The kinematical roots of dynamics

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Whenever the total kinetic energy of an isolated collection of interacting bodies changes, the amount of its *change* as measured from different inertial frames has identical values. This remarkable invariance implies two of the principal conservation laws of Newtonian as well as relativistic mechanics. Moreover, together with the relevant rule of velocity addition, it even *determines* the exact form of the kinetic energy as a scalar function of the speed of motion in either context.

I. AN INTRODUCTORY EXAMPLE

It is a well-publicized fact that the chemical energy of propellants transforms into the *kinetic* energy of a rocket. Imagine two inertial frames of reference S and S', the latter coasting past S with some large-enough velocity. Let a rocket R be fired from S in the direction in which S' is moving. Observers in S now find R gaining speed all right, but those in S' do not. To the latter R is indeed a *retro*-rocket. It keeps *losing* speed and losing kinetic energy in its rearward motion relative to S'. Yet the residents of S' do notice R's propellants burning on with undiminished vigor. What *can* be happening in their reckoning, then, to the energy shed by all that fiery stuff?

The upshot of these thoughts is that the changes occurring in the kinetic energy of a *single* body viewed in isolation are not properly auditable. Suppose the burning propellants release a total amount of energy E. A considerable part H of E goes to waste as unutilized heat. Even so, the rocket vehicle *alone* cannot appropriate all the rest of the energy $E-H$. Both the vehicle and the exhaust gases rushing out of its nozzles must share this $E-H$ as acquisitions of kinetic energy. (We understand acquisitions as signed magnitudes here—positive if gains and negative if losses.) Taken individually, these acquisitions perforce depend on any particular frame of reference in which they are measured, yet summed up together they do turn out to be invariant. Thus, if the additional amounts of kinetic energy acquired by the vehicle and the gases have the values ΔK_V and ΔK_G in the frame S, and the values $\Delta K'_V$ and $\Delta K'_G$ in S', then indeed $\Delta K_V \neq \Delta K'_V$ and $\Delta K_G \neq \Delta K'_G$. Nevertheless, the algebraic sum of the two Δ's figuring in this context emerges as a truly frame-independent quantity:

$$\Delta K_V + \Delta K_G = \Delta K'_V + \Delta K'_G = E - H. \tag{1}$$

Striking as this invariance is, few textbooks seem to have ever made any mention of it in the past.

II. THE PROOF: A PRE-RELATIVISTIC PRESENTATION

In nonrelativistic classical mechanics, the kinetic energy K of a mass M moving with a velocity $\mathbf{V}$ is given by the familiar formula

$$2K = M\mathbf{V}\cdot\mathbf{V} = MV^2. \tag{2}$$

We now imagine that a number of ''bodies'' are interacting with one another, and that an observer O at rest with respect to some inertial frame of reference S watches the process. This is a generalization of the example of the rocket we discussed in Sec. I. At some instant of time, the observer O measures the masses and the velocities of all the bodies, say N in number. Suppose (s)he finds these to be M_ρ and $\mathbf{V}_\rho$, respectively, for the ρth body $(\rho = 1,2,...,N)$. As interactions progress, some of the ''bodies'' may merge together, some may break up, yet others could stay intact. It is thus possible that O finds fewer or more ''bodies'' at another time. Let the mass and the velocity of the ρth body then be m_ρ and $\mathbf{v}_\rho$, respectively $(\rho = 1,2,...,n)$. It now follows that O has observed an increment $\Sigma k - \Sigma K$ accruing to the total kinetic energy of all the bodies, where

$$\Sigma K = \frac{1}{2}\sum_{\varrho=1}^{N} M_\rho \mathbf{V}_\rho^2, \quad \Sigma k = \frac{1}{2}\sum_{\varrho=1}^{n} m_\rho \mathbf{v}_\rho^2. \tag{3}$$

Suppose there is a second inertial observer O' who concurrently carries out the same measurements as O does. Let the velocity of the *first* observer O relative to O' be $\mathbf{u}$. We employ *primed* variables to designate all of O''s other measurements. It now looks proper to assume

$$M'_\rho = M_\rho \quad (\rho = 1,2,...,N), \quad m'_\rho = m_\rho \quad (\rho = 1,2,...,n), \tag{4}$$

and

$$\mathbf{V}'_\rho = \mathbf{V}_\rho + \mathbf{u} \quad (\rho = 1,2,...,N), \quad \mathbf{v}'_\rho = \mathbf{v}_\rho + \mathbf{u}, \quad (\rho = 1,2,...,n). \tag{5}$$

Equation (4) says that *masses* are invariant—a hypothesis which holds even in special relativity if (as is customary) we understand the term to mean the *rest* masses always. Equation (5) expresses just the *vector* law of the addition of velocities. Its applicability is limited, of course, to prerelativistic mechanics only.

Obvious analogues of Eq. (3) define the primed total kinetic energies $\Sigma k'$ and $\Sigma K'$ of the bodies observed. A simple computation using Eqs. (4) and (5) now unveils the following crucial result:

$$\begin{aligned}(\Sigma k' - \Sigma K') &- (\Sigma k - \Sigma K)\\ &= \frac{1}{2}u^2\left\{\sum_{\varrho=1}^{n} m_\rho - \sum_{\varrho=1}^{N} M_\rho\right\}\\ &\quad + \mathbf{u}\cdot\left\{\sum_{\varrho=1}^{n} m_\rho \mathbf{v}_\rho - \sum_{\varrho=1}^{N} M_\rho \mathbf{V}_\rho\right\}.\end{aligned} \tag{6}$$

If the interacting bodies are free from outside influences, classical mechanics tells us that these bodies must together conserve their total *mass*, as well as their total *momentum*. Accordingly, the differences appearing within the two pairs of curly braces on the right-hand side of Eq. (6) vanish, thereby giving

$$\Sigma k' - \Sigma K' = \Sigma k - \Sigma K. \tag{7}$$

Equation (7) is a generalized form of Eq. (1) cited earlier in the context of the rocket and its exhaust gases. It affirms the remarkable invariance we alluded to at the outset: Any *increment* accruing to the total kinetic energy of an isolated system of bodies has identical values under measurements carried out from different inertial frames. Were it not for this fortunate result, calculations of kinetic energy would have amounted to just fanciful numerology.

The vector **u** of Eq. (6) is arbitrary. This makes the steps used in our deduction of Eq. (7) fully reversible. Thus Eq. (7) implies back to the laws of mass conservation and momentum conservation as forcefully as it is itself implied by them. Equation (7) should therefore serve as a legitimate point of departure for some further excursions into the foundations of mechanics. As we shall see, these investigations do in fact open up a new gateway toward understanding the subtle dynamics of special relativity as well.

III. SOME HISTORICAL NOTES

The quantity $2K$ of Eq. (2) originates from Gottfried Wilhelm von Leibniz (1646–1716), who gave it the name *vis viva*[1] way back in 1695. Decades before, René Descartes (1596–1650) and his followers had been inclined toward the seemingly more natural $K=MV$ as the correct representation of the ''*vis*'' of a mass M moving with the speed V. Understandably, this idea of the *vis* (or vitality) was steeped in slippery metaphysics in those days (as it perhaps is even today). All the same, Leibniz could still come up[2] with some surprisingly ingenious and rational arguments in support of his *vis viva*. At bottom, his thinking was based on a clearly perceived notion of interconvertibility of the energies of falling bodies, potential and kinetic. (These exact terms weren't in vogue in Leibniz's time, though.) In view of its profound historical significance, we give an outline of Leibniz's actual reasoning in the Appendix.

Another concept that seems to have evolved in time from Leibniz's thoughts is *work*. The term *work* was introduced[3] in its current technical sense only in 1826 by Jean Victor Poncelot (1788–1867), and it finds no place in the writings of Isaac Newton (1642–1727),[4,5] or Leibniz himself. Yet today this word is so firmly rooted in our technical vocabulary that we even have learned to think of the kinetic energy $K=\frac{1}{2}MV^2$ as *just* the ''work done in raising the speed of the mass M from zero to V.'' Historically, however, it is the definition of *work* that has been fashioned from kinetic energy's *already* known form, rather than the other way around. In any case, the much flaunted equality of K with the work done on M by an obligingly effective Newtonian force $M\,d\mathbf{V}/dt$ in a hypothetical process of imparting speed,

$$K=\int d\mathbf{r}\cdot M d\mathbf{V}/dt=\int (d\mathbf{r}/dt)\cdot M\,d\mathbf{V}$$
$$=\int \mathbf{V}\cdot M d\mathbf{V}=\frac{1}{2}\,MV^2$$

to wit, looks more tautological than telling.

IV. THE UNIQUE CHARACTER OF KINETIC ENERGY

None of the historical considerations just surveyed envisages observing dynamical events from different frames of reference. For this reason, none of these can do justice to kinetic energy's relevance to mechanics at its foundational level.

As might be expected, this profound relevance stems from just the invariance $\Sigma k' - \Sigma K' = \Sigma k - \Sigma K$ which, as we saw, regulates interactions within an isolated system of bodies, and ensures compliance of two basic conservation laws. As a matter of fact, it turns out that there is *essentially only one* way of defining kinetic energy so as to support this invariance. In classical mechanics, that unique way proves to be just what Leibniz formulates, namely his Eq. (2). In contrast, the Cartesian preference $K=MV$ (for example) does *not* uphold our grand invariance $\Sigma k' - \Sigma K' = \Sigma k - \Sigma K$. And finally, when we switch to special relatively, the invariance approach once more homes in on just the correct expression for kinetic energy in the new context as well.

We do need to make our premises more precise, of course, before we can prove these claims. We assume that every material ''body'' interacting with its neighbors has an invariant mass M, and a frame-dependent velocity **V**. We look for a well-behaved nonconstant scalar function $F(X)$ such that the alternative expression

$$K=MF(\mathbf{V}\cdot\mathbf{V})=MF(V^2), \tag{8}$$

to be accepted as the *new* kinetic energy of M henceforth, *also* upholds our basic invariance

$$\Sigma k' - \Sigma K' = \Sigma k - \Sigma K, \tag{7}$$

when masses like M interact with one another. We assume that $F(X)$ possesses *continuous* first and second derivatives in some interval $0<X<a$. We also assume that $F(0)$ is defined, but leave[6] the behavior of the function at $X=0$ unspecified.

V. OBSERVING A FISSION EVENT FROM TWO FRAMES

We are now ready for a proof of the assertions made in Sec. IV. It would be adequate for our limited purpose at the moment to base our arguments on a simple special case only. Consider the *fission* of a single isolated mass M into just *two* fragments m_1 and m_2 ($M\neq 0$, $m_1\neq 0$, $m_2\neq 0$). To elaborate, our first observer O finds that an *initially* stationary mass M at rest in his/her frame S breaks up spontaneously, and that its fragments m_ρ fly away with some velocities $\mathbf{v}_\rho$ ($\rho=1,2$) ($\mathbf{v}_\rho\neq\mathbf{0}$ and $\mathbf{v}_1\neq\mathbf{v}_2$). And, as before, the second observer O' finds that his/her companion's frame S has itself been in motion with a uniform velocity **u** all along.

The observer O' measures the velocity of the original mass M to be **u**, the same as the velocity of the frame S in which it was at rest. Let O' also measure the velocities of the flying fragments m_ρ, and find them to be $\mathbf{v}'_\rho$ (ρ=1,2). With the kinetic energy redefined as $K=MF(V^2)$ through Eq. (8), our crucial invariance $\Sigma k' - \Sigma K' = \Sigma k - \Sigma K$ takes on the simplified form

$$M[F(u^2)-F(0)]=m_1[F(\nu_1'^2)-F(\nu_1^2)] + m_2[F(\nu_2'^2)-F(\nu_2^2)]. \qquad (9)$$

Rather than designating the derivatives of $F(X)$ with the customary primes (which are currently reserved for signaling O''s *measurements* instead), let us employ the notation $F^{(1)}(X)$, $F^{(2)}(X)$, etc., for the derivatives in question. Since $F(X)$ is not a constant, $F^{(1)}(X)\neq 0$ except possibly for a set of isolated X's. We now study *some* arbitrary fission event (only one!) in which the fragment velocities (each nonzero) are unequal vectorially[7] ($\mathbf{v}_1\neq\mathbf{v}_2$), and in which $F^{(1)}(\nu_\rho^2)\neq 0$. In this circumstance, because of $F^{(1)}$'s continuity, $F^{(1)}(X)\neq 0$ in a whole neighborhood of $X=\nu_\rho^2$. When the velocity $\mathbf{u}$ of the observer O relative to O' is small, the fragment velocity $\mathbf{v}_\rho'$ measured by O' stays close to the corresponding velocity $\mathbf{v}_\rho$ measured by O. Accordingly, $F^{(1)}(\nu_\rho'^2)\neq 0$ as long as $\mathbf{u}$ is sufficiently small. [The reader can see from Eqs. (14) and (14-R) appearing in the sequel (Secs. VI and VII) that once these statements hold for $\rho=1$ *or* 2, they must automatically be true for the other ρ as well.]

In what follows, it would certainly be possible to deal with both the relativistic and the pre-relativistic situations concurrently, since the latter is, after all, a limiting special case of the first. Yet the size of relativistic expressions might at the start keep readers from seeing the forest for the trees. We would prefer, therefore, to discuss only the nonrelativistic theme to begin with.

VI. THE NONRELATIVISTIC CASE

In nonrelativistic mechanics, we have, for plugging into our invariance relation (9), the simple vectorial law

$$\mathbf{v}_\rho'=\mathbf{u}+\mathbf{v}_\rho \quad (\rho=1,2). \qquad (10)$$

We note in this connection

$$\nu_\rho'^2=(u_x+\nu_{\rho x})^2+(u_y+\nu_{\rho y})^2+(u_z+\nu_{\rho z})^2 \quad (\rho=1,2),$$
$$u^2=u_x^2+u_y^2+u_z^2, \qquad (11)$$

where the subscripts x, y, and z designate components with respect to an arbitrarily chosen rectangular Cartesian axes. We may insert these values in Eq. (9), differentiate the result partially with respect to u_x, simplify and rearrange terms thus:

$$\{MF^{(1)}(u^2)-m_1F^{(1)}(\nu_1'^2)-m_2F^{(1)}(\nu_2'^2)\}u_x = \{m_1F^{(1)}(\nu_1'^2)\}\nu_{1x}+\{m_2F^{(1)}(\nu_2'^2)\}\nu_{2x}. \qquad (12)$$

Similar manipulations of Eq. (9) with $\partial/\partial u_y$ and $\partial/\partial u_z$ give two more analogous results which when adjoined to Eq. (12) yield the vector relation

$$\{MF^{(1)}(u^2)-m_1F^{(1)}(\nu_1'^2)-m_2F^{(1)}(\nu_2'^2)\}\mathbf{u} = m_1F^{(1)}(\nu_1'^2)\mathbf{v}_1+m_2F^{(1)}(\nu_2'^2)\mathbf{v}_2. \qquad (13)$$

In other words, Eq. (13) is derived from Eq. (11) as just the *gradient with respect* to $\mathbf{u}$ of our invariance relation (9). Now if the coefficient of $\mathbf{u}$ in Eq. (13) were nonzero, $\mathbf{u}$ would become a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, and so should lie in the plane/line ξ (say) spanned by the latter. As such, $\mathbf{u}$'s coefficient *must* be zero when $\mathbf{u}$ doesn't lie in ξ, and *because of its continuity, this coefficient must continue to be zero even when* $\mathbf{u}$ *does lie in* ξ. For all possible $\mathbf{u}$'s, therefore, Eq. (13) reduces to

$$\{m_1F^{(1)}(\nu_1'^2)\}\mathbf{v}_1=-\{m_2F^{(1)}(\nu_2'^2)\}\mathbf{v}_2. \qquad (14)$$

According to the assumptions we made at the start, the coefficients of the $\mathbf{v}_\rho$'s here must be nonzero (for small enough $\mathbf{u}$'s), and these $\mathbf{v}_\rho$'s must consequently *lie in a single straight line*. In Eq. (14), the $\mathbf{v}_\rho$'s and the m_ρ's do not depend on $\mathbf{u}$. Accordingly, the quotient $F^{(1)}(\nu_1'^2)/F^{(1)}(\nu_2'^2)$ must also be independent of $\mathbf{u}$, so the gradient of this quotient with respect to $\mathbf{u}$ must vanish. Taking note of Eq. (11), this consideration results in a relationship of the form $A\mathbf{u}=B\mathbf{v}_1-C\mathbf{v}_2$ in which

$$A=F^{(1)}(\nu_1'^2)F^{(2)}(\nu_2'^2)-F^{(1)}(\nu_2'^2)F^{(2)}(\nu_1'^2), \qquad (15A)$$

$$B=F^{(1)}(\nu_2'^2)F^{(2)}(\nu_1'^2), \quad C=F^{(1)}(\nu_1'^2)F^{(2)}(\nu_2'^2). \qquad (15B)$$

On the other hand, as in the case of Eq. (13), $\mathbf{u}$'s coefficient must identically vanish in $A\mathbf{u}=B\mathbf{v}_1-C\mathbf{v}_2$, so from Eq. (15A)

$$F^{(1)}(\nu_1'^2)F^{(2)}(\nu_2'^2)=F^{(1)}(\nu_2'^2)F^{(2)}(\nu_1'^2). \qquad (16)$$

Since now $B\mathbf{v}_1-C\mathbf{v}_2=\mathbf{0}$, relations (15B) and (16) lead to

$$F^{(1)}(\nu_2'^2)F^{(2)}(\nu_1'^2)(\mathbf{v}_1-\mathbf{v}_2)=\mathbf{0}, \qquad (17)$$

implying that F's second derivative $F^{(2)}(\nu_1'^2)=0$. (Note that by earlier assumptions $F^{(1)}\neq 0$ and $\mathbf{v}_1\neq\mathbf{v}_2$.) Successive integrations now give

$$F(\nu_1'^2)=\alpha\nu_1'^2+\beta \quad (\alpha,\beta \text{ constants}), \qquad (18)$$

which is essentially the function giving rise to Leibniz's *vis viva*.

VII. THE RELATIVISTIC CASE

The foregoing proof for nonrelativistic mechanics has in fact been modeled in such a manner that *exactly* analogous steps will settle the relativistic case as well.

It proves convenient to set c, the speed of light *in vacuo*, equal to unity in special relativity. The relativistic analogue of Eq. (11) for compounding velocity magnitudes can then be written as

$$\nu_\rho'^2=1-w_\rho^2(1-\nu_\rho^2)(1-u^2), \qquad (11\text{-R})$$

where

$$w_\rho=1/(1+\mathbf{v}_\rho\cdot\mathbf{u}). \qquad (11\text{-r})$$

Here we have, of course,

$$u^2=u_x^2+u_y^2+u_z^2, \quad \mathbf{v}_\rho\cdot\mathbf{u}=\nu_{\rho x}u_x+\nu_{\rho y}u_y+\nu_{\rho z}u_z.$$

The gradient with respect to $\mathbf{u}$ of the invariance relation (9) now turns out to be

$$\{MF^{(1)}(u^2)-m_1(1-\nu_1^2)w_1^2F^{(1)}(\nu_1'^2) - m_2(1-\nu_2^2)w_2^2F^{(1)}(\nu_2'^2)\}\mathbf{u} = (1-u^2)\{m_1(1-\nu_1^2)w_1^3F^{(1)}(\nu_1'^2)\mathbf{v}_1 + m_2(1-\nu_2^2)w_2^3F^{(1)}(\nu_2'^2)\mathbf{v}_2\}. \qquad (13\text{-R})$$

Again $\mathbf{u}$'s coefficient is necessarily zero, the $\mathbf{v}_\rho$'s are necessarily collinear from the dependence

$$m_1(1-\nu_1^2)w_1^3F^{(1)}(\nu_1'^2)\mathbf{v}_1=-m_2(1-\nu_2^2)w_2^3F^{(1)}(\nu_2'^2)\mathbf{v}_2, \tag{14-R}$$

the gradient of the quotient $w_1^3F^{(1)}(\nu_1'^2)/w_2^3F^{(1)}(\nu_2'^2)$ with respect to $\mathbf{u}$ necessarily vanishes, and this last condition leads to a relationship of the form $A\mathbf{u}=B\mathbf{v}_1-C\mathbf{v}_2$. On performing the necessary calculations, *and looking back at the crucial relation (11-R)*, one finds

$$\begin{aligned} A(1-u^2)&=2\{(1-\nu_2'^2)F^{(1)}(\nu_1'^2)F^{(2)}(\nu_2'^2)-(1-\nu_1'^2)\\ &\quad\times F^{(1)}(\nu_2'^2)F^{(2)}(\nu_1'^2)\}\\ &=0 \quad \text{identically},\\ B&=\{2(1-\nu_1'^2)F^{(2)}(\nu_1'^2)-3F^{(1)}(\nu_1'^2)\}F^{(1)}(\nu_2'^2)w_1,\\ C&=\{2(1-\nu_2'^2)F^{(2)}(\nu_2'^2)-3F^{(1)}(\nu_2'^2)\}F^{(1)}(\nu_1'^2)w_2. \end{aligned} \tag{15-R}$$

The analogues of Eqs. (16) and (17) now turn out to be

$$(1-\nu_2'^2)F^{(1)}(\nu_1'^2)F^{(2)}(\nu_2'^2) =(1-\nu_1'^2)F^{(1)}(\nu_2'^2)F^{(2)}(\nu_1'^2) \tag{16-R}$$

and

$$\{2(1-\nu_1'^2)F^{(2)}(\nu_1'^2)-3F^{(1)}(\nu_1'^2)\} \times F^{(1)}(\nu_2'^2)(w_1\mathbf{v}_1-w_2\mathbf{v}_2)=\mathbf{0}. \tag{17-R}$$

Since $F^{(1)}(\nu_2'^2)\neq 0$, and $w_1\mathbf{v}_1\neq w_2\mathbf{v}_2$ (it turns out[8] that if $\mathbf{v}_1\neq\mathbf{v}_2$ then $w_1\mathbf{v}_1\neq w_2\mathbf{v}_2$ as well), Equation (17-R) reduces to

$$\frac{F^{(2)}(X)}{F^{(1)}(X)}=\frac{3}{2(1-X)} \quad (X=\nu_1'^2). \tag{17-r}$$

Remembering that $F^{(\rho)}$ is the ρth derivative of F, this equation integrates into

$$F(X)=\alpha\gamma(X)-\beta \quad (\alpha,\beta \text{ constants}), \tag{18-R}$$

where the function $\gamma(X)$ is given by

$$\gamma(X)=\frac{1}{\sqrt{1-X}}. \tag{19-R}$$

Equation (18-R)'s $F(X)$ is just the function used in the definition of kinetic energy in special relativity, except that it appears here in a generalized form. A large number of earlier derivations of the result do exist,[9] of course, but they all assume *imprecisely formulated conservation laws* to begin with. This makes the use of some fuzzy reasoning indispensable in their overall approach.

VIII. SPECIAL RELATIVITY'S CONSERVATION LAWS

We have not as yet established the *existence* of a well-behaved function $F(X)$ which upholds the invariance stated in Eq. (9). What we have proved is that *if* such a function does exist, it must, in relativistic mechanics, be the one given by Eq. (18-R). Can Eq. (18-R)'s $F(X)$ really ensure compliance of the invariance without running into mathematical inconsistencies?

It can and it does. Its strength in this connection stems from the transformation property

$$\gamma(\nu'^2)=\gamma(u^2)\gamma(\nu^2)(1+\mathbf{u}\cdot\mathbf{v}) \tag{20-R}$$

of the function $\gamma(X)$ defined in Eq. (19-R). Here $\mathbf{v}$ and $\mathbf{v}'$ are a pair of corresponding velocities of a body as measured by the observers O and O', respectively, and $\mathbf{u}$ as usual is the velocity of O relative to O'. Equation (20-R) is no more than a restatement of Eqs. (11-R/r) used earlier. Where, as in Sec. II, a number of interacting bodies are involved, $\mathbf{v}$ and $\mathbf{v}'$ can range over $\mathbf{V}_\rho$ and $\mathbf{V}'_\rho$ $(\rho=1,2,...,N)$ on one occasion, and over $\mathbf{v}_\rho$ and $\mathbf{v}'_\rho$ $(\rho=1,2,...,n)$ on another. Recalling now the kinetic energy expressions $k=mF(\nu^2)$, $k'=mF(\nu'^2)$, etc., with $F(X)$ given by Eq. (18-R), our basic invariance $\Sigma k'-\Sigma K'=\Sigma k-\Sigma K$ takes on the form

$$\begin{aligned} 0&=(\Sigma k'-\Sigma k)-(\Sigma K'-\Sigma K)\\ &=\alpha[\gamma(u^2)-1]\left(\sum_{\varrho=1}^{n} m_\rho\gamma(\nu_\rho^2)-\sum_{\varrho=1}^{N} M_\rho\gamma(V_\rho^2)\right)\\ &\quad+\alpha\gamma(u^2)\mathbf{u}\cdot\left(\sum_{\varrho=1}^{n} m_\rho\gamma(\nu_\rho^2)\mathbf{v}_\rho-\sum_{\varrho=1}^{N} M_\rho\gamma(V_\rho^2)\mathbf{V}_\rho\right), \end{aligned} \tag{21-R}$$

thanks to the property (20-R) of $\gamma(X)$. The factors $\gamma(u^2)-1$ and $\gamma(u^2)\mathbf{u}$ appearing here are arbitrary. Equation (21-R) is a handsome analogue of Sec. II's Eq. (6). It conveniently reduces the invariance relation $\Sigma k'-\Sigma K'=\Sigma k-\Sigma K$ involving an *infinity* of different inertial frames to only *two* conservation laws holding in any *single* frame of reference, namely: (i) the law of conservation of a scalar quantity known as the ''*total relativistic energy*''

$$\sum_{\varrho=1}^{n} m_\rho\gamma(\nu_\rho^2)=\sum_{\varrho=1}^{N} M_\rho\gamma(V_\rho^2), \tag{22-R}$$

and (ii) the law of conservation of a vector quantity called the ''*total relativistic momentum*''

$$\sum_{\varrho=1}^{n} m_\rho\gamma(\nu_\rho^2)\mathbf{v}_\rho=\sum_{\varrho=1}^{N} M_\rho\gamma(V_\rho^2)\mathbf{V}_\rho. \tag{23-R}$$

Exactly as in the Newtonian case (Sec. II), the invariance implies the conservation laws while the latter in turn imply the invariance back. Our fission event being a special case with $N=1$ and $n=2$, the corresponding invariance relation (9) is guaranteed to hold *in all inertial frames* once the two relations (22-R) and (23-R) are satisfied in only a single inertial frame. This ensures against mathematical inconsistencies.

Before we close this discussion, we must note in passing that the set of measurements $(\mathbf{V}_\rho,\mathbf{V}'_\rho)$ can never be simultaneous in both the frames S and S' $(\rho=1,2,...,N)$, nor can $(\mathbf{v}_\rho,\mathbf{v}'_\rho)$ be either $(\rho=1,2,...,n)$. As is well known, simultaneity is not an invariant condition in the relativistic context. On the other hand, no conservation law can be expected to hold if the timings of the individual measurements are left entirely unrestricted. A simple way to resolve this difficulty consists in replacing simultaneity with a softer but fully invariant condition to which the name *contemporaneity* seems apt. *A contemporaneous set of events is one in which each event is separated from every other event by only a spacelike interval*. Our observers O and O' can in principle ensure that the set of measurements they both carry out on any single occasion are indeed contemporaneous, and we may *assume* that once they do so, they will find the invariance and the conservation laws unfailingly obeyed under all circumstances.

IX. TOTAL ''RELATIVISTIC'' ENERGY AND TOTAL KINETIC ENERGY

Turning back to Eqs. (18-R) and (22-R), we discover a particularly simple expression for the change in the total kinetic energy of an isolated system of bodies as measured by *any* inertial observer:

$$\Sigma k - \Sigma K = \beta\left(\sum_{\varrho=1}^{N} M_\rho - \sum_{\varrho=1}^{n} m_\rho\right). \qquad \text{(24-R)}$$

The expression within the parentheses is called the *mass defect*. The incremental kinetic energy is thus equal to β times the mass defect always.

Quite frequently, we find physicists accustomed to setting $\beta=0$. The change in the kinetic energy $\Sigma k - \Sigma K$ is then *identically* zero, and is therefore *trivially* invariant. The kinetic energy determined in this way by setting $\beta=0$ essentially coincides with the ''total *relativistic* energy'' defined in the context of the conservation law (22-R) of Sec. VIII. But as something that never changes its value in spite of all the ongoing interactions, this quantity doesn't quite conform to what one normally understands by the term *kinetic* energy. It does have other virtues, however—the most notable one being that it can serve as the time component of a dynamical system's momentum–energy four vector with precisely the right transformation properties.

When we look for all the features that we intuitively associate with kinetic energy (such as variability), the only way out is to choose $\beta>0$ in Eq. (18-R). The actual value chosen (together with α's value) determines the size of the unit of mass (or energy) adopted for measurements, and is therefore immaterial from a theoretical point of view. Equation (24-R) shows that any change occurring in the total *mass* of an isolated dynamical system always manifests itself as a proportional change in the total *kinetic* energy of the system. This delineates the true content of the most widely quoted formula of early relativists, $E=mc^2$.

Unlike total relativistic energy, the total kinetic energy of an isolated system of bodies (obtained by setting $\beta\neq 0$) doesn't behave like a vector component. But as repeatedly stressed, the total kinetic energy possesses the remarkable property that *changes* occurring in it are *invariant* across all inertial frames in which they are measured. And this single property serves to uniquely identify it as well.

X. CONCLUSION

As a concept, energy—or kinetic energy—can be taken to be primitive and irreducible. Trying to describe it as, say, ''the capacity to do work by virtue of...'' doesn't really carry us very far. The unsophisticated view that kinetic energy—or for that matter any energy, since the latter could be just hidden kinetic energy in a bound state—merely consists in how *vigorously* bodies are on the whole moving about in a dynamical system seems to work much better after all. Though the motions of the bodies taken individually are frame dependent and therefore seem to compromise this line of thinking, we have demonstrated that the changes occurring in them can be turned truly *invariant* by quantifying them with a properly constructed unique kinetic energy function $F(X)$ and summing up over the whole system. Basically, the function $F(X)$ is determined by the law of composition of *velocities* applicable in any given context. The kinematical roots of dynamics are thus deep and strong, and can provide valuable insight into the mechanisms by which nature keeps bodies big and small incessantly moving around in the universe.

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The lines on which this article originally tackled its mathematical issues were more involved and less elegant. The author owes the ideas leading to the simpler proofs now presented to the reviewers' discerning vision.

APPENDIX: LEIBNIZ'S DEFENSE OF HIS *VIS VIVA* (CF. SEC. III)

Imagine a mass m near the surface of the earth with a speed ν vertically upwards. The investigations of Galileo Galilei (1564–1642) show that this mass ascends to a height $h=\nu^2/(2g)$ before losing all its speed. Let it then be connected to a second mass $M\gg m$ by a suitably designed lever, and allowed to fall with *negligible speed* through a short vertical distance d. As a result, M is raised through a height D given by $MD=md$. Finally, put the lever away and allow the masses m and M to drop freely under gravity. When they return to their original locations, M will have fallen through a height D gaining a speed $\surd(2gD)=\surd(2gmd/M)$ in the process, and m through a height $(h-d)$ gaining a corresponding speed $\surd\{2g(h-d)\}=\surd(\nu^2-2gd)$. Leibniz now demands

$$mF(\nu^2)=mF(\nu^2-2gd)+MF(2gmd/M). \qquad (25)$$

Equation (25) is clearly satisfied if $F(X)=X$. Leibniz argues, on the other hand, that if $F(X)=\surd X$ as imagined by Descartes, the right-hand side of Eq. (25) would outweigh its left-hand side. This would potentially turn the arrangement of the masses and the lever into a *perpetuum mobile*, an impossibility.

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[1]Max von Laue, ''Inertia and Energy'' in *Albert Einstein: Philosopher-Scientist*, edited by P. A. Schilpp (Harper Torchbooks, New York, 1959), p. 510.

[2]Rev. Fr. P. Castabel, *Leibniz and Dynamics*, English translation (Hermann, Paris, 1973).

[3]See Ref. 1, p. 511.

[4]S. Chandrasekhar, *Newton's Principia for the Common Reader* (Clarendon, Oxford, 1995).

[5]J. Haneweaver, *The World of Physics* (Simon and Schuster, New York, 1987), Vol. 1.

[6]This takes account of the fact that our base variable is $X=V^2$ rather than $X=V$, and that, as opposed to a direct function of V quite well-behaved at $V=0$, its corresponding V^2 function could turn out to be nondifferentiable at $V^2=0$.

[7]If $\mathbf{v}_1=\mathbf{v}_2$, the ''fragments'' would fly away together in one piece like a magic carpet, and observers wouldn't notice any fission at all. Magic carpets violate Newton's first law, of course, but we *don't* have to invoke *this* law here. If $\mathbf{v}_1=\mathbf{v}_2$, Eq. (9) reduces to $M[F(u^2)-F(0)]=(m_1+m_2)\times[F(\nu_1'^2)-F(\nu_1^2)]$, so $F(u^2)$'s nonconstancy implies $m_1+m_2\neq 0$. Equations (14)/(14-R) now assume the form $(m_1+m_2)C\mathbf{v}_1=\mathbf{0}$ *with* $(m_1+m_2)C\neq 0$. As such, $\mathbf{v}_1(=\mathbf{v}_2)=\mathbf{0}$: the only possible magic carpet is one which never takes off.

[8]If $w_1\mathbf{v}_1=w_2\mathbf{v}_2$ then $(1+\mathbf{u}\cdot\mathbf{v}_2)\mathbf{v}_1=(1+\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2$. This implies $\mathbf{v}_1-\mathbf{v}_2=(\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_2-(\mathbf{u}\cdot\mathbf{v}_2)\mathbf{v}_1=\mathbf{u}\times(\mathbf{v}_2\times\mathbf{v}_1)=\mathbf{0}$, since the $\mathbf{v}_\rho$'s are collinear.

[9]See, for example, A. P. French, *Special Relativity*, MIT Introductory Physics Series (MIT, Cambridge, 1968), pp. 169–175.