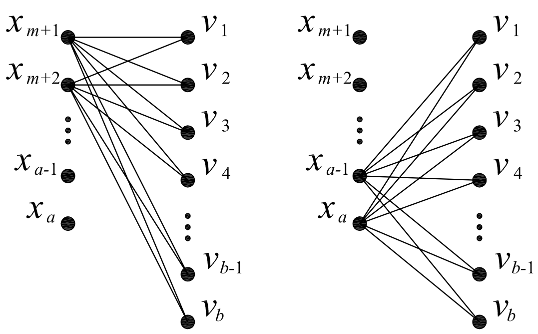

Figure 4: Sequence V''.

These sequences of vertices create edge-disjoint closed trails in G. Let G' be a subgraph of G induced by the set of edges in V'. Hence V' is an Eulerian trail for G' which is a complete bipartite subgraph of G. Let us part V' into disjoint subsequences $V_1 := (x_m, ..., x_{m_1})$, $V_i := (x_{m_1+...+m_{i-1}}, ..., x_{m_1+...+m_i})$ for $i \in \{2, ..., l-1\}$ and $V_l := \{x_{m_1+...+m_{l-1}}, ..., x_m)$. The sets of vertices of V_i create edge-disjoint open trails T_1,...,T_l of lengths t_1,...,t_l (see fig. 3).

Let G'' be a graph described by the sequence of vertices of V''. Observe that G'' is also a complete bipartite subgraph of G with two disjoint sets of vertices $C := A \setminus \{x_1, ..., x_m\}$ and B. The edges of V'' induce an Eulerian trail for G'' so we can define the edge-disjoint open trails T_{l+1},...,T_p of lengths t_{l+1},...,t_p in G'' analogously as in case I (see fig 4). Obviously, T_1,..,T_p are edge-disjoint open trails in G.

Let us assume now that $l = 1$. Hence, $t_1 = m \cdot b$ for some even integer m and t_i is not an even multiplicity of b for any $i \in \{2, ..., p\}$. Let us consider sequences V''' (see fig. 5) and V^{IV} (see fig. 6) such that:

$$V''' := (v_1, x_1, v_2, x_2, v_3, x_1, v_4, x_2, v_5..., x_{m-1}, v_b, x_a, v_{b-1})$$

and

$$\begin{gathered} V^{IV} := (v_{b-1}, x_{a-1}, v_{b-2}, x_a, v_{b-3}, x_{a-1}, v_{b-4}, x_a, ..., v_1, x_{a-1}, v_b \\ x_{a-2}, v_{b-1}, x_{a-3}, v_{b-2}, \ldots, x_{a-2}, v_1, x_{a-3}, v_b, \ldots \\ x_{m+2}, v_{b-1}, x_{m+1}, v_{b-2}, \ldots, x_{m+2}, v_1, x_{m+1}, v_b, x_m, v_1). \end{gathered}$$

Observe that the sequence of vertices of V''' creates an open trail T_1 of length t_1. Moreover, with the single exception of $\{v_1, x_{m+1}, v_b, x_m\}$, every other subsequence which contains consecutive vertices of V^{IV} and start and finish at the same vertex v_i for any $i \in \{1, ..., b\}$ induces a closed trail of length $k \cdot b$ for some even integer k. The only exception is the set of last four vertices $\{v_1, x_{m+1}, v_b, x_m\}$ which induces a closed trail of length four in G. Suppose now that there exists $j \in \{2, ..., p\}$ such that $t_j \neq 4$. Without loss of generality we can assume that $t_p \neq 4$. For such admissible sequence τ, applying analogous methods as in case I, we can define the open trails T_2,...,T_p of length t_2,...,t_p in G such that T_1,...,T_p are edge-disjoint subgraphs of G. Assume now that $t_i = 4$ for any $i \in \{2, ..., p\}$ and $b > 4$. The edges of a sequence

$$V^V := (v_2, x_2, v_3, x_1, v_4, x_2, v_5, ..., x_{m-1}, v_b, x_a, v_{b-1}, x_{a-1}, v_{b-2})$$

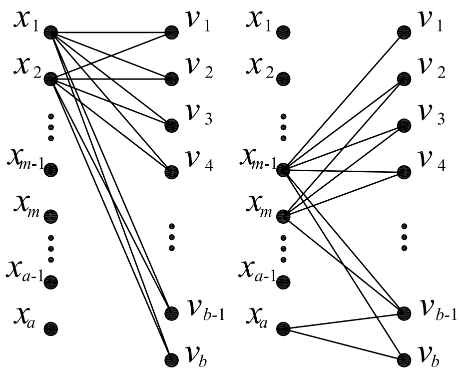

Figure 5: Sequence V^{III}.

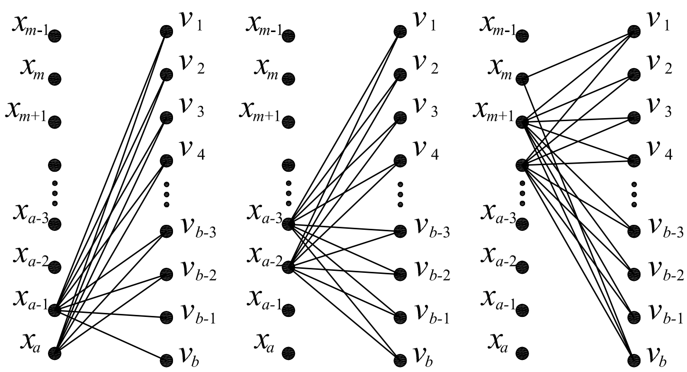

Figure 6: Sequence V^{IV}.

induce an open trail of length t_1 in G. Consider a sequence

$$V^{VI} := (v_{b-2}, x_a, v_{b-3}, x_{a-1}, v_{b-4}, x_a, \dots$$

$$v_1, x_{a-1}, v_b, x_{a-2}, v_{b-1}, x_{a-3}, v_{b-2}, \dots, x_{m+1}, v_b, x_m, v_1, x_1, v_2).$$

(see fig. 7 and 8) Let us part V^{VI} into $(p-1)$ sets, each of them containing five consecutive elements of it. Then these sets induce edge-disjoint open trails of length four in G. A decomposition of $G = K_{4,4}$ into edge-disjoint open trails for $\tau = (8, 4, 4)$ we show in the figure 9.

B. Suppose now that some of elements of τ are odd. Without loss of generality we can assume that t_1,...,t_l are odd for some $l \leq p$ and t_{l+1},...,t_p are even. Observe that l is even so there exists a positive number k such that

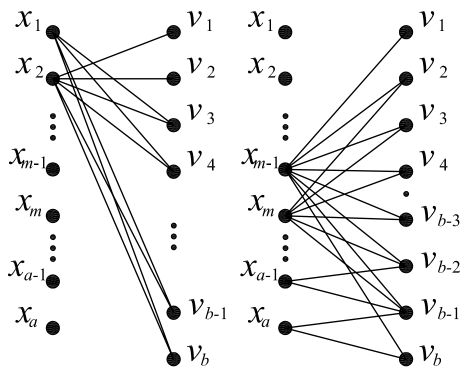

Figure 7: Sequence V^V.

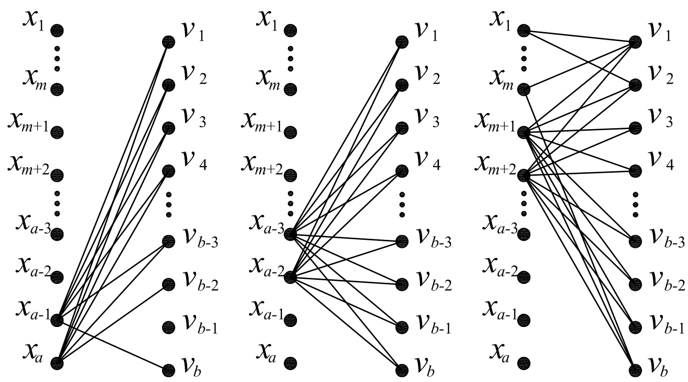

Figure 8: Sequence V^{VI}.

$l = 2k$. Let us define $d_i := t_{2i-1} + t_{2i}$ for $i \in \{1, ..., k\}$. Consider a sequence $\tau' := (d_1, ..., d_k, t_{2k+1}, ..., t_p\}$. Applying the same arguments as above G is decomposable into open trails D_1,...,D_k, T_{2k+1},... T_p of lengths d_1,...,d_k, t_{2k+1},...,t_p. It is easy to observe that each open trail D_j we can part into two edge-disjoint open trails T_{2j-1}, T_{2j} of lenghts t_{2j-1}, t_{2j}. Hence, $T_1, ..., T_p$ is a G-realization of τ and the proof is finished. □

Let us consider now complete bipartite graphs $K_{a,a}$ for an odd a. By the previous theorem such graphs are not arbitrarily decomposable into open trails but we can prove the following theorem:

Theorem 5 *For any odd a the graph $K'_{a,a}$ is decomposable into open trails*

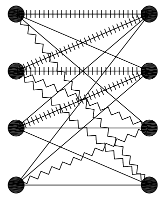

Figure 9: A decomposition of $K_{4,4}$ into edge-disjoint open trails for $\tau = (8, 4, 4)$.

of lengths t_1,...,t_p *for each admissible sequence* $\tau = (t_1, ..., t_p)$.

Let G be a bipartite graph $K'_{a,a}$ with odd a. Observe that $p > 1$, because there does not exist an open trail of length $(a^2 - a)$ in $K'_{a,a}$. Let $A := \{x_1, ..., x_a\}$ and $B := \{v_1, ..., v_a\}$. Let I_a be the matching such that $x_i v_j \in I_a$ if and only if $j = i$. Let $\tau = (t_1, ..., t_p)$ be a sequence of positive integers such that $\sum_{i=1}^{p} t_i = a^2 - a$ and $p \geq 2$. The proof of this theorem is analogous to the proof of Theorem 4.
Let us suppose first that t_i is even for any $i \in \{1, ..., p\}$. We consider two cases:
Case I. Assume now that t_i is not an even multiplicity of a for any $i \in \{1, ..., p\}$. Let us consider a sequence (see fig. 10)

$$U := (v_1, x_a, v_2, x_1, v_3, x_2, \ldots, v_1, v_a, x_2, v_1, x_3, v_2, x_4, v_3, x_a, v_4, ..., x_3, v_a, x_4, ...,$$

$$v_1, x_i, v_2, x_{i+1}, \ldots, x_{i+1}, v_i, x_a, v_{i+1}, x_i, v_{i+2}, x_{i+1}, \ldots, x_i, v_a, x_{i+1}, \ldots,$$

$$v_1, x_{a-2}, v_2, x_{a-1}, \ldots, x_{a-1}, v_{a-2}, x_a, v_{a-1}, x_{a-2}, v_a, x_{a-1}).$$

This sequence of vertices creates an Eulerian trail in $K_{a,a} - I_a$. Let $W = (w_1, ..., w_k) \subset V$ be a subsequence of consecutive elements of U such that $w_1 = w_k = v_i$ for some $i \in \{1, ..., a\}$. The sequence of vertices in W describes a closed trail in $K_{a,a} - I_a$ of length $m \cdot a$ for some even $m \leq a$.

We will define subsequences V_1,...,V_p of U analogously like in the proof of Theorem 4. So let V_1 contain $(t_1 + 1)$ first elements of U. Hence it starts at v_1 and its next elements are the consecutive elements of U up to $(t_1 + 1)$-th element. Let us denote this element of B by v^2. Let V_2 start at v^2 and let it contain next $(t_2 + 1)$ elements of sequence U and so on. For each $i \in \{1, ..., p\}$ the sequence of vertices of V_i creates an open trail T_i of length t_i in G and T_1,...,T_p are edge-disjoint subgraphs of G.

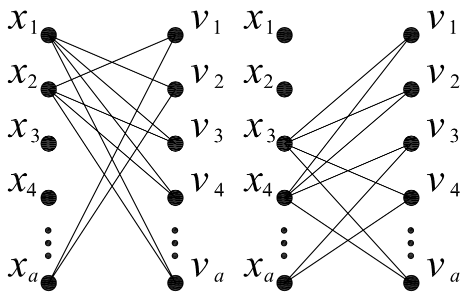

Figure 10: Sequence U.

Case II. Let $t_1 = m_1 \cdot a$,...,$t_l = m_l \cdot a$ for some $l \in \{1, ..., p\}$ and for some even integers $m_1, ..., m_l$. Suppose first that $l \geq 2$ and let $m := m_1 + ... + m_l$. Then, consider sequences

$$U' := (x_m, v_1, x_a, v_2, x_2, v_3, x_1, ..., x_1, v_a, x_2, v_1, x_3, v_2, x_4, v_3, x_a, v_4, x_3, \dots,$$

$$x_3, v_a, x_4, ..., v_1, x_i, v_2, x_{i+1}, \dots, x_{i+1}, v_i, x_a, v_{i+1}, x_i, v_{i+2}, x_{i+1}, \dots, x_i, v_a, x_{i+1}, \dots,$$

$$v_1, x_{m-1}, v_2, x_m, \dots, x_m, v_{m-1}, x_a, v_m, x_{m-1}, v_{m+1}, x_m, \dots, x_{m-1}, v_a, x_m)$$

and

$$U'' := (v_1, x_{m+1}, v_2, x_{m+2}, v_3, x_{m+1}, \dots, x_{m+2}, v_{m+1}, x_a, v_{m+2}, x_{m+1}, v_{m+3}, x_{m+2}, \dots,$$

$$x_{m+1}, v_a, x_{m+2}, \dots, v_1, x_i, v_2, x_{i+1}, \dots,$$

$$x_{i+1}, v_i, x_a, v_{i+1}, x_i, v_{i+2}, x_{i+1}, \dots, x_i, v_a, x_{i+1}, \dots, v_1, x_{a-2}, v_2, x_{a-1}, \dots,$$

$$x_{a-1}, v_{a-2}, x_a, v_{a-1}, x_{a-2}, v_a, x_{a-1}, ..., v_1, x_{a-1}, ..., x_{a-1}, v_b, x_a, v_1).$$

These sequences of vertices create edge-disjoint closed trails in G. Let G' be a subgraph of G created by the sequence of vertices of U'. Hence U' is an Eulerian trail for G'. Let us part U' into disjoint subsequences $V_1 := (x_m, ..., x_{m_1})$, $V_i := (x_{m_1+...+m_{i-1}}, ..., x_{m_1+...+m_i})$ for $i \in \{2, ..., l-1\}$ and $V_l := \{x_{m_1+...+m_{l-1}}, ..., x_m)$. The squences V_i create the sets of vertices of edge-disjoint open trails T_1,...,T_l of lengths t_1,...,t_l (see fig. 11).

Now, let G'' be a graph described by the sequence of vertices of U''. Observe that G'' is a bipartite subgraph of G with two disjoint sets of vertices $C := A \setminus \{x_1, ..., x_m\}$ and B (see fig. 12). The edges of U'' induce an Eulerian trail for G'' so we can define the edge-disjoint open trails T_{l+1},...,T_p of lengths

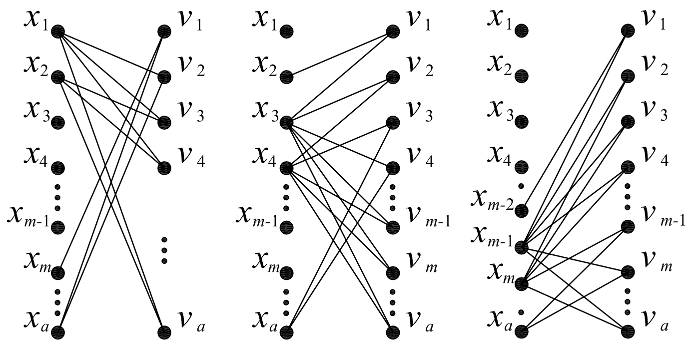

Figure 11: Sequence U'.

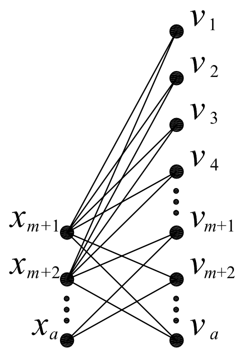

Figure 12: Sequence U''.

t_{l+1},...,t_p in G''.

Let us assume now that $l = 1$. Hence, $t_1 = m \cdot a$ for some even integer m and t_i is not an even multiplicity of a for any $i \in \{2, ..., p\}$. Let us consider sequences

$$U''' := (v_1, x_a, v_2, x_2, v_3, x_1, v_4, x_2, v_5, \ldots, x_{m-1}, v_a, x_{a-2}, v_{a-1})$$

and

$$U^{IV} := (v_{a-1}, x_a, v_{a-2}, x_{a-1}, v_{b-3}, \ldots, v_1, x_{a-1}, v_a, \ldots$$
$$x_{a-4}, v_{a-1}, x_{a-3}, v_{a-2}, x_{a-4}, v_{a-3}, x_a, v_{a-4}, x_{a-3}, v_{a-5}, \ldots,$$
$$x_{m+1}, v_{a-1}, x_{m+2}, v_{a-2}, x_{m+1}, \ldots, v_1, x_{m+2}, v_a, x_m, v_1).$$

Observe that the edges of U''' induce an open trail T_1 of length t_1 (see

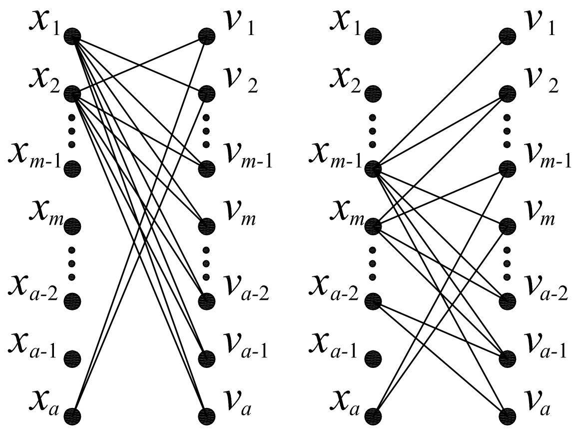

Figure 13: Sequence U'''.

fig. 13). Moreover, with the single exception of $\{v_1, x_{m+2}, v_a, x_m\}$, every other subsequence which contains consecutive vertices of U^{IV} and start and finish at the same vertex v_i for any $i \in \{1, ..., b\}$ induces a closed trail of length $k \cdot a$ for some even integer k (see fig. 14). The only exception is the set of last four vertices $\{v_1, x_{m+2}, v_a, x_m\}$ which induces a closed trail of length four in G. Suppose that there exists $j \in \{2, ..., p\}$ such that $t_j \neq 4$. Without loss of generality we can assume that $t_p \neq 4$. For such admissible sequence τ we can define the open trails T_2,...,T_p of length t_2,...,t_p in G such that T_1,...,T_p are edge-disjoint subgraphs of G. Assume now that $t_i = 4$ for any $i \in \{2, ..., p\}$. The vertices in a sequence

$$U^V := (v_2, x_1, v_3, x_2, v_4, x_1, v_5, ..., x_{m-1}, v_a, x_{a-2}, v_{a-1}, x_a, v_{a-2})$$

create an open trail of length t_1 in G (see fig. 15). Consider a sequence

$$U^{VI} := (v_{a-2}, x_{a-1}, v_{a-3}, x_{a-2}, v_{a-4}, \ldots,$$

$$x_{a-2}, v_1, x_{a-1}, v_a, x_{a-4}, v_{a-1}, x_{a-3}, v_{a-2}, x_{a-4}, v_{a-3}, \ldots,$$

$$x_{m+1}, v_1, x_{m+2}, v_a, x_m, v_1, x_a, v_2).$$

Let us part U^{VI} into $(p-1)$ sets, each of them containing five consecutive elements of it. Then these sets induce edge-disjoint open trails of length four in G (see fig. 16).

Suppose now that some of elements of τ are odd. It is obvious that there

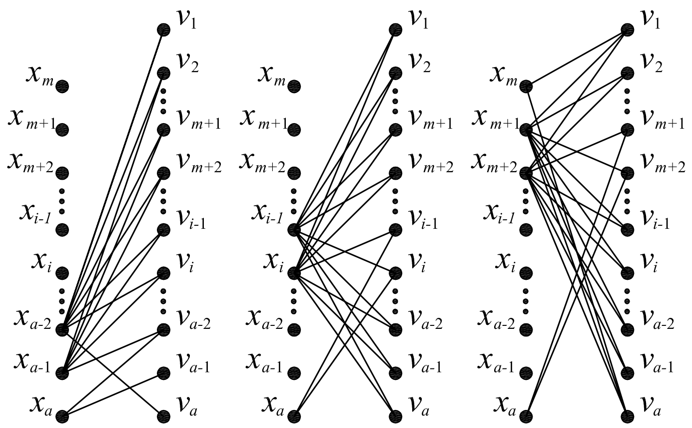

Figure 14: Sequence U^{IV}.

is an even number of odd elements in τ. Analogously like in Theorem 4 we can "glue" odd parts creating an element of even length. Hence the proof is finished. $\square$

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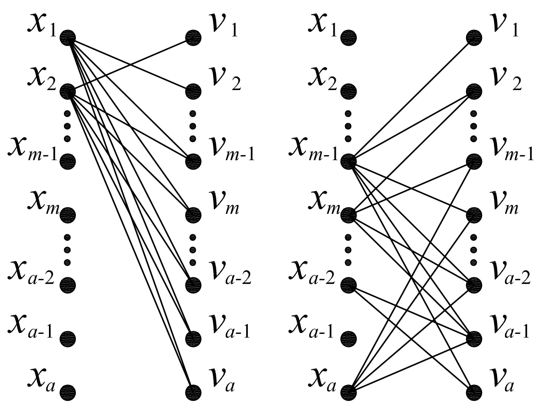

Figure 15: Sequence U^V.

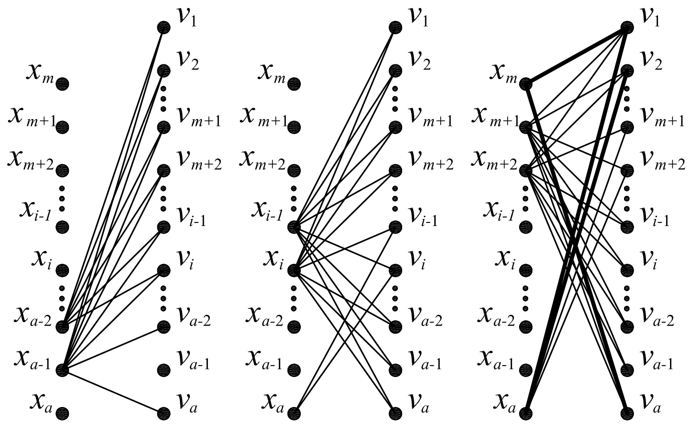

Figure 16: Sequence U^{VI}.