Algebra – Zadanie domowe nr 5

Diagonalizacja endomorfizmu

Zad. 1 Czy diagonalizowalny $\varphi \in End(\mathbb{R}_2[x])$ $\varphi(p)(x) = 3x\,p''(x) + (x^2-1)p(1) + x\,p(2)$

$$\varphi(1) = 3x \cdot 0 + (x^2-1)\cdot 1 + x \cdot 1 = -1 + x + x^2$$
$$\varphi(x) = 3x \cdot 0 + (x^2-1)\cdot 1 + x \cdot 2 = -1 + 2x + x^2$$
$$\varphi(x^2) = 3x \cdot 2 + (x^2-1)\cdot 1 + x \cdot 4 = -1 + 10x + x^2$$

$$A = M_\varphi = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 2 & 10 \\ 1 & 1 & 1 \end{bmatrix}$$

$$0 = \det(A - t\cdot I) = \begin{vmatrix} -1-t & -1 & -1 \\ 1 & 2-t & 10 \\ 1 & 1 & 1-t \end{vmatrix} \underset{\substack{w_1+w_2 \\ w_3-w_2}}{=} \begin{vmatrix} -t & 1-t & 9 \\ 1 & 2-t & 10 \\ 0 & t-1 & -t-9 \end{vmatrix} \underset{w_1+w_3}{=} \begin{vmatrix} -t & 0 & -t \\ 1 & 2-t & 10 \\ 0 & t-1 & -t-9 \end{vmatrix} =$$

$$= -t(2-t)(-t-9) - t(t-1) + 10t(t-1) = t\left[(2-t)(t+9) - t + 1 + 10t - 10\right] = t\left[-t^2 + 2t + 9\right]$$

$\Delta = 4 + 36 = 40$, $\sqrt{\Delta} = 2\sqrt{10}$

$t = 0$, $t = 1-\sqrt{10}$, $t = 1+\sqrt{10}$, $t = \frac{-2 \pm 2\sqrt{10}}{-2}$

$Spec(\varphi) = \{0, 1-\sqrt{10}, 1+\sqrt{10}\}$

widmo proste → diagonal.

Zad. 2 Niech $\varphi \in End(\mathbb{R}^3)$ t.że

$$\varphi(1,0,0) = (1,0,0)$$
$$\varphi(1,1,0) = (-1,-1,0)$$
$$\varphi(1,1,1) = (0,0,0)$$

Oblicz $\varphi^{100}(3,6,9)$.

$Spec(\varphi) = \{1, -1, 0\}$

Baza wektorów własnych $C = \{c_1 = (1,0,0),\ c_2 = (1,1,0),\ c_3 = (1,1,1)\}$

B - baza kanoniczna

$$P = P_{B \to C} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{macierz przejścia}$$

$$D = M_{CC}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I SPOSÓB:

$A = M_{BB}(\varphi)$ $\quad A = P \cdot D \cdot P^{-1}$, $\quad A^{100} = P \cdot D^{100} \cdot P^{-1}$

$P^{-1} = ?$

$$[P|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[w_2-w_3]{w_1-w_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{w_1-w_2} \left[I \,\middle|\, \underbrace{\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}}_{P^{-1}}\right]$$

$\varphi^{100}(3,6,9) = ?$

$$A^{100} \cdot \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = P \cdot D^{100} \cdot P^{-1} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{100} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 0 \end{bmatrix}$$

$\varphi^{100}(3,6,9) = (-6,-3,0)$

II SPOSÓB

$(3,6,9) = [-3,-3,9]_C$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{array}\right] \to \left[I \,\middle|\, \begin{array}{c} -3 \\ -3 \\ 9 \end{array}\right]$$

$$D^{100} \cdot \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix}$$

$$\varphi^{100}(3,6,9) = [-3,-3,0]_C = -3(1,0,0) - 3(1,1,0) = (-6,-3,0)$$

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Zad. 3 Diagonalizacja
Baza.

$\varphi : \mathbb{R}^3 \to \mathbb{R}^3$

$\varphi(x,y,z) = (x+2y-2z,\ x+3z,\ x+3y)$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & -\lambda & 3 \\ 1 & 3 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) - 6 + 6 - 2\lambda - 9(1-\lambda) + 2\lambda = (1-\lambda)(\lambda^2-9)$$

$\operatorname{Spec}(\varphi) = \{-3, 1, 3\}$ widmo proste $\Rightarrow \varphi$ - diagonalizowalny

• $\lambda_1 = 1$

$(A-\lambda_1)X = 0$

$$(A-I)X = 0 \iff \begin{bmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 1 & 3 & -1 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow{w_2-w_1} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & -4 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow[w_2/:4]{w_3-w_2} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1+w_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\begin{cases} a + 2c = 0 \\ b - c = 0 \end{cases}$

$c = t$

$b = c = t$

$a = -2c = -2t$

$E_1 = \{(-2t, t, t) : t \in \mathbb{R}\} = \operatorname{lin}_{\mathbb{R}}\{(-2,1,1)\}$

• $\lambda_2 = 3$

$(A-3I)X = 0$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \xrightarrow{w_1:(-2)} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \xrightarrow[w_3-w_1]{w_2-w_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & -4 \end{bmatrix} \xrightarrow{w_2:(-2)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1+w_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\begin{cases} a = 0 \\ b - c = 0 \end{cases}$ $b = c$

$E_3 = \{(0, t, t), t \in \mathbb{R}\} = \operatorname{lin}_{\mathbb{R}}\{(0,1,1)\}$

• $\lambda_3 = -3$

$(A+3I)X = 0$

$$\begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix} \xrightarrow[w_1/2]{w_3-w_2} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2-w_1\cdot 2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

$\begin{cases} a + 3b + 3c = 0 \\ -5b + 7c = 0 \end{cases}$

$a = -3b - 3c = \frac{21}{5}c - 3c = \frac{6}{5}c$

$b = -\frac{7}{5}c$

$c = t$

$E_{-3} = \{(\frac{6}{5}t, -\frac{7}{5}t, t), t \in \mathbb{R}\}$

$\operatorname{lin}_{\mathbb{R}}\{\frac{1}{5}(6, -7, 5)\}$

baza

$\mathcal{B} = \{(-2,1,1), (0,1,1), \frac{1}{5}(6,-7,5)\}$

$$M_{\mathcal{B}\mathcal{B}}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Zadanie domowe – Diagonalizacja

Zad. 3 b) $\varphi \in End(\mathbb{C}^3)$, $\varphi(z_1, z_2, z_3) = (iz_1 - z_1 + 2z_2 + z_3,\ 2iz_2 + 3z_3,\ iz_3 - z_3)$

$A = M_\varphi(\mathcal{B}_k, \mathcal{B}_k)$ $\quad A = \begin{bmatrix} i-1 & 2 & 1 \\ 0 & 2i & 3 \\ 0 & 0 & i-1 \end{bmatrix}$

$\mathcal{B}_k = ((1,0,0),(0,1,0),(0,0,1))$

$\chi_A(t) = \det(A - tI) = (2i - t)(i-1-t)^2$ $\qquad Spec(A) = \{\lambda_1 = -1+i,\ \lambda_2 = 2i\}$, $k_1 = 2$, $k_2 = 1$

$\lambda_1 = -1+i$ $\quad E_{\lambda_1} = ?$ $\quad \begin{bmatrix}0\\0\\0\end{bmatrix} = (A - \lambda_1 I)\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1+i & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} \Rightarrow \begin{cases} 2z_2 + z_3 = 0 \\ (1+i)z_2 + 3z_3 = 0 \end{cases} \Rightarrow z_3 = -2z_2$

$(1+i)z_2 - 6z_2 = 0$

$z_2 = z_3 = 0$

$E_{-1+i} = \{(z_1, 0, 0);\ z_1 \in \mathbb{C}\} = lin_{\mathbb{C}}\{(1,0,0)\}$ $\qquad \dim E_{-1+i} = 1 \neq k_1 = 2$

Macierz **nie** jest diagonalizowalna

$\lambda_2 = 2i$ $\quad E_{2i} = ?$ $\quad \begin{bmatrix}0\\0\\0\end{bmatrix} = (A - 2iI)\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} = \begin{bmatrix} -1-i & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -1-i \end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} \Rightarrow \begin{cases} z_3 = 0 \\ 2z_2 - (1+i)z_1 = 0 \end{cases}$

$E_{2i} = \{(z_1, -\frac{1}{2}(1+i)z_1, 0);\ z_1 \in \mathbb{C}\} = lin_{\mathbb{C}}\{(1, -\frac{1}{2} - \frac{1}{2}i, 0)\}$ $\qquad \dim E_{2i} = 1$

$= lin_{\mathbb{C}}\{(2, -1-i, 0)\}$

Zad. 4

$a \in \mathbb{R}$ $\qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1+a^2 & 0 \end{bmatrix}$

$\chi_A(t) = \det(A - tI) = \begin{vmatrix} -t & 1 & 0 & 0 \\ 0 & -t & -a^2 & 0 \\ 0 & 0 & -t & 1 \\ 1 & 0 & 1+a^2 & -t \end{vmatrix} = -t\begin{vmatrix} -t & -a^2 & 0 \\ 0 & -t & 1 \\ 0 & 1+a^2 & -t \end{vmatrix} + 1\cdot(-1)^5\begin{vmatrix} 1 & 0 & 0 \\ -t & -a^2 & 0 \\ 0 & -t & 1 \end{vmatrix} =$

$= (-t)^2\begin{vmatrix} -t & 1 \\ 1+a^2 & -t \end{vmatrix} - \begin{vmatrix} -a^2 & 0 \\ -t & 1 \end{vmatrix} = t^2(t^2 - 1 - a^2) - (-a^2) = t^4 - t^2 - t^2a^2 + a^2 = t^4 - (1+a^2)t^2 + a^2$

$t^2 = u$ $\qquad u^2 - (1+a^2)u + a^2 = 0$ $\qquad \Delta = (1+a^2)^2 - 4\cdot1\cdot a^2 = 1 + 2a^2 + a^4 - 4a^2 = a^4 - 2a^2 + 1 = (a^2-1)^2$

$\Delta \geq 0$

- $\Delta = 0$ gdy $a^2 = 1$, $a = \pm 1$ $\quad$ wówczas $u = t^2 = \frac{1+a^2}{2} > 0$

$a = \pm 1 \longrightarrow A - tI = \begin{bmatrix} -t & 1 & 0 & 0 \\ 0 & -t & -1 & 0 \\ 0 & 0 & -t & 1 \\ 1 & 0 & 2 & -t \end{bmatrix}$ $\qquad \chi_A(t) = t^4 - 2t^2 + 1 = (t^2-1)^2 = (t-1)^2(t+1)^2$

$\lambda_1 = 1$, $k_1 = 2$, $\lambda_2 = -1$, $k_2 = 2$

$r(A - I) = r\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 2 & -1 \end{bmatrix} = r\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} = r\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$r(A - I) = 3$

$\dim E_1 = 4 - 3 = 1 \neq k_1 = 2$

nie diagonalizowalna

$r(A + I) = r\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} = r\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 2 & 1 \end{bmatrix} = r\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$r(A+I) = 3$

$\dim E_{-1} = 4 - 3 = 1 \neq k_2 = 2$

nie diagonalizowalna

- $\Delta > 0$ gdy $a^2 \neq 1$ $\Rightarrow u_1 \neq u_2$

Aby $u_1 > 0, u_2 > 0$; $\quad \begin{matrix} u_1 \cdot u_2 > 0 \\ u_1 + u_2 > 0 \end{matrix}$ $\overset{\text{wzory Viete'a}}{\Longleftrightarrow}$ $\begin{cases} 1 + a^2 > 0 & \text{zawsze} \\ a^2 > 0 & \text{dla } a \neq 0 \end{cases}$

$x_1 + x_2 = -\frac{b}{a}$

$x_1 \cdot x_2 = \frac{c}{a}$

$u^2 - (1+a^2)u + a^2$

$a = 0$ $\quad A - tI = \begin{bmatrix} -t & 1 & 0 & 0 \\ 0 & -t & 0 & 0 \\ 0 & 0 & -t & 1 \\ 1 & 0 & 1 & -t \end{bmatrix}$ $\quad \det(A - tI) = -t\cdot(-t^3 + t) + (-1)\cdot 0 = t^2(t^2 - 1)$ $\qquad \lambda = 0$ $\ k = 2$

$r(A - 0\cdot I) = r\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = 3$ $\quad \dim E_0 = 1$ nie diag.

- $a \notin \{0, 1, -1\} \rightarrow A$ diagonalizowalna

Zadanie domowe – Diagonalizacja

Zad. 5 $a_1 = a_2 = 2$, $a_3 = 4$, $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, $k=3$

Wzór ogólny?
$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_{n-1} \\ a_{n-2} \\ a_{n-3} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A}^{n-3} \cdot \begin{bmatrix} a_3 \\ a_2 \\ a_1 \end{bmatrix}$$
($n-k$)

$$\chi_A(t) = \det(A - tI) = \begin{vmatrix} 2-t & 1 & -2 \\ 1 & -t & 0 \\ 0 & 1 & -t \end{vmatrix} = t^2(2-t) - 2 + t = -t^2(t-2) + (t-2) = (t-2)(1-t^2) = (t-2)(1-t)(1+t)$$

$Spec(A) = \{-1, 1, 2\}$ widmo proste $\Rightarrow$ A diagonalizowalna

$\lambda_1 = -1$ $(A+I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} b+c=0 \\ a+b=0 \end{matrix}$ $E_{-1} = \{(-b, b, -b)\, b \in \mathbb{R}\}$ $v_1 = (1, -1, 1)$

$\lambda_2 = 1$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (A - I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow a = b = c$ $E_1 = \{(a, a, a);\, a \in \mathbb{R}\}$ $v_2 = (1,1,1)$

$\lambda_3 = 2$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (A - 2I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{matrix} a - 2b = 0 \\ b - 2c = 0 \end{matrix}$ $E_2 = \{(4c, 2c, c);\, c \in \mathbb{R}\}$ $v_3 = (4, 2, 1)$

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = P \cdot D^{n-3} \cdot P^{-1}\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \qquad P^{-1} = ?$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 2 & 6 & 1 & 1 & 0 \\ 0 & 0 & -3 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/3 & 0 & 4/3 \\ 0 & 2 & 0 & -1 & 1 & 2 \\ 0 & 0 & -3 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/3 & 0 & 4/3 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1 \\ 0 & 0 & 1 & 1/3 & 0 & -1/3 \end{array}\right] \to \left[\begin{array}{c|ccc} I_3 & 1/6 & -1/2 & 1/3 \\ & -1/2 & 1/2 & 1 \\ & 1/3 & 0 & -1/3 \end{array}\right]$$
(P^{-1})

$$P^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & 6 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^{n-3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{n-3} \end{bmatrix}\frac{1}{6}\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & 6 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} (-1)^{n-3} & 1 & 2^{n-1} \\ (-1)^{n-2} & 1 & 2^{n-2} \\ (-1)^{n-3} & 1 & 2^{n-3} \end{bmatrix}\begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = ?$$

$$a_n = \frac{1}{6} \cdot \left[2 \cdot (-1)^{n-3} + 6 + 4 \cdot 2^{n-1}\right] = \frac{1}{3} \cdot (-1)^{n-3} + \frac{1}{3} \cdot 2^n + 1$$

Zad. 6

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^5 - A^4 = ? \qquad B = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix} \qquad B^{77} = ?$$

$$\chi_A(t) = \begin{vmatrix} 2-t & 3 & 1 \\ 0 & -t & 0 \\ 0 & -1 & 2-t \end{vmatrix} = -t(2-t)^2 = -t(4 - 4t + t^2) = -t^3 + 4t^2 - 4t \Rightarrow / -A^3 + 4A^2 - 4A = 0$$

$$\begin{array}{l} \quad t^2 + 3t + 8 \\ \hline (t^5 - t^4) : (t^3 - 4t^2 + 4t) \\ t^5 - 4t^4 + 4t^3 \\ \hline 3t^4 - 4t^3 \\ 3t^4 - 12t^3 + 12t^2 \\ \hline 8t^3 - 12t^2 \\ 8t^3 - 32t^2 + 32t \\ \hline 20t^2 - 32t \end{array}$$

$A^5 - A^4 = 20A^2 - 32A$:

$\chi_B(t) = (i - t)^2 \Rightarrow (i \cdot I - B)^2 = 0 \quad -I - 2iB + B^2 = 0$

$$\begin{cases} t^{77} = q(t) \cdot (i - t)^2 + \overbrace{at + b}^{r(t)} \\ 77\, t^{76} = q'(t) \cdot (i-t)^2 + q(t) \cdot 2(i-t) \cdot (-1) + a \end{cases}$$

$t = i$ $\begin{cases} i^{77} = ai + b \\ 77\, i^{76} = a \end{cases}$ $\quad a = (i^4)^{19} \cdot 77 = 77$, $b = i^{77} - ai = i - 77i = -76i$

$B^{77} = 77B - 76iI = \ldots$ policzyć

Zad. 7

$A \in M_3(\mathbb{C})$, $tr A = 2$, $\lambda_1 = 7 \in Spec(A)$, $v = (1, 2, 1)$, $w = (1, 1, 0)$ odpow. λ_1

v, w – lin. niezależne $\Rightarrow \dim E_7 \geq 2$, $k_1 \geq 2$

$2 = tr A = 7 + 7 + \lambda_2 \Rightarrow \lambda_2 = -12$, $k_2 = 1$

$\det A = 7 \cdot 7 \cdot (-12) = -588$