Algebra – Zad. domowe nr. 2 – Macierze i ukł. równań liniowych

① Rozwiąż układ równań

$$\begin{cases} x + y + 2z + 3w = 0 \\ 3x + 4y + 5z + 4w = 4 \\ 4x + 5y + 6z + 3w = 5 \\ 5x + 3y + 9z + 9w = -9 \end{cases}$$

$$[A|U] = \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 3 & 4 & 5 & 4 & 4 \\ 4 & 5 & 6 & 3 & 5 \\ 5 & 3 & 9 & 9 & -9 \end{array}\right] \xrightarrow[\substack{W_2-3W_1 \\ W_3-4W_1 \\ W_5-5W_1}]{} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & -5 & 4 \\ 0 & 1 & -2 & -9 & 5 \\ 0 & -2 & -1 & -6 & -9 \end{array}\right] \xrightarrow[\substack{W_3-W_2 \\ W_4+2W_2}]{} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & -5 & 4 \\ 0 & 0 & -1 & -4 & 1 \\ 0 & 0 & -3 & -16 & -1 \end{array}\right] \xrightarrow[\substack{W_4-3\cdot W_3 \\ W_3\cdot(-1)}]{}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & -5 & 4 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & -4 & -4 \end{array}\right] \xrightarrow[W_4\cdot(-\frac{1}{4})]{} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & -5 & 4 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[\substack{W_3-4W_4 \\ W_2+5W_4 \\ W_1-3W_4}]{} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & -3 \\ 0 & 1 & -1 & 0 & 9 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[\substack{W_2+W_3 \\ W_1-2W_3}]{} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[W_1-W_2]{} \left[\begin{array}{c|c} I_4 & \begin{matrix} 3 \\ 4 \\ -5 \\ 1 \end{matrix} \end{array}\right]$$

Odp. $x=3,\ y=4,\ z=-5,\ w=1$

ukł. oznaczony

② Ilości rozwiązań w zależności od $p \in \mathbb{R}$

$$U = [A|B] = \left[\begin{array}{cccc|c} 3p & 6 & 3 & 0 & 6 \\ p & p+1 & 3 & 1 & 6 \\ p & 2 & p-1 & 1 & 1 \\ 2p & 4 & 2 & p-3 & 2p-2 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} p & p+1 & 3 & 1 & 6 \\ p & 2 & p-1 & 1 & 1 \\ 2p & 4 & 2 & p-3 & 2p-2 \\ 3p & 6 & 3 & 0 & 6 \end{array}\right] \xrightarrow[\substack{W_2-W_1 \\ W_3-2W_1 \\ W_4-3W_1}]{}$$

$$\rightarrow \left[\begin{array}{cccc|c} p & p+1 & 3 & 1 & 6 \\ 0 & 1-p & p-4 & 0 & -5 \\ 0 & 2-2p & -4 & p-5 & 2p-14 \\ 0 & 3-3p & -6 & -3 & -12 \end{array}\right] \xrightarrow[\substack{W_3-2W_2 \\ W_4-3W_2}]{} \left[\begin{array}{cccc|c} p & p+1 & 3 & 1 & 6 \\ 0 & 1-p & p-4 & 0 & -5 \\ 0 & 0 & 4-2p & p-5 & 2p-4 \\ 0 & 0 & 6-3p & -3 & 3 \end{array}\right] \xrightarrow[\substack{W_3\cdot\frac{1}{2} \\ W_4\cdot\frac{1}{3}}]{}$$

$$\rightarrow \left[\begin{array}{cccc|c} p & p+1 & 3 & 1 & 6 \\ 0 & 1-p & p-4 & 0 & -5 \\ 0 & 0 & 2-p & \frac{1}{2}p-\frac{5}{2} & p-2 \\ 0 & 0 & 2-p & -1 & 1 \end{array}\right] \xrightarrow[W_4-W_3]{} \left[\begin{array}{cccc|c} p & p+1 & 3 & 1 & 6 \\ 0 & 1-p & p-4 & 0 & -5 \\ 0 & 0 & 2-p & \frac{1}{2}p-\frac{5}{2} & p-2 \\ 0 & 0 & 0 & -\frac{1}{2}p+\frac{3}{2} & 3-p \end{array}\right]$$

$p \in \mathbb{R} \setminus \{0, 1, 2, 3\}$ układ oznaczony $\quad r(U) = r(A) = 4 = n$

jedno rozw.

$$p=0 \quad \left[\begin{array}{cccc|c} 0 & 1 & 3 & 1 & 6 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 2 & -\frac{5}{2} & -2 \\ 0 & 0 & 0 & \frac{3}{2} & 3 \end{array}\right] \xrightarrow[\substack{W_2-W_1 \\ W_4\cdot\frac{2}{3}}]{} \left[\begin{array}{cccc|c} 0 & 1 & 3 & 1 & 6 \\ 0 & 0 & -7 & -1 & -11 \\ 0 & 0 & 2 & -\frac{5}{2} & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[\substack{W_2+W_4 \\ W_3+\frac{5}{2}W_4}]{} \left[\begin{array}{cccc|c} 0 & 1 & 3 & 1 & 6 \\ 0 & 0 & -7 & 0 & -9 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$-2 + \frac{5}{2}\cdot 2 = 3 \qquad 2z = 3$

$7z = 9$

sprzeczny!

$p=1$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 6 \\ 0 & 0 & -3 & 0 & -5 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{w_2 \leftrightarrow w_3} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 6 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & -3 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{w_3+3w_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 6 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -6 & -8 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{cases} w=2 \\ 6w=8 \end{cases}$$ sprzeczny

$p=2$

$$\left[\begin{array}{cccc|c} 2 & 3 & 3 & 1 & 6 \\ 0 & -1 & -2 & 0 & -5 \\ 0 & 0 & 0 & -3/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \end{array}\right]$$

sprzeczny $\begin{cases} z=0 \\ \frac{1}{2}z=1 \end{cases}$ brak rozw.

$p=3$

$$\left[\begin{array}{cccc|c} 3 & 4 & 3 & 1 & 6 \\ 0 & -2 & -1 & 0 & -5 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$r(U)=r(A)=3 < n=4$

ukł. nieoznaczony

Nieskończenie wiele rozw. zależnych od 1 parametru

(3)

$$(X+I)^T \cdot A = 2A - I \quad / \cdot A^{-1}$$

$$(X+I)^T = (2A-I)A^{-1} = 2I - A^{-1} \quad /^T$$

$$X+I = (2I-A^{-1})^T = 2I - (A^{-1})^T$$

$$X = I - (A^{-1})^T$$

$A \in M_4(\mathbb{R})$

$A = [a_{ij}]$

$$a_{ij} = \begin{cases} 2i+j & ; \ i=j \\ 3 & ; \ i \neq j \end{cases}$$

$$A = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 6 & 3 & 3 \\ 3 & 3 & 9 & 3 \\ 3 & 3 & 3 & 12 \end{bmatrix} \qquad A = 3 \cdot B = 3 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

$$A^{-1} = (3B)^{-1} = 3^{-1} \cdot B^{-1} = \frac{1}{3} \cdot B^{-1} \qquad B^{-1} = ?$$

$$[B|I] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{w_3-w_1 \\ w_4-w_1}]{w_2-w_1} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[w_4 \cdot \frac{1}{3}]{w_3 \cdot \frac{1}{2}} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{w_1-w_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 4/3 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & -1/3 & 0 & 0 & 1/3 \end{array}\right] \xrightarrow{w_1-w_2-w_3} \left[\begin{array}{c|c} I & \underbrace{\begin{matrix} 17/6 & -1 & -\frac{1}{2} & -1/3 \\ -1 & 1 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ -1/3 & 0 & 0 & 1/3 \end{matrix}}_{B^{-1}} \end{array}\right]$$

$$X = I - \left(\frac{1}{3}B^{-1}\right)^T = I - \frac{1}{3} \cdot (B^{-1})^T = I - \begin{bmatrix} 17/18 & -1/3 & -1/6 & -1/9 \\ -1/3 & 1/3 & 0 & 0 \\ -1/6 & 0 & 1/6 & 0 \\ -1/9 & 0 & 0 & 1/9 \end{bmatrix} = \begin{bmatrix} \frac{1}{18} & \frac{1}{3} & \frac{1}{6} & \frac{1}{9} \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{6} & 0 & 5/6 & 0 \\ \frac{1}{9} & 0 & 0 & 8/9 \end{bmatrix}$$