Zadanie domowe nr 1

Zad. 1

a) $\forall n \in \mathbb{N} \quad \sum_{k=1}^{n} k(k+1) = \frac{1}{3} n(n+1)(n+2)$

$1^\circ \quad n=1 \qquad L = \sum_{k=1}^{1} k(k+1) = 1 \cdot 2 \qquad P = \frac{1}{3} \cdot 1 \cdot 2 \cdot 3 = 2 \qquad L=P$

2° Niech $n \in \mathbb{N}$ $\quad \underbrace{\sum_{k=1}^{n} k(k+1) = \frac{1}{3} n(n+1)(n+2)}_{\text{zał. indukcyjne}} \overset{?}{\Rightarrow} \sum_{k=1}^{n+1} k(k+1) = \frac{1}{3}(n+1)(n+2)(n+3)$

$$L = \sum_{k=1}^{n+1} k(k+1) = \sum_{k=1}^{n} k(k+1) + (n+1)(n+2) \overset{\text{zał. ind.}}{=} \frac{1}{3} n (n+1)(n+2) + (n+1)(n+2) =$$

$$= \frac{1}{3}(n+1)(n+2)(n+3) = P$$

3° Na mocy zasady indukcji matematycznej wzór jest prawdziwy dla każdej liczby naturalnej.

b) $\forall n \in \mathbb{N} \quad 6 / 10^n - 4 \qquad$ "6 dzieli 10^n-4"

$1^\circ \quad n=1 \qquad 10^1 - 4 = 10-4 = 6 = 6 \cdot 1$

2° Niech $n \in \mathbb{N} \qquad 6/10^n-4 \overset{?}{\Rightarrow} 6/10^{n+1}-4 \qquad 6/10^n-4 \iff \exists l \in \mathbb{N} \ 10^n-4 = 6 \cdot l$

$$10^{n+1}-4 = 10 \cdot 10^n - 4 \overset{\text{zał. ind.}}{=} 10 \cdot (6l+4) - 4 = 10 \cdot 6l + 40 - 4 = 10 \cdot 6l + 36 = 6(10l+6)$$

3° Na mocy zasady indukcji mat. stwierdzenie jest prawdziwe dla każdego $n \in \mathbb{N}$.

Algebra - Zad. domowe nr. 1 - Struktury alg. Liczby zespolone

(1)

a) Czy "o" łączne w zbiorze $\mathbb{Z}$ $\quad \forall x,y\in\mathbb{Z} \quad x\circ y = x+(-1)^x\cdot y$

$x\circ y\in\mathbb{Z} \qquad x\circ y = x\pm y\in\mathbb{Z}$

Dla $x,y,z\in\mathbb{Z}$

$L=(x\circ y)\circ z = (x+(-1)^x\cdot y)\circ z = x+(-1)^x\cdot y + (-1)^{x+(-1)^x\cdot y}\cdot z =$

$= x+(-1)^x\cdot y + (-1)^{x\pm y}\cdot z$

$P = x\circ(y\circ z) = x\circ(y+(-1)^y\cdot z) = x+(-1)^x\cdot(y+(-1)^y\cdot z) =$

$= x+(-1)^x\cdot y + (-1)^{x+y}\cdot z$

$L=P$ bowiem $(-1)^{x+y} = (-1)^{x-y+2y} = (-1)^{x-y}\cdot\underbrace{(-1)^{2y}}_{=1} = (-1)^{x-y}$

TAK ŁĄCZNE

b) Czy $(\mathbb{R}\setminus\{-1\},\circ)$ to półgrupa / monoid / grupa (przemienna)?

$\forall x,y\in\mathbb{R}\setminus\{\,\} \quad x\circ y := x+y+xy$

"o" wewnętrzne $\quad x\neq -1, y\neq -1 \Rightarrow x+y+xy = x(1+y)+y = x(1+y)+y+1-1$

$= \underset{\neq 0}{(1+y)}\underset{\neq 0}{(x+1)}-1 \neq -1$

przemienne $\quad x\circ y = x+y+xy = y+x+yx = y\circ x$

łączne $\quad x,y,z\in\mathbb{R}\setminus\{-1\}$

$L=(x\circ y)\circ z = (x+y+xy)\circ z = x+y+xy+z+(x+y+xy)\cdot z =$

$= x+y+z+xy+xz+yz+xyz$

$P = x\circ(y\circ z) = x\circ(y+z+yz) = x+y+z+yz+x\cdot(y+z+yz) =$

$= x+y+z+yz+xy+xz+xyz$

$L=P$

el. neutralny $\quad x\circ e = e\circ x = x$

przemienne! $\quad x+e+xe = x, \quad e(1+x)=0 \quad \underline{e=0}$

$e\in\mathbb{R}\setminus\{-1\}$

(symetryczny)
el. odwrotny $\quad x\circ y = e = 0$

$x+y+xy=0, \quad y(1+x) = -x, \quad y=\frac{-x}{1+x} \quad x\neq -1$

$y\neq -1$ bo $\frac{-x}{1+x} = -1$, tj. $y\in\mathbb{R}\setminus\{-1\} \Rightarrow \forall x\neq -1 \ \exists y\neq -1 : y\circ x = x\circ y = e$

⇕

$-x = -1-x$

$0=-1$ sprzeczność

(2) Rozwiąż równania:

a) $z^4 = \dfrac{i-1}{\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}} \qquad z^4 = \dfrac{\sqrt2\left(-\frac{\sqrt2}{2}+\frac{\sqrt2}{2}i\right)}{\cos\frac{\pi}{3}+i\cdot\sin\frac{\pi}{3}} = \dfrac{\sqrt2\left(\cos\frac34\pi+i\sin\frac34\pi\right)}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$

$r^4(\cos4\varphi+i\sin4\varphi) = \sqrt2\left(\cos\left(\frac34\pi-\frac{\pi}{3}\right)+i\sin\left(\frac34\pi-\frac{\pi}{3}\right)\right)$

$r^4=\sqrt2, \quad 4\varphi = \frac{5}{12}\pi+2k\pi \quad k=0,1,2,3$

$r=\sqrt[8]{2}, \quad \varphi_k = \frac{5}{48}\pi + k\cdot\frac{\pi}{2} \quad k=0,1,2,3$

$z_k = r\cdot(\cos\varphi_k + i\sin\varphi_k) \quad k=0,1,2,3$

(2)

b) $9i \cdot z^3 = \bar{z}^5$ $\qquad z = re^{i\varphi}$, $r>0$ $\quad \vee \quad z=0$ ← jest rozwiązaniem

$$9i\, r^3 e^{3i\varphi} = r^5 e^{-5i\varphi} \quad , \quad r^3\left[9e^{\frac{\pi}{2}i}\cdot e^{3i\varphi} - r^2 e^{-5i\varphi}\right]=0 \;/ : r\neq 0$$

$$9e^{\frac{\pi}{2}i} e^{3i\varphi} = r^2 e^{-5i\varphi}$$

$$9e^{8i\varphi} = r^2 e^{-\frac{\pi}{2}i}$$

$r^2 = 9 \;\wedge\; 8\varphi = -\frac{\pi}{2} + 2k\pi$

$r=3 \qquad \varphi_k = -\frac{\pi}{16} + k\cdot\frac{\pi}{4} \qquad k=0,1,\dots,7$

9 rozwiązań: $\quad z=0 \;,\; z_k = 3e^{\varphi_k i} \quad \varphi_k$ j.w.

c) $z^3 - 3z^2 + 6z - 4 = 0$

	1	-3	6	-4
1	1	-2	4	0

$z_0 = 1$

$(z-1)(z^2-2z+4)=0 \;,\; (z-1)\cdot\left[(z-1)^2+3\right]=0$

$(z-1)\left[(z-1)^2-(-3)\right] = (z-1)\left[(z-1)^2-(\sqrt{3}i)^2\right]=0$

$(z-1)(z-1-\sqrt{3}i)(z-1+\sqrt{3}i)$

$z_0=1 \qquad z_1 = 1+\sqrt{3}i \qquad z_2 = 1-\sqrt{3}i$

(3) Zaznacz na płaszczyźnie zbiory

a) $A = \{z\in\mathbb{C}: \; Im(\sqrt{2}\cdot e^{\frac{\pi}{4}i}) \leq |\frac{1}{2}i\cdot\bar{z} - 2 + 6i| < |\sqrt{3}+i|^2$

$\sqrt{2}e^{\frac{\pi}{4}i} = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = 1+i \qquad Im(\sqrt{2}\cdot e^{\frac{\pi}{4}i}) = 1$

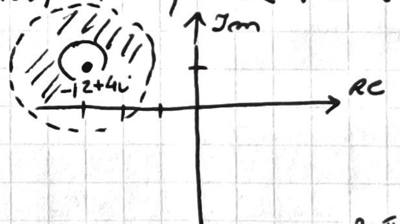
$|\sqrt{3}+i| = \sqrt{3+1} = 2$

tj: $\quad 1 \leq |\frac{1}{2}i\bar{z} - 2 + 6i| < 4$

$|\frac{1}{2}i\bar{z} - 2+6i| = |\frac{1}{2}i(\bar{z} - \frac{4}{i} + 12)| = |\frac{1}{2}|\cdot|i|\cdot|\bar{z}+4i+12| =$

$= \frac{1}{2}\cdot|\overline{z + 12 - 4i}| = \frac{1}{2}|z+12-4i| = \frac{1}{2}|z-(-12+4i)|$

$1 \leq \frac{1}{2}|z-(-12+4i)| < 4 \;,\quad 2 \leq |z-(-12+4i)| < 8$



b) $B = \{z\in\mathbb{C}: \; Im(z^6) < 0\}$

$z^6 = r^6(\cos 6\varphi + i\sin 6\varphi)$

$Im(z^6) = \underbrace{r^6}_{>0} \sin 6\varphi < 0 \iff \sin 6\varphi < 0$

$\pi + 2k\pi < 6\varphi < 2\pi + 2k\pi$

$\frac{\pi}{6} + k\cdot\frac{\pi}{3} < \varphi < \frac{\pi}{3} + k\cdot\frac{\pi}{3}$

$k\in\mathbb{Z}$