Classical Lifetime of a Bohr Atom

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(March 7, 2005)

1 Problem

In the Bohr model of the hydrogen atom's ground state, the electron moves in a circular orbit of radius $a_0 = 0.53 \times 10^{-10}$ m around the proton, which is assumed to be rigidly fixed in space. Since the electron is accelerating, a classical analysis suggests that it will continuously radiate energy, and therefore the radius of the orbit would shrink with time.

Considerations such as these in 1903 by J.J. Thomson, Phil. Mag. **6**, *673 (1903),*[1] *led him to note that there is no radiation if the charge is distributed in space so as to form steady currents. While a spinning shell or ring of charge provides a model for the magnetic moment of an atom, such charge configurations provide no restoring force against displacements of the center of the shell/ring from the nucleus. The (continuous) charge distribution must extend all the way to the nucleus if there is to be any possibility ofclassical electrostatic stability. A version of these insights was incoporated in Thomson's (not entirely self-consistent) model of the atom as a kind of "plum pudding" where the nucleus had a continuous, extended charge distribution in which more pointlike electrons were embedded. In this context, Rutherford's measurements of* α*-particle scattering, which showed that the nucleus was compact, came as something of a surprise, and re-opened the door to models such as that of Bohr in which pointlike electrons orbited pointlike nuclei, and radiation was suppressed by a "quantum" rule.*

Note that the model of the atom that emerged following Schrödinger contains some classically agreeable features if (ground states of) atoms are not to radiate: The electron in an atom is considered to have a spatially extended wave function that extends to the origin. The electric current associated with the electron is steady, and hence would not radiate if this were a classical current.

a) Assuming that the electron is always in a nearly circular orbit and that the rate of radiation of energy is sufficiently well approximated by classical, nonrelativistic electrodynamics, how long is the fall time of the electron, *i.e.*, the time for the electron to spiral into the origin?

b) The charge distribution of a proton has a radius of about 10^{-15} m, so the classical calculation would be modified once the radius of the electron's orbit is smaller than this. But even before this, modifications may be required due to relativistic effects.

Based on the analysis of part a), at what radius of the electron's orbit would its velocity be, say $0.1c$, where c is the speed of light, such that relativistic corrections

[1] http://puhep1.princeton.edu/~mcdonald/examples/EM/thomson_pm_45_673_03.pdf
Several subsequent authors have claimed the existence of radiationless orbital motion of classcial charges. However, these claims all appear to have defects. See P. Pearle, *Absence of Radiationless Motions of Relativistically Rigid Classical Electron*, Found. Phys. **7**, 931 (1977),
http://puhep1.princeton.edu/~mcdonald/examples/EM/pearle_fp_7_931_77.pdf

become significant? What fraction of the electron's fall time remains according to part a) when the velocity of the electron reaches $0.1c$?

c) Do the relativistic corrections increase or decrease the fall time of the electron?

It suffices to determine the sign of the leading correction as the radial velocity of the radiating electron approaches the speed of light.

A question closely related to the present one is whether the rate of decay of the orbit of a binary pulsar system due to gravitational radiation is increased or decreased by special-relativistic "corrections" as the orbital velocity becomes relativistic.

2 Solution

a) The dominant energy loss is from electric dipole radiation, which obeys the Larmor formula (in Gaussian units),

$$\frac{dU}{dt} = -\langle P_{E1} \rangle = -\frac{2e^2a^2}{3c^3}, \tag{1}$$

where a is the acceleration of the electron. For an electron of charge $-e$ and (rest) mass m_0 in an orbit of radius r about a fixed nucleus of charge $+e$, the radial component of the nonrelativistic force law, $\mathbf{F} = m_0\mathbf{a}$, tells us that

$$\frac{e^2}{r^2} = m_0 a_r \approx m_0 \frac{v_\theta^2}{r}, \tag{2}$$

in the adiabatic approximation that the orbit remains nearly circular at all times. In the same approximation, $a_\theta \ll a_r$, *i.e.*, $a \approx a_r$, and hence,

$$\frac{dU}{dt} = -\frac{2e^6}{3r^4 m_0^2 c^3} = -\frac{2}{3}\frac{r_0^3}{r^4} m_0 c^3. \tag{3}$$

where $r_0 = e^2/m_0c^2 = 2.8 \times 10^{-15}$ m is the classical electron radius. The total nonrelativistic energy (kinetic plus potential) is, using eq. (2),

$$U = -\frac{e^2}{r} + \frac{1}{2}m_0 v^2 = -\frac{e^2}{2r} = -\frac{r_0}{r} m_0 c^2. \tag{4}$$

Equating the time derivative of eq. (4) to eq. (3), we have

$$\frac{dU}{dt} = \frac{r_0}{2r^2}\dot{r} m_0 c^2 = -\frac{2}{3}\frac{r_0^3}{r^4} m_0 c^3, \tag{5}$$

or

$$r^2\dot{r} = \frac{1}{3}\frac{dr^3}{dt} = -\frac{4}{3}r_0^2 c. \tag{6}$$

Hence,

$$r^3 = a_0^3 - 4r_0^2 ct. \tag{7}$$

The time to fall to the origin is

$$t_{\rm fall} = \frac{a_0^3}{4r_0^2 c}\,. \tag{8}$$

With $r_0 = 2.8 \times 10^{-15}$ m and $a_0 = 5.3 \times 10^{-11}$ m, $t_{\rm fall} = 1.6 \times 10^{-11}$ s.

This is of the order of magnitude of the lifetime of an excited hydrogen atom, whose ground state, however, appears to have infinite lifetime.

b) The velocity v of the electron has components

$$v_r = \dot r = -\frac{4}{3}\frac{r_0^2}{r^2}c, \tag{9}$$

using eq. (6), and

$$v_\theta = r\dot\theta = \sqrt{\frac{e^2}{m_0 r}} = \sqrt{\frac{r_0}{r}}c, \tag{10}$$

according to eq. (2).

The azimuthal velocity is much larger than the radial velocity so long as $r \gg r_0$. Hence, $v/c \approx v_\theta/c$ equals 0.1 when $r_0/r \approx 0.01$, or $r \approx 100r_0$.

When $r = 100r_0$ the time t is given by eq. (7) as

$$t = \frac{a_0^3 - r^3}{4r_0^2 c}\,, \tag{11}$$

so that

$$\frac{t_{\rm fall} - t}{t_{\rm fall}} = \frac{r^3}{a_0^3} = \left(\frac{2.8 \times 10^{-13}}{5.3 \times 10^{-11}}\right)^3 \approx 1.5 \times 10^{-7}. \tag{12}$$

For completeness, we record other kinematic facts in the adiabatic approximation.

The angular velocity $\dot\theta$ follows from eq. (10) as

$$\dot\theta = \sqrt{\frac{r_0}{r^3}}c. \tag{13}$$

The second time derivatives are thus

$$\ddot r = \frac{8}{3}\frac{r_0^2}{r^3}\dot r c = -\frac{32}{9}\frac{r_0^4}{r^5}c^2, \qquad \ddot\theta = -\frac{3}{2}\sqrt{\frac{r_0}{r^5}}\dot r c = 2\sqrt{\frac{r_0}{r}}\frac{r_0^2}{r^4}c^2. \tag{14}$$

The components of the acceleration are

$$a_r = \ddot r - r\dot\theta^2 = -\frac{32}{9}\frac{r_0^4}{r^5}c^2 - \frac{r_0}{r^2}c^2 \approx -\frac{r_0}{r^2}c^2 = -r\dot\theta^2, \tag{15}$$

$$a_\theta = 2\dot r\dot\theta + r\ddot\theta = -\frac{8}{3}\sqrt{\frac{r_0}{r}}\frac{r_0^2}{r^3}c^2 \ll a_r. \tag{16}$$

c) We now examine the leading relativistic corrections to the nonrelativistic analysis of part a).

First, we recall that the lab frame rate of radiation by an accelerated charge obeys the Larmor formula (1) provided we use the acceleration in the instantaneous rest frame rather than in the lab frame. This is true because both dU and dt transform like the time components of a four-vector, so their ratio is invariant.

In the adiabatic approximation, the acceleration is transverse to the velocity. That is, $v_r \ll v_\theta$ from eqs. (9) and (10), while $a_r \gg a_\theta$ from eqs. (15) and (16). Therefore,

$$a^\star = \gamma^2 a, \tag{17}$$

where a is the lab-frame acceleration, $a^\star$ is the acceleration in the instantaneous rest frame, and $\gamma = 1/\sqrt{(1 - v^2/c^2)} = 1/\sqrt{1-\beta^2}$. Equation (17) holds because $a^\star = d^2l^\star/dt^{\star 2}$, and $dl^\star = dl$ for motion transverse to the velocity of the electron, while the time-dilation is $dt^\star = dt/\gamma$. Thus, the rate of radiation of energy by a relativistic orbiting electron is

$$\frac{dU}{dt} = -\left\langle P_{E1} \right\rangle = -\frac{2e^2 a^{\star 2}}{3c^3} = -\frac{2\gamma^4 e^2 a_r^2}{3c^3}\,. \tag{18}$$

[If the acceleration were parallel to the velocity, $a^\star * = \gamma^3 a$ since now there would also be the Lorentz contraction, $dl = dl^\star/\gamma$.]

The adiabatic orbit condition (2) for a relativistic electron becomes

$$\frac{e^2}{r^2} = \gamma m_0 a_r = \gamma m_0 \frac{v_\theta^2}{r} \approx \gamma m_0 \frac{v^2}{r}\,. \tag{19}$$

This can be thought of as the transform of the rest-frame relation $eE_r^\star = dP_r^\star/dt^\star$ upon noting that $E_r^\star = \gamma E_r$ since the electric field is tranverse to the velocity, $dt* = dt/\gamma$, and $dP_r^\star = dP_r = \gamma m_0 dv_r$.

Combining eq. (18) with the first form of eq. (19), we have

$$\frac{dU}{dt} = -\frac{2\gamma^2 e^6}{3m_0^2 c^3 r^4} = -\frac{2}{3}\gamma^2 \frac{r_0^3}{r^4} m_0 c^3\,. \tag{20}$$

We also rewrite eq. (19) as

$$\frac{e^2}{m_0 c^2 r} = \frac{r_0}{r} = \gamma \frac{v^2}{c^2} = \gamma\beta^2 \approx \gamma\left(1 - \frac{1}{\gamma^2}\right), \tag{21}$$

and hence,

$$\gamma^2 - \gamma\frac{r_0}{r} - 1 = 0, \tag{22}$$

$$\gamma = \frac{\frac{r_0}{r} + \sqrt{\frac{r_0^2}{r^2} + 4}}{2} = \sqrt{1 + \frac{r_0^2}{4r^2}} + \frac{r_0}{2r} \approx 1 + \frac{r_0}{2r} + \frac{r_0^2}{8r^2}\,. \tag{23}$$

The total lab-frame energy is now

$$U = \gamma m_0 c^2 - \frac{e^2}{r} = \left(\gamma - \frac{r_0}{r}\right) m_0 c^2 \approx \left(1 - \frac{r_0}{2r} + \frac{r_0^2}{8r^2}\right) m_0 c^2, \qquad (24)$$

using eq. (23). Then,

$$\begin{aligned} \frac{dU}{dt} &\approx \left(\frac{r_0}{2r^2} - \frac{r_0^2}{4r^3}\right) \dot{r} m_0 c^2 = \left(1 - \frac{r_0}{2r}\right) \frac{r_0}{2r^2} \dot{r} m_0 c^2 \\ &= -\frac{2}{3}\gamma^2 \frac{r_0^3}{r^4} m_0 c^3 \,, \end{aligned} \qquad (25)$$

using eq. (20). Finally,

$$\dot{r} \approx -\frac{4}{3}\gamma^2 \frac{r_0^2}{r^2} \frac{c}{1 - \frac{r_0}{2r}} \approx -\frac{4}{3}\frac{r_0^2}{r^2} c \left(1 + \frac{3r_0}{2r}\right), \qquad (26)$$

which is larger than the nonrelativistic result (6) by the factor $1 + 3r_0/2r$. Hence, the fall time of the electron is decreased by the relativistic corrections.

We note that the relativistic corrections increased the rate of radiation, and decreased the factor A in the relation $dU/dt = A\dot{r}$. Hence, both of these corrections lead to an increase in the radial velocity $\dot{r}$, and to a decrease in the corresponding fall time of the electron.